ECE 329 | Introduction to Electromagnetic Fields | Spring 13

University of Illinois | Gilbert, Kim, Schutt-Aine, Waldrop

Exam 1

Thursday, Feb. 14, 2013 — 7:00-8:15 PM

Name:	
Section:	9 AM 12 Noon 1 PM 2 PM

Please clearly PRINT your name in CAPITAL LETTERS and CIRCLE YOUR SECTION in the above boxes.

This is a closed book exam and calculators are not allowed. Please show all your work and make sure to include your reasoning for each answer. **All answers should include units wherever appropriate**.

Problem 1 (25 points)	
Problem 2 (25 points)	
Problem 3 (25 points)	
Problem 4 (25 points)	
TOTAL (100 points)	

1. Consider a static volumetric charge density $\rho(x, y, z) = 6\epsilon_0\delta(z) + \rho_s\delta(z-4)$ C/m^3 in a given region of free space (having permittivity ϵ_0), where the displacement field in the region $0 < z < 4$ m is known to be $\mathbf{D} = \hat{x}\epsilon_0 + \hat{z}3\epsilon_0$ C/m^2, and $D_z = 2\epsilon_0$ for $z > 4$ m. Furthermore, field $\mathbf{D}$ is uniform in each of the regions $z < 0$, $0 < z < 4$, and $z > 4$.

   a) (3 pts) Determine ρ_s.

   b) (3 pts) Determine $\mathbf{D}$ in the region $z > 4$ m.

   c) (4 pts) Determine $\mathbf{D}$ in the region $z < 0$.

   d) (6 pts) Determine $\mathbf{E}$ in all three regions ($z < 0$, $0 < z < 4$, and $z > 4$).

   e) (5 pts) What is the voltage drop from the point $(0, 0, 0)$ to the point $(0, 0, 4)$?

   f) (4 pts) Is the plane at $z = 4$ m an equipotential surface? Explain.

2. An electric field in free space is given as $\mathbf{E} = \hat{x}yz + \hat{y}zx + \hat{z}xy$ V/m.

a) (5 pts) Determine if the electric field given above is conservative.

b) (10 pts) Determine the electrostatic potential function, $V(x, y, z)$, corresponding to the field $\mathbf{E}$ assuming that the potential at the origin is $V(0, 0, 0) = 0$ V.

c) (5 pts) Determine the total charge contained in a cubic volume $V = 1$ m^3 with vertices at $(x, y, z) = (0, 0, 0)$ and $(1, 1, 1)$ m.

d) (5 pts) Determine the charge density ρ in the region corresponding to the electric field.

3. The region between two infinite, plane parallel, perfectly conducting plates at $z = 0$ and $z = 1$ m is filled with two slabs of perfect dielectric materials having constant electric permittivities $\epsilon_1 = 2\epsilon_o$ for $0 < z < d$ m (region 1) and ϵ_2 for $d < z < 1$ (region 2). The bottom plate is held at constant potential $V(0) = 0$ and the electrostatic field between the plates is known to be

$$\mathbf{E}(z) = \begin{cases} -\dfrac{3\epsilon_2}{8\epsilon_0}\hat{z}, & 0 < z < d \\ -\dfrac{3\epsilon_1}{8\epsilon_0}\hat{z}, & d < z < 1 \end{cases} \quad \text{V/m}$$

a) (4 pts) Verify that the above field satisfies Maxwell's boundary condition regarding **D** at the boundary between the two dielectric slabs.

b) (7 pts) Write the expression for the electrostatic potential $V(z)$ for $0 < z < 1$ m in terms of ϵ_1, ϵ_2, and d.

c) (5 pts) Determine ϵ_2 if the surface charge density on the top plate (at $z = 1$ m) is $\rho_s = 3\epsilon_o$ C/m^2.

d) (3 pts) Does $V(z)$ determined in part (b) satisfy Laplace's equation in the region $0 < z < 1$ m? Explain your answer.

e) (5 pts) What would be the capacitance C of the structure described above if the parallel plates at $z = 0$ and $z = 1$ m were constrained to have finite areas $A = W^2$ (where $W \gg 1$ m) facing one another. In this calculation ignore the fringing fields, and express C as a function of ϵ_1, ϵ_2, d, and A.

4. Consider a co-axial cable of length L that is centered on the origin and lies along the $\hat{z}$ axis. The inner conducting wire has a radius $R = 10$ mm and carries a surface charge density of $-3\epsilon_0$ C/m^2, while the thin, outer conducting shell has a radius $3R = 30$ mm and carries a surface charge density of ϵ_0 C/m^2. The region between the two conductors is filled with a perfect dielectric material characterized by permittivity $\epsilon = 6\epsilon_0$.

   a) (8 pts) Assuming that $R \ll L$ (such that fringing fields at the edges of the cable can be neglected), use Gauss' Law in integral form to determine the electric field in the region between the conductors ($R < r < 3R$) and in the region outside of both conductors ($r > 3R$).

   b) (5 pts) What are the magnitude and unit direction, *in Cartesian coordinates*, of the displacement field $\mathbf{D}$ between the conductors at the *Cartesian* point $P = (-15\,\text{mm}, 20\,\text{mm}, 10\,\text{mm})$?

   c) (8 pts) Assuming the inner conducting wire is held at a constant potential of $V_R = 10$ mV, what is the electrostatic potential V_{3R} on the outer conducting shell? **Hint**: there are two independent ways of solving for V_{3R}.

   d) (4 pts) If the dielectric material between the conductors is not perfect, such that $\sigma \neq 0$, indicate whether the following statements are true or false:

      i. **TRUE/FALSE:** Free electrons inside the material will be accelerated toward the inner conductor via the Lorentz force.

      ii. **TRUE/FALSE:** Current will flow in the $+\hat{r}$ direction.

      iii. **TRUE/FALSE:** The surface charge density on the outer conductor will decrease initially.

      iv. **TRUE/FALSE:** The capacitance of the system will decrease initially.

6. Consider a static volumetric charge density $\rho(x; y; z) = 6\varepsilon_o\ \delta(z)+\rho_s(z - 4)$ C/m^3 in a given region of free space (having permittivity ε_o), where the displacement field in the region $0 < z < 4$ m is known to be $\mathbf{D} = \hat{x}\varepsilon_o + \hat{z}3\varepsilon_o$ C/m^2 for and $D_z = 2\varepsilon_o$ C/m^2 for $z > 4$ m. Furthermore, field **D** is uniform in each of regions $z < 0$, $0 < z < 4$ m, and $z > 4$ m.

a) Determine ρ_s.

b) Determine **D** for the region $z > 4$ m.

c) Determine **D** for the region z < 0.

d) Determine **E** in all three regions ($z < 0$, $0 < z < 4$, and $z > 4$).

e) What is the voltage drop from the point (0,0,0) and the point (0,0,4)?

f) Is the plane at z=4 m an equi-potential surface? Explain.

Solutions

6. Consider a static charge density ρ $\rho(x; y; z) = 6\varepsilon_o\ \delta(z)+\rho_s(z - 4)$ C/m^3 in a given region of free space (having permittivity ε_o), where the displacement field is known to be $\mathbf{D} = \hat{x}\varepsilon_o + \hat{z}3\varepsilon_o$ C/m^2 for $0 < z < 4$ m and $D_z = 2\varepsilon_o$ C/m^2 in the region z>4 m. The volume charge density corresponds to two infinite surfaces at z = 0m and z = 4m with surface charges of 6C/m^3 and ρ_s C/m^3, respectively.

a) Apply boundary conditions at the interface: $\rho = \hat{n}\cdot\left[\mathbf{D}_1 - \mathbf{D}_2\right]$At z = 4m, we have that the surface charge density ρ_s must be equal to the difference between the normal components of **D** on each side of the interface. Then, with $\hat{n} = \hat{z}$, we have:

$$\rho_s = D_z\big|_{z=4^+} - D_z\big|_{z=4^-} = 2\varepsilon_o - 3\varepsilon_o = -\varepsilon_o \frac{C}{m^2}$$

b) In the region z > 4m, we know that $D_z = 2\varepsilon_o$ C/m^2. In addition, since the tangential components of **E** at the interface at z = 4m must be continuous, i.e., $E_x\big|_{z=4^+} = E_x\big|_{z=4^-}$, and assuming that the charged sheets are in a vacuum, we can verify that $D_x\big|_{z=4^+} = D_x\big|_{z=4^-} = \varepsilon_o$ C/m^2. Extending the field to the region z > 4 m, we find that

$$\mathbf{D} = \varepsilon_o \hat{x} + 2\varepsilon_o \hat{z} \ \frac{\text{C}}{\text{m}^2} \quad \text{for } z > 4 \text{ m.}$$

c) Applying boundary conditions at the interface at z = 0 m, we have that $D_z\big|_{z=0^+} - D_z\big|_{z=0^-} = 6$ thus, $D_z\big|_{z=0^-} = 3\varepsilon_o - 6\varepsilon_o = -3\varepsilon_o$ C/m². In addition, since $E_x\big|_{z=0^+} = E_x\big|_{z=0^-}$ and given that the space is a vacuum, we can verify that $D_x\big|_{z=0^-} = D_x\big|_{z=0^+} = \varepsilon_o$. Extending the fields to the region z < 0 m, we find that

$$\mathbf{D} = \varepsilon_o \hat{x} - 3\varepsilon_o \hat{z} \ \frac{\text{C}}{\text{m}^2} \quad \text{for } z < 0 \text{ m.C}$$

d) The displacement field, **D**, is defined to be equal to $\varepsilon_{medium}\mathbf{E}$. Since the region is in free space $\varepsilon_{medium} = \varepsilon_0$. The electric field in the region where z < 0 m is

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o} = \frac{\varepsilon_o \hat{x} + (3\varepsilon_o - 6)\hat{z}}{\varepsilon_o} = \hat{x} - 3\hat{z} \ \frac{\text{V}}{\text{m}}$$

The electric field in the region from z = 0m to z = 4m is

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o} = \frac{\varepsilon_o \hat{x} + 3\varepsilon_o \hat{z}}{\varepsilon_o} = \hat{x} + 3\hat{z} \ \frac{\text{V}}{\text{m}}$$

The electric field in the region where z > 4m is

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o} = \frac{\varepsilon_o \hat{x} + 2\varepsilon_o \hat{z}}{\varepsilon_o} = \hat{x} + 2\hat{z} \ \frac{\text{V}}{\text{m}}$$

e) We can calculate the voltage drop from *(0,0,0)* m to *(0,0,4)* with the following equation

$$\Delta V = V_{(0,0,4)} - V_{(0,0,4)} = -\int_0^4 \mathbf{E} \cdot \mathbf{dl} = -\int_0^4 \frac{\mathbf{D}}{\varepsilon_o} \cdot \mathbf{dl}$$

$$= -\int_0^4 (\hat{x} + 3\hat{z}) \cdot \hat{z} dz = -3z\Big|_0^4 = -12 \ V$$

f) The plane at z=4 m is NOT an equipotential surface. Because of the x-component of the electric field, integration on the top plane in the horizontal direction gives a non-zero result indicating a potential drop on the surface.

2. $\vec{E} = \hat{x}\, yz + \hat{y}\, zx + \hat{z}\, xy$ V/m   free space : $\varepsilon = \varepsilon_0$

ⓐ conservative = curl free :

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix} = \hat{x}(x - x) + \hat{y}(0-0) + \hat{z}(z - z)$$

$$= 0 \quad \therefore \ \vec{E} \text{ is conservative}$$

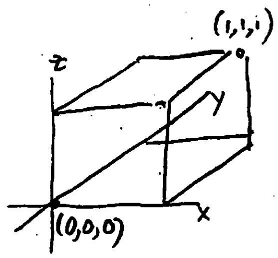
ⓑ $$V_{(x,y,z)} - V_{(0,0,0)} = V(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} -\vec{E}\cdot d\vec{l}$$

$$= \int_{(0,0,0)}^{(x,0,0)} -E_x\Big|_{\substack{y=0\\z=0}} dx + \int_{(x,0,0)}^{(x,y,0)} -E_y\Big|_{\substack{x=x\\z=0}} dy + \int_{(x,y,0)}^{(x,y,z)} -E_z\Big|_{\substack{x=x\\y=y}} dz$$

$$= \int_0^x 0\, dx + \int_0^y 0\, dy + \int_0^z (-xy)\, dz$$

$$= -xyz\Big|_0^z = -xyz \ [V] \ /\!/$$

ⓒ $$\int_V \rho\, dV = ? = \oint_S \vec{D}\cdot d\vec{s}$$



$$= \oint_S \varepsilon_0 \vec{E}\cdot d\vec{s}$$

$$= \int \varepsilon_0 E_x\Big|_{x=1} \hat{x}\cdot\hat{x}\, dy\, dz + \int \varepsilon_0 E_x\Big|_{x=0} \hat{x}\cdot(-\hat{x})\, dy\, dz$$
$$+ \int \varepsilon_0 E_y\Big|_{y=1} \hat{y}\cdot\hat{y}\, dx\, dz + \int \varepsilon_0 E_y\Big|_{y=0} \hat{y}\cdot(-\hat{y})\, dy\, dz$$
$$+ \int \varepsilon_0 E_z\Big|_{z=1} \hat{z}\cdot\hat{z}\, dx\, dy + \int \varepsilon_0 E_z\Big|_{z=0} \hat{z}\cdot(-\hat{z})\, dy\, dz$$

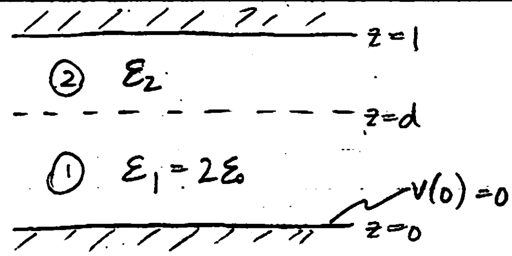
$$\left.\begin{aligned} &= \varepsilon_0\left[\int (yz)\,dy\,dz - (yz)\,dy\,dz\right] \\ &+ \varepsilon_0\left[\int (zx)\,dx\,dz - (zx)\,dx\,dz\right] \\ &+ \varepsilon_0\left[\int (xy)\,dx\,dy - (xy)\,dx\,dy\right] \end{aligned}\right\} = 0 = \int_V \rho\, dV \ /\!/$$

ⓓ $$\rho(x,y,z) = ? = \nabla\cdot\vec{D} = \nabla\cdot(\varepsilon_0\vec{E}) = \frac{\partial}{\partial x}\varepsilon_0 E_x + \frac{\partial}{\partial y}\varepsilon_0 E_y + \frac{\partial}{\partial z}\varepsilon_0 E_z$$

$$= (0) + (0) + (0)$$

$\rho = 0$ everywhere //

3.



$\vec{E}_1 = -\dfrac{3\varepsilon_2}{8\varepsilon_0}\hat{z}$

$\vec{E}_2 = -\dfrac{3\varepsilon_1}{8\varepsilon_0}\hat{z}$

ⓐ boundary condition equ: $\hat{n}\cdot(\vec{D}^+ - \vec{D}^-) = \rho_s = 0$ (since $\sigma = 0$ in both regions)

let $\hat{n} = \hat{z}$: $D_{z_1} - D_{z_2} = 0$

so $\varepsilon_1 E_{z_1} - \varepsilon_2 E_{z_2} = 0 = \dfrac{\varepsilon_1(-3\varepsilon_2)}{8\varepsilon_0} - \dfrac{\varepsilon_2(-3\varepsilon_1)}{8\varepsilon_0}$ ✓

ⓑ $\vec{E}_1 = -\nabla V_1 = \text{constant} \therefore V_1(z) = A_1 z + B_1$ linear function

$\vec{E}_2 = -\nabla V_2 = \text{constant} \therefore V_2(z) = A_2 z + B_2$.

$V(0) = V_1(0) = B_1 = 0$ (given)

$-\nabla V_1 = -A_1\hat{z} = E_{1z}\hat{z}$ so $A_1 = \dfrac{3\varepsilon_2}{8\varepsilon_0}$

$-\nabla V_2 = -A_2\hat{z} = E_{2z}\hat{z}$ so $A_2 = \dfrac{3\varepsilon_1}{8\varepsilon_0}$

$V_1(d) = V_2(d)$ so $A_1 d = A_2 d + B_2 \Rightarrow B_2 = (A_1 - A_2)d$.

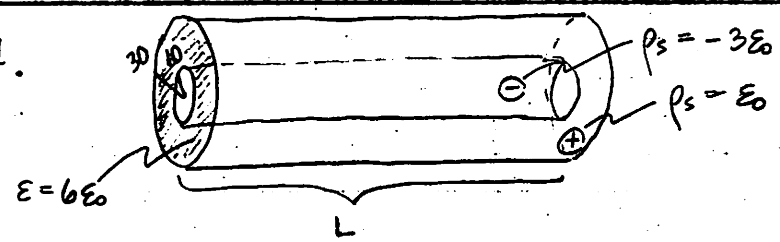
$$V(z) = \begin{cases} \dfrac{3\varepsilon_2}{8\varepsilon_0} z & 0 < z < d \\ \dfrac{3\varepsilon_1}{8\varepsilon_0} z + \left(\dfrac{3\varepsilon_2 - 3\varepsilon_1}{8\varepsilon_0}\right) d & d < z < 1 \end{cases}$$

ⓒ $\dfrac{//\vec{E}=0//}{\vec{E}_2}$ $3\varepsilon_0$ $\quad \hat{z}\cdot(0 - \varepsilon_2\vec{E}_2) = 3\varepsilon_0 = +\dfrac{3\varepsilon_1\varepsilon_2}{8\varepsilon_0}$ $\quad (\varepsilon_1 = 2\varepsilon_0)$

so $\varepsilon_2 = \dfrac{8\varepsilon_0^2}{\varepsilon_1} = 4\varepsilon_0$ //

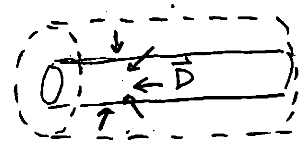
ⓓ NO, not across $0 < z < 1$ since $\nabla^2 V \neq 0$ @ $z = d$.

ⓔ $C = \dfrac{Q}{V} = \dfrac{\rho_s A}{V} = \dfrac{3\varepsilon_0 A}{V}$ where $V = \underbrace{E_{1z}d}_{\text{voltage rise across region ①}} + \underbrace{E_{2z}(1-d)}_{\text{voltage rise across region ②}}$

4. 

ⓐ Gauss law: $\oint_S \vec{D}\cdot d\vec{s} = \int_V \rho\, dV = Q_{encl}$ = inner charge only.

Know $\vec{D}$ is radial: so (S ↖) cylindrical surface of radius $3R < r < R$, length L, centered



$$\oint \vec{D}\cdot d\vec{S} = \int D_r \hat{r}\cdot\hat{r}\, dS = D_r \int dS = D_r(2\pi r L)$$

$$Q_{encl} = \rho_s\, 2\pi R L \quad [C]$$

so $\vec{D} = D_r\hat{r} = \hat{r}\,\rho_s \dfrac{2\pi R L}{2\pi r L} = \hat{r}\,\dfrac{-3\varepsilon_0 R}{r} = \varepsilon E_r \hat{r} = 6\varepsilon_0 E_r \hat{r}$

so $\vec{E} = \hat{r}\left(\dfrac{-R}{2r}\right) = -\hat{r}\dfrac{1}{200r}$ V/m $\quad (R = 10^{-2}\text{m})$

ⓑ $\vec{D} = \varepsilon\vec{E} = -\dfrac{3\varepsilon_0 R}{r}\hat{r} = -\dfrac{3\varepsilon_0(10^{-2})}{r}\hat{r}$

@ (−15 mm, 20 mm, 10 mm) $\quad r = |\vec{r}| = \sqrt{(15)^2 + (20)^2}$ mm

$= \sqrt{(3\cdot5)^2 + (4\cdot5)^2} = 5\sqrt{25} = 25$ mm

$\hat{r} = \dfrac{\vec{r}}{r} = \left(\dfrac{-15}{25}, \dfrac{20}{25}, 0\right)$. cylindrical radius!

so $\vec{D} = -\dfrac{3\varepsilon_0(10^{-2})}{25(10^{-3})}\left(\dfrac{-15}{25}, \dfrac{20}{25}, 0\right) \Rightarrow |\vec{D}| = \dfrac{6}{5}\varepsilon_0$ C/m²

unit dir $= \left(\dfrac{3}{5}, \dfrac{-4}{5}, 0\right)$

ⓒ $V_{3R} - V_R = \displaystyle\int_R^{3R} -\vec{E}\cdot d\vec{l} = \int_R^{3R} \frac{1}{200r}\hat{r}\cdot\hat{r}\, dr = \frac{1}{200}\left(\ln r\Big|_R^{3R}\right) = \frac{1}{200}\ln\left(\frac{3R}{R}\right) = \frac{\ln 3}{200}$ [V]

so $V_{3R} = \dfrac{\ln(3)}{200} + \dfrac{1}{100}$ [V]

$= \varepsilon_0(2\pi(3R)L) = +3\varepsilon_0(2\pi R L)$

–or– $C = \dfrac{Q}{\Delta V} = \varepsilon L \dfrac{2\pi}{\ln\left(\frac{3R}{R}\right)}$ so $\Delta V = \dfrac{\ln(3)\, Q}{6\varepsilon_0 L 2\pi} = \ln(3)\dfrac{R}{2} = \dfrac{\ln(3)}{200}$ [V]

ⓓ (i) FALSE. $\vec{F} = -e\vec{E} \propto +\hat{r}$ $\quad$ (ii) FALSE. $\vec{J} = \sigma\vec{E} \propto -\hat{r}$

(iii) TRUE. +Q gains −e $\quad$ (iv) FALSE. $C = \dfrac{Q}{V}$ is constant. (see above)