1. **Capacitance problem:** Consider an air-filled parallel plate capacitor with capacitance $C = \frac{1}{36\pi}$ nF (n=nano=10^{-9}).

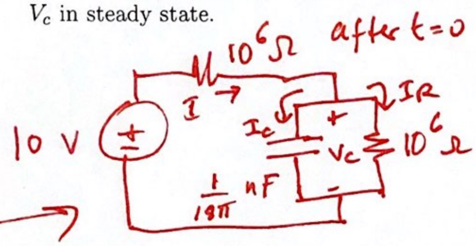
   a) (5 pts) What would be C if the filling of the capacitor were altered by some dielectric with permittivity $\epsilon = 2\epsilon_o$?

   $C \propto \varepsilon \Rightarrow C_{new} = 2C_{old} = 2\times\frac{1}{36\pi}\,nF = \frac{1}{18\pi}\,nF$ //.

   b) (5 pts) What would be conductance of the same capacitor if the conductivity of the dielectric were $\sigma = 10^{-6}$ S/m?

   $\frac{G}{C} = \frac{\sigma}{\varepsilon}$, $\sigma = 10^{-6}$, $\varepsilon = 2\varepsilon_0 = \frac{10^{-9}}{18\pi}$, $C = \frac{10^{-9}}{18\pi}$ $\Rightarrow G = 10^{-6}\,S$ // $\Rightarrow R = \frac{1}{G} = 10^{6}\,\Omega$

   c) (6 pts) If the physical capacitor with the C and G given in (a) and (b) were connected across a 10 V DC source with an internal resistance of $R_s = 10^6\,\Omega$, what would be the capacitor voltage V_c in steady state.



   In steady state $I_c = 0$, $I = I_R$ and by voltage division

   $V_c = 10\,V\,\frac{10^6}{10^6+10^6} = 5\,V$ //.

   d) (3 pts) Given V_c from part (c), what additional information about the capacitor would you need in order to calculate the steady state electric field of the capacitor between its plates?

   $|\bar{E}| = \frac{V_c}{d}$ ← plate separation ← we need place separation to obtain $\bar{E}$ from V_c.

   e) (3 pts) Given V_c from part (c), what would be the steady state charge Q stored on one of the capacitor plates?

   $Q = CV_c = \frac{10^{-9}}{18\pi}\times 5$ Coul //.

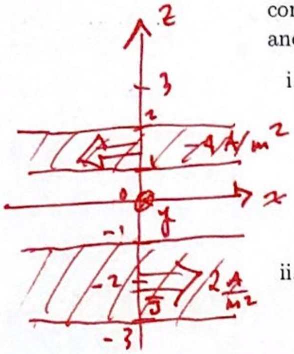
   f) (3 pts) If 10 V source in (c) were disconnected from the capacitor at time $t = 0$, at what later time t_1 would $V_c(t_1) = V_c(0)e^{-1}$? **Hint:** what is the decay time constant of the physical capacitor described above?

   With 10 V source disconnected, capacitor will discharge across its shunt resistor with time constant

   $\tau = RC = (10^6\Omega)\left(\frac{10^{-9}}{18\pi}F\right) = \frac{1}{18\pi}\times 10^{-3}\,s = \frac{1}{18\pi}\,ms$ //.

2. Two parts of this question are independent:

a) We have two infinite slabs of finite widths in the z-direction with $\hat{x}$-directed current flows with constant volumetric current densities of $\mathbf{J}_i = J_{ix}\hat{x}$. Slab 1 occupies the region $-3 < z < -1$ m and has $J_{1x} = 2$ A/m^2 while Slab 2 occupies the region $1 < z < 2$ m and has a $J_{2x} = -4$ A/m^2.



i. (5 pts) What is the magnetic field intensity vector $\mathbf{H}$ on the z-axis at $z = -3$, -1.5, 0, 1.5, 3 m due these current slabs?

$\bar{H}(z=-3) = 0$ because no "net current" above (or below) $z=-3$.

$\bar{H}(z=-1) = -\hat{y}\,4\,\frac{A}{m}$ $\bar{H}(z=-1.5) = -\hat{y}\,3\frac{A}{m}$ $\bar{H}(z=0) = -\hat{y}\,4\,A/m$

ii. (5 pts) Plot $H_y(z)$ as a function of z for $-3 < z < 3$ m — consistent with answers to part (i) — and justify your plot?

$\bar{H}(z=+1.5) = -\hat{y}\,2\,A/m$

$\bar{H}(z=3) = 0$

iii. (5 pts) Re-plot $H_y(z)$ if an additional surface current sheet with -4$\hat{x}$ A/m current density is placed on $z = 1$ m surface?

Add $\hat{y}\,2\frac{A}{m}$ of $\bar{H}$ for $z>1$ and subtract $\hat{y}\,2\frac{A}{m}$ for $z<0$.

See dashed lines above including these changes

b) Consider magnetic vector potential $\mathbf{A}(x, y, z) = -\hat{y}|x-1|$ Wb/m in a free space region containing possible current sheets.

i. (3 pts) Explain why $\nabla \cdot \mathbf{A} = 0$.

$$\nabla\cdot\bar{A} = \cancel{\frac{\partial A_x}{\partial x}}^{\,0} + \frac{\partial A_y}{\partial y} + \cancel{\frac{\partial A_z}{\partial z}}^{\,0} = \frac{\partial A_y}{\partial y} = 0$$

$A_y = -|x-1|$

ii. (4 pts) Determine $\mathbf{B} = \nabla \times \mathbf{A}$. **Hint**: consider $x > 1$ and $x < 1$ cases separately to handle $|x-1|$ in simple terms.

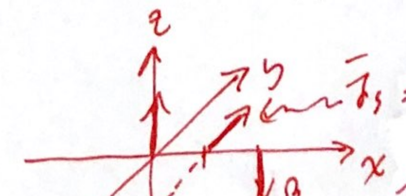
$x>1 \Rightarrow |x-1| = x-1 \Rightarrow \bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 1-x & 0 \end{vmatrix} = \hat{z}\frac{\partial}{\partial x}(1-x) = -\hat{z}$ for $x>1$ m

$x<1 \Rightarrow |x-1| = 1-x$ $\sim \hat{z}\frac{\partial}{\partial x}(x-1) = \hat{z}$ for $x<1$ m

in Wb/m^2

$\bar{B} = -\mathrm{sgn}(x-1)\hat{z}$ Wb/m^2

iii. (3 pts) What would be $\mathbf{J}(x, y, z)$ that would produce $\mathbf{B}$ of (ii)?



$\bar{J}_s = \frac{2}{\mu_0}\hat{y}\frac{A}{m} \Rightarrow \bar{H} = \mp\frac{1}{\mu_0}\hat{z}\frac{A}{m}$

$\bar{B} = \mu_0\bar{H} = \mp\hat{z}\,\frac{Wb}{m^2}$ ✓

By inspection, a surface current at $x=1$

$\Rightarrow \bar{J} = \frac{2}{\mu_0}\delta(x-1)\hat{y}\,\frac{A}{m^2}$

3. Two parts of this question are independent:

a) Given a loop of wire with radius a_1 on the xy-plane and current I in counterclockwise direction when viewed from the top, the magnetic field evaluated on the z-axis is given as

$$\mathbf{B} = \hat{z}\frac{\mu_o I a_1^2}{2z^3} \Rightarrow B_z\Big|_{z=100m} = \frac{\mu_o \frac{2}{\mu_o} 1^2}{2z^3} = \frac{1}{z^3}\Big|_{z=100m} = 10^{-6}$$

for $z \gg a_1$. Let a_1= 1 m, $I = 2/\mu_o$, and use this magnetic field vector to answer the following questions for a small loop with radius a_2=1 cm, encircling the z-axis, and moving upwards with velocity $dz/dt = 1$ m/s in the vicinity of $z = 100$ m.

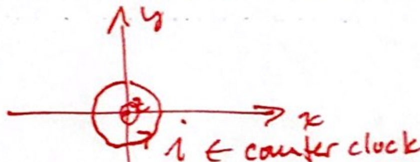
i. (4 pts) Estimate the magnetic flux Ψ linked to the small loop when its height $z = 100$ m. **Hint:** the product of B_z at $z = 100$ m and the area of the small loop would be a good estimate.

$$\Psi(z=100m) = B_z \pi a_2^2 = 10^{-6}\times\pi\times(10^{-2})^2 = \pi\times10^{-10}\ \text{Wb}$$

ii. (4 pts) Estimate the emf $-\frac{d\Psi}{dt}$ of the small loop when it is at $z = 100$ m height?

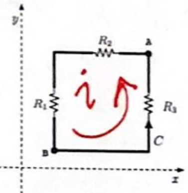
$$\mathcal{E} = -\frac{d}{dt}\Psi = -\pi\times10^{-4}\frac{d}{dt}\frac{1}{z^3} = -\pi\times10^{-4}\frac{(-3)}{z^4}\frac{dz}{dt} = 3\pi\times10^{-4}\times10^{-8} = 3\pi\times10^{-12}\ \text{V}$$

iii. (4 pts) Make a sketch showing (unambiguously) the flow direction of the (positive) current around the small loop when it is moving upwards at $z = 100$ m height? Explain your reasoning.



i ← counter clockwise for the loop moving up into region of weaker Ψ.

b)



Consider the circuit depicted on the left. Resistors $R_1 = 3\,\Omega$, $R_2 = 2\,\Omega$, and $R_3 = 1\,\Omega$ constitute a square loop of 1 m^2 physical area in a region where there is a uniform magnetic field $\mathbf{B} = B_o\hat{z}$. It is known that $dB_o/dt = -6$ T/s. Answer the following questions:

i. (3 pts) What is the emf $\mathcal{E}$ in the circuit in the circulation direction C indicated on the diagram.

$$\mathcal{E} = -\frac{d}{dt}\Psi = -\frac{d}{dt}\left[1m^2\times B_0\right] = -\frac{dB_0}{dt} = +6V$$

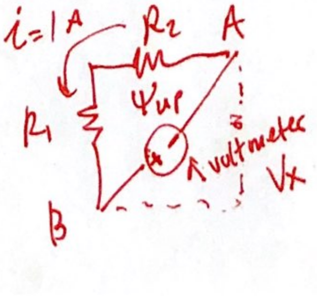
ii. (3 pts) What is the loop current I induced in the circuit in the circulation direction C indicated on the diagram.

$$i = \frac{\mathcal{E}}{R_{tot}} = \frac{6V}{6\Omega} = 1\,A$$

iii. (3 pts) What is the voltage drop across resistor $R_3 = 1\,\Omega$ in the direction of the current flow?

$$V_3 = iR_3 = (1A)(1\Omega) = 1V$$

iv. (4 pts) If a voltmeter is connected *diagonally* between points A and B with the (+) lead contacting point B (and in effect splitting the square loop into a pair of equal area triangles) what will be the voltmeter reading?



$$V_x + \underbrace{iR_2 + iR_1}_{5V} = -\frac{d}{dt}\Psi_{up} = 3V$$

$$\Rightarrow V_x = -2V$$