ECE 329 | Introduction to Electromagnetic Fields | Fall 12

University of Illinois | Kim, Heil, Waldrop, Wasserman

Exam 2

Thursday, Oct. 25, 2012 — 7:00-8:15 PM

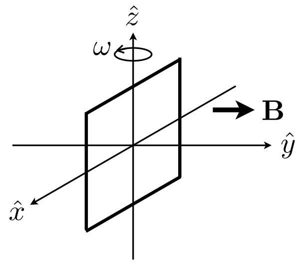
Name:	
Section:	8 AM 10 AM 12 Noon 1 PM

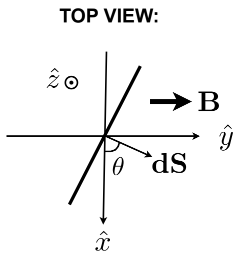
Please clearly PRINT your name in CAPITAL LETTERS and CIRCLE YOUR SECTION in the above boxes.

This is a closed book exam and calculators are not allowed. Please show all your work and make sure to include your reasoning for each answer. All answers should include units wherever appropriate.

Problem 1 (25 points)	
Problem 2 (25 points)	
Problem 3 (25 points)	
Problem 4 (25 points)	
TOTAL (100 points)	

1. Consider the following simple AC generator, where a square loop of wire having surface area A is mechanically rotated around the $\hat{z}$ axis at a constant frequency $\omega > 0$. In the presence of a static magnetic field $\mathbf{B} = B_0\hat{y}$ Wb/m^2, where $B_0 > 0$, the changing magnetic flux through the loop induces an alternating electric current around the loop (see figure below):





a) (5 pts) Using the surface vector orientation specified in the "top view" diagram on the right (where $\theta = \omega t$ is defined to be the time-varying angle of $\mathbf{dS}$ with respect to the $\hat{x}$ axis), write the expression for the magnetic flux Φ through the loop.

b) (5 pts) Write the expression for the electromotive force $\mathcal{E}$ around the loop that is induced by the time-varying magnetic flux through it.

c) (2 pts) What loop orientation angle θ corresponds to $\mathcal{E} = 0$?

(i) $\theta = 0$ **(ii)** $\theta = \frac{\pi}{2}$ **(iii)** $\theta = \pi$ **(iv)** $\theta = \frac{3\pi}{2}$ **(v)** *both* (i) *and* (iii) **(vi)** *both* (ii) *and* (iv)

d) (3 pts) For the loop orientation shown in the "top view" diagram on the right, where $\frac{d\theta}{dt} > 0$, in what direction will the induced current flow? Indicate your answer by drawing an arrow along the top segment of the loop shown in the figure that is consistent with current flow in that segment.

e) (2 pts each) Under which of the following conditions would **the direction of the induced current that you found in part (d)** ***remain the same*** (assuming the loop orientation remains the same as in the "top view" figure above)? Circle all of the following conditions that apply:

i. ... if the magnetic field direction is reversed, i.e., $\mathbf{B}$ now points along $-\hat{y}$

ii. ... if the orientation of the $\mathbf{dS}$ vector is reversed from that shown in the figure

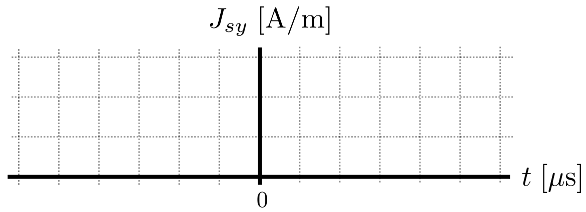
iii. ... if θ is redefined as the angle between the $\mathbf{dS}$ shown and the $\hat{y}$ axis

iv. ... if the loop rotates in the opposite direction, such that $\omega = \frac{d\theta}{dt} < 0$

v. ... if the loop is not rotating, but B_0 is increasing in time

2. On the $z = 0$ plane in a perfect, non-magnetic, dielectric having $\epsilon = 9\epsilon_0$, a current sheet density $\mathbf{J_s}(t) = f(t)\hat{y}$ A/m is generated, where $f(t) = 3\Delta(\frac{t}{\tau})u(t)$, and $\tau = 2$ μs.

   a) (3 pts) Plot $J_{sy}(t)$ below. Be sure to appropriately label all axes!



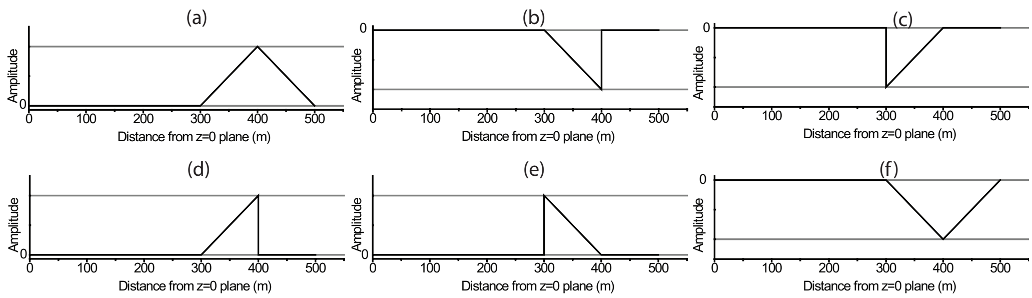
   b) (2 pts) At what time t_0 does a sensor located at at $z = 200$ m detect the electromagnetic wavefront that is emitted from the current sheet at $z = 0$?

   c) (6 pts) Write the expression for the electric and magnetic fields ($\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$, respectively) as a function of time and position above and below the $z = 0$ plane.

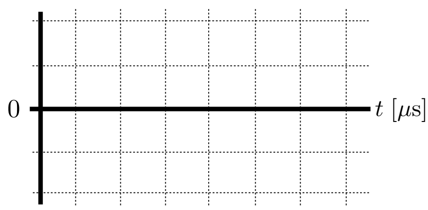
   d) (8 pts) Which of the six plots below – labeled (a) through (f) – depicts the following waveforms given in (i) through (iv)? Note that the horizontal axis in all plots corresponds to $|z|$, the distance from the $z = 0$ plane. Also, labels can be used more than once!

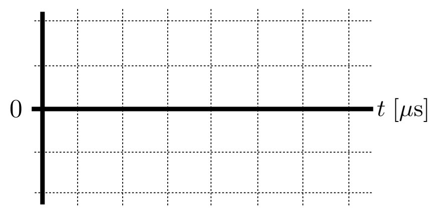
   (i) $\mathbf{H}^+(z > 0,\ t = 4\ \mu\text{s})$: _____ (iii) $\mathbf{E}^+(z > 0,\ t = 4\ \mu\text{s})$: _____

   (ii) $\mathbf{H}^-(z < 0,\ t = 4\ \mu\text{s})$: _____ (iv) $\mathbf{E}^-(z < 0,\ t = 4\ \mu\text{s})$: _____



   e) (6 pts) Plot the electric and magnetic field strength as a function of time at $z = 400$ m. Be sure to appropriately label all axes!





3. Consider an infinite slab (extending in y and z directions) of a finite width $W = 3$ m described by $-1 < x < 2$. The slab is electrically neutral but it conducts a uniform current density of $\mathbf{J} = 2\hat{y}$ A/m^2. Outside the slab, that is for $x > 2$ and $x < -1$, the charge and current densities are zero.

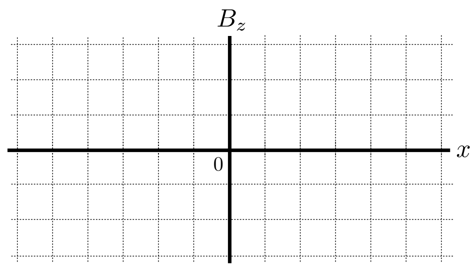
   a) (5 pts) Using the right-hand-rule and Biot-Savart law, discuss why the current slab should generate equal and opposite directed magnetic fields in $\pm\hat{z}$ directions in front of and behind the plane of symmetry of the slab.

   b) (5 pts) What is $\mathbf{B}$ on the $x = 0.5$ m plane? Briefly explain the reasoning behind your answer.

   c) (5 pts) Make use of the integral form of Ampere's law, and the deductions of parts (a) and (b), to find $B_z(x)$ in the regions outside of the slab.

   d) (5 pts) Use Ampere's law to find $B_z(x)$ within the current slab.

   e) (5 pts) Plot B_z as a function of x over $-3 < x < 3$. Label all relevant values of B_z and x.



4. (25 points) For each of the four plane waves (in free space) described by

$$(1)\ \mathbf{E_1} = -4\cos(\omega t + \beta z)\hat{y}\ \mathrm{V/m}$$

$$(2)\ \mathbf{E_2} = E_0 \sin(\omega t + \beta x)\hat{y} - E_0 \cos(\omega t + \beta x)\hat{z}\ \mathrm{V/m}$$

$$(3)\ \mathbf{H_3} = \sin(\omega t + \beta z + \tfrac{\pi}{3})\hat{x} + \cos(\omega t + \beta z - \tfrac{\pi}{6})\hat{y}\ \mathrm{A/m}$$

$$(4)\ \mathbf{H_4} = \sin(\omega t - \beta z - \tfrac{\pi}{2})\hat{x} + \cos(\omega t - \beta z)\hat{y}\ \mathrm{A/m}$$

a) (8 pts) Determine the phasor expression for the given four waves.

b) (8 pts) Determine the expression for **H** or **E** that accompanies the given four waves.

c) (3 pts) Find the instantaneous power that crosses a 1 m^2 area in the xy-plane from $-z$ to $+z$ for $\mathbf{E_1}$ and $\mathbf{H_4}$.

d) (6 pts) Find the time averaged power that crosses a 1 m^2 area in the xy-plane from $-z$ to $+z$ for $\mathbf{E_2}$ and $\mathbf{H_3}$.

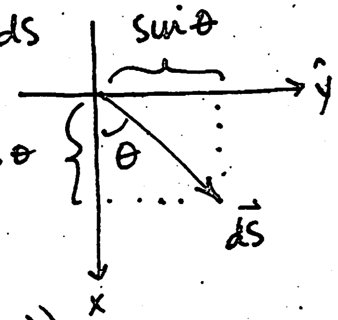
#1 AC generator via rotating loop in static $\vec{B} = B_0\hat{y}$

ⓐ $\Phi = \int \vec{B}\cdot\vec{ds}$ where $\vec{ds} = (\hat{x}\cos\theta + \hat{y}\sin\theta)ds$

$= \int B_0\hat{y}\cdot\hat{y}\sin\theta\, ds$

$= B_0 \sin\theta \int ds$

$= B_0 \sin\theta\, A$ [Wb]

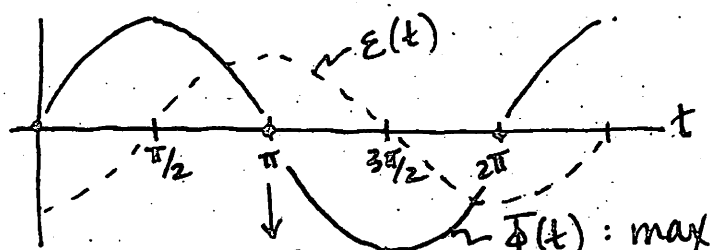


ⓑ $\mathcal{E} = -\frac{d\Phi}{dt} = -B_0 A \frac{d}{dt}(\sin\theta) = -B_0 A \frac{d}{dt}(\sin(\omega t))$ since $\theta = \omega t$.

$= -B_0 A \cos(\omega t)\,\omega$ [V]

ⓒ $\mathcal{E} = 0$ when $\cos(\omega t) = \cos(\theta) = 0$. ... at $\theta = \pi/2$ and $3\pi/2$ (iv)

NOTE! $\Phi(t)$ is constant at this angle w/ respect to time:



* assuming $\theta(t=0) = 0$

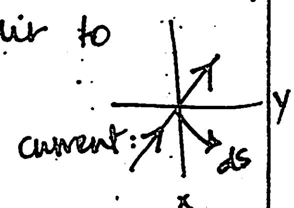
$\Phi(t)$: max slope at π, 2π, etc.

max $\mathcal{E}$ at max $\frac{d\Phi}{dt}$.

ⓓ for $0 < \theta < \pi/2$ (as in figure) and $\frac{d\theta}{dt} > 0$, $\mathcal{E} < 0$.

so induced current is <u>opposite</u> to contour (ccw given $\vec{ds}$)

Lenz Law: need to generate induced $\vec{B}$ in $-\hat{y}$ dir to counteract the increase in flux...



ⓔ (i) FALSE. if $B_0 < 0$ then $\mathcal{E} > 0$.

* (ii) TRUE. now $\mathcal{E} > 0$ but contour is cw not ccw.

* (iii) TRUE. assuming dir. of rotation is same (so $\frac{d\theta}{dt} < 0$)

(iv) FALSE. opp. rotation ... $\mathcal{E} > 0$ now.

* (v) TRUE. need current to generate $\vec{B}$ to <u>oppose</u> increase in flux.

(#2) current generating TEM wave : $\vec{J}_s(t) = \hat{y}\, 3\Delta\left(\frac{t}{\tau}\right)u(t)$

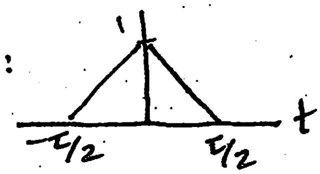
propagating in material: (where $\tau = 2\,\mu s$)

$\mu = \mu_0$ (non-magnetic) and $\varepsilon = 9\varepsilon_0$ (perf. dielectric)

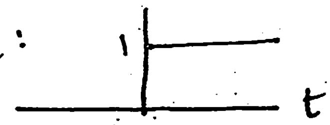
so: $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0 \cdot 9\varepsilon_0}} = \frac{1}{3}\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{c}{3} = 100\ ^{m}/_{\mu s}$.

$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{9\varepsilon_0}} = \frac{1}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{120\pi}{3} = 40\pi\ \Omega$

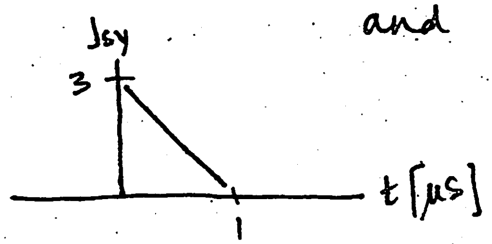
(a) plot $J_{sy}(t)$: NOTE: $\Delta\left(\frac{t}{\tau}\right)$ looks like :



and $u(t)$ looks like :



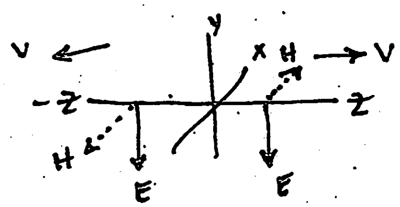
so



(b) $t_0 = z/v = 200/100 = 2\ \mu s$.

(c) $\vec{E}(z,t) = -\hat{y}\frac{\eta}{2}J_s(t \mp z/v) = -\hat{y}\, 20\pi \cdot 3\,\Delta\left(\frac{t \mp z/v}{\tau}\right)u(t \mp z/v)$ [V/m]

$\vec{H}(z,t) = \pm\hat{x}\,\frac{1}{2}J_s(t \mp z/v) = \pm\hat{x}\,\frac{3}{2}\,\Delta\left(\frac{t \mp z/v}{\tau}\right)u(t \mp z/v)$ [A/m]

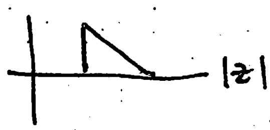


$\hat{e} \times \hat{h} = \hat{v}$

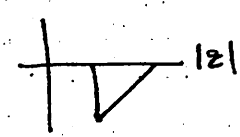
$-\hat{y} \times \pm\hat{x} = \pm\hat{z}$

(d) what happened first at the origin propagated farthest

so all waveforms with zero magnitude leading edge are invalid :



and



also, all symmetric Δ functions are invalid.

(i) $H^+(z>0, t_0)$ in $+\hat{x}$ dir $\longrightarrow$ (d)

(ii) $H^-(z<0, t_0)$ in $-\hat{x}$ dir $\longrightarrow$ (b)

(iii) $E^+(z>0, t_0)$ in $-\hat{y}$ dir $\longrightarrow$ (b)

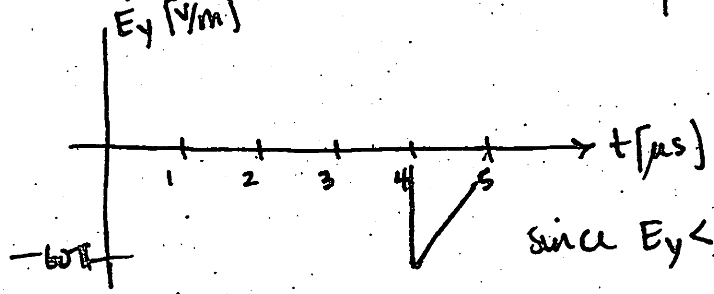
(iv) $E^-(z<0, t_0)$ in $-\hat{y}$ dir $\longrightarrow$ (b)

#2 con't

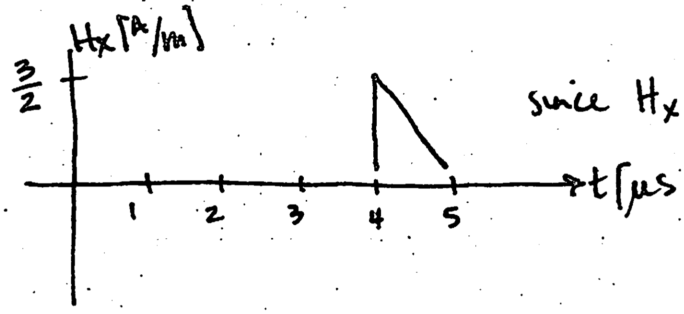
(e) what happened <u>first</u> at the source will happen <u>first</u> away from the source:

prop. to 400 m:

takes $\frac{400\,m}{100\,m/\mu s} = 4\,\mu s$.



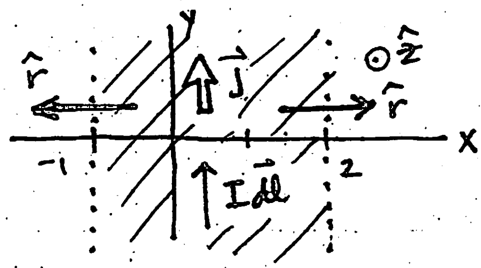
since $E_y < 0$ for $z \geq 0$.



since $H_x > 0$ for $z > 0$

#3 current slab. $\vec{J} = 2\hat{y}$ for $-1 < x < 2$.

(a) Biot Savart Law: field due to infinitesimal current filament: $d\vec{B} = \dfrac{\mu_0 I \vec{dl} \times \hat{r}}{4\pi r^2}$



for $x > 1/2$, $\hat{r} = +\hat{x}$

$x < 1/2$, $\hat{r} = -\hat{x}$

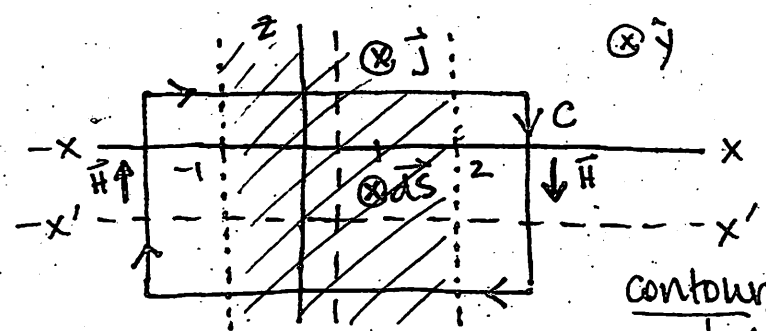
so $\vec{dB}$ is along $\hat{y} \times (\pm\hat{x}) = \mp\hat{z}$ for $x \gtrless 1/2$ (RH rule)

For r = distance along $\pm\hat{x}$ from $x = 1/2$ (midplane), then $|\vec{dB}|$ is the same on both sides.

(b) $\vec{B}(x = 1/2) = 0$ since current is equal on two sides of the midplane $x = 1/2$. superposition of field due to two sides cancels

(c) Ampere's Law: $\oint_c \vec{H} \cdot \vec{dl} = \int_s \vec{J} \cdot \vec{ds} + \cancel{\frac{d}{dt}\int \vec{D} \cdot \vec{ds}}$ → 0 since $\vec{B}$ is static

choose centered contour: define $x' = x - 1/2$.



to find field outside, need $x' > 3/2$ $(x > 2)$ & $x' < 3/2$ $(x < -1)$

<u>contour dimensions</u>:

height: Δz

width: $2|x'|$

$\vec{ds} = \hat{y}\, ds$.

current enclosed:

$I_{encl} = \int \vec{J} \cdot \vec{ds} = \int (2\hat{y}) \cdot (\hat{y}\, ds)$

$= 2\Delta z (2 - (-1)) = 6\Delta z$

$\oint_c \vec{H} \cdot \vec{dl} = \int |H_z(x' > 3/2)| (-\hat{z}) \cdot (-\hat{z}\, dz) + \int |H_z(z' < 3/2)| (\hat{z}) \cdot (\hat{z}\, ds)$

$= \Delta z \left[|H_z(x' > 3/2)| + |H_z(x' < 3/2)| \right] = \Delta z\, 2\, |H_z(x')|$ since $|H_z(x')| = |H_z(-x')|$

so $|H_z(x')| = \dfrac{6\Delta z}{2\Delta z} = 3$ so $H(x) = \begin{cases} -\hat{z}\, 3 & \text{for } x' > 3/2 \ (x > 2) \\ +\hat{z}\, 3 & \text{for } x' < 3/2 \ (x < -1) \end{cases}$ [A/m]

and $B_z(x) = H_z(x) \mu_0$ $\left[\frac{Wb}{m^2}\right]$

#3 cont

(d) Ampere's law: as with diagram on prev. page, use centered contour, but extends inside slab only.

still define $x' = x - 1/2$. need $0 < x' < 3/2 \rightarrow 1/2 < x < 2$

$-3/2 < x' < 0 \rightarrow -1 < x < 1/2$

as before,

$$\oint_c \vec{H} \cdot \vec{dl} = \Delta z \, 2 \, |H_z(x')|$$

but now current enclosed is smaller (function of $|x'|$):

$$\int \vec{J} \cdot \vec{ds} = 2 \Delta z (|x'| 2)$$

so $|H_z(x')| = 2|x'|$ so $H_z(x') = \begin{cases} -\hat{z}\, 2x' & 0 < x' < 3/2 \\ -\hat{z}\, 2x' & -3/2 < x' < 0 \end{cases}$

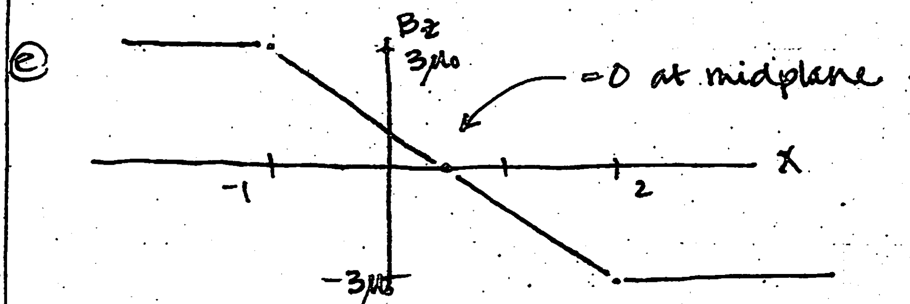
as before $B_z = H_z \mu_0$

change coordinates

$$H_z(x) = \begin{cases} -\hat{z}\, 2(x - 1/2) & 1/2 < x < 2 \\ -\hat{z}\, 2(x - 1/2) & -1 < x < 1/2 \end{cases}$$

[NOTE. $H_z > 0$ for $-1 < x < 1/2$.]

(e)



* double check shifted solution given in formula sheet:

$$\vec{B} = \begin{cases} -\hat{z}\, \mu_0 \underbrace{(2)}_{J_0} \underbrace{\left(\tfrac{3}{2}\right)}_{w/2} & x > [\underbrace{3/2}_{w/2} + \underbrace{1/2}_{\text{shifted}}] \text{ or } (x - 1/2) > \underbrace{3/2}_{w/2} \\ -\hat{z}\, \mu_0 (2)(x - 1/2) & -3/2 < (x - 1/2) < 3/2 \text{ or } -1 < x < 2 \\ +\hat{z}\, \mu_0 (2)\left(\tfrac{3}{2}\right) & (x - 1/2) < 3/2 \end{cases}$$

<u>verified</u>.

#4

(a) $\tilde{E}_1 = -4e^{j\beta z}\hat{y}$ V/m

$\tilde{E}_2 = -jE_o e^{j\beta x}\hat{y} - E_o e^{j\beta x}\hat{z}$ V/m

$\tilde{H}_3 = -je^{j\beta z}e^{j\pi/3}\hat{x} + e^{j\beta z}e^{j\pi/6}\hat{y}$ A/m $\quad -j = e^{-j\pi/2}$

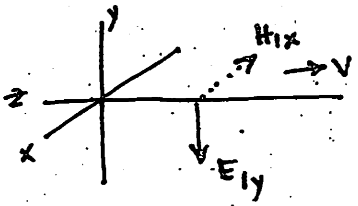
$\quad\quad = e^{j\beta z}\left(e^{-j\pi/2}e^{j\pi/3}\hat{x} + e^{j\pi/6}\hat{y}\right)$

$\quad\quad = e^{j\beta z}e^{j\pi/6}(\hat{x}+\hat{y})$

$\tilde{H}_4 = -je^{-j\beta z}e^{j\pi/2}\hat{x} + e^{-j\beta z}\hat{y}$ A/m $\quad -je^{j\pi/2} = e^{-j\pi} = -1$

$\quad\quad = e^{-j\beta z}(-\hat{x}+\hat{y})$

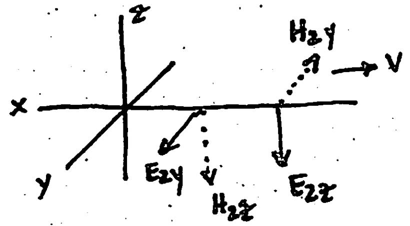
(b) (1) : $\hat{v} = -\hat{z}$



$\tilde{H}_1 = \frac{4}{\eta_o}e^{j\beta z}(-\hat{x})$ A/m

$\vec{H} = -\frac{4}{\eta_o}\cos(\omega t + \beta z)\hat{x}$ A/m

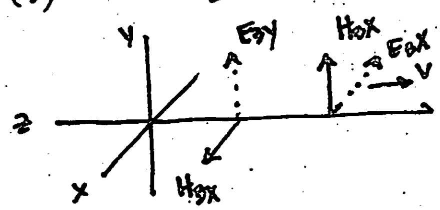
(2) : $\hat{v} = -\hat{x}$



$\tilde{H}_2 = -j\frac{E_o}{\eta_o}e^{j\beta x}(-\hat{z}) + \frac{E_o}{\eta_o}e^{j\beta x}(-\hat{y})$ A/m

$\vec{H}_2 = -\frac{E_o}{\eta_o}\sin(\omega t + \beta x)\hat{z} - \frac{E_o}{\eta_o}\cos(\omega t + \beta x)\hat{y}$ A/m

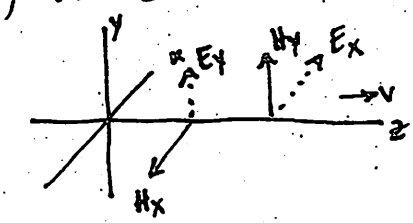
(3) : $\hat{v} = -\hat{z}$



$\tilde{E}_3 = \eta_o e^{j\beta z}e^{j\pi/6}(\hat{y}-\hat{x})$ V/m

$\vec{E}_3 = \eta_o\cos(\omega t + \beta z - \pi/6)\hat{y} - \eta_o\cos(\omega t + \beta z - \pi/6)\hat{x}$

(4) $\hat{v} = +\hat{z}$



$\tilde{E}_4 = \eta_o e^{-j\beta z}(\hat{y}+\hat{x})$ V/m

$\vec{E}_4 = \eta_o\cos(\omega t - \beta z)\hat{x} + \eta_o\underbrace{\cos(\omega t - \beta z)}_{= -\sin(\omega t - \beta z - \pi/2)}\hat{y}$ V/m

#4 cont

(c) instaneous power (time dependent) crossing surface:

$= \int \vec{S}\cdot d\vec{s}$ (where $d\vec{s} = \hat{z}\,dx\,dy$ and $\int ds = 1\ m^2$).

$$(1):\quad \vec{S}_1 = \underbrace{\left(-4\cos(\omega t+\beta z)\hat{y}\right)}_{\vec{E}_1(z,t)} \times \underbrace{\left(\frac{-4}{\eta_0}\cos(\omega t+\beta z)\hat{x}\right)}_{\vec{H}_1(z,t)} \quad [W/m^2]$$

$$= \frac{16}{\eta_0}\cos^2(\omega t+\beta z)\underbrace{(\hat{y}\times\hat{x})}_{-\hat{z}} \quad (\text{check}: = \hat{v}?\ \text{YES.})$$

$$\text{so } \int \vec{S}_1\cdot d\vec{s} = \int \frac{16}{\eta_0}\cos^2(\cdots)(-\hat{z})\cdot +\hat{z}\,dx\,dy$$

$$= -\frac{16}{\eta_0}\cos^2(\omega t+\beta z)(1)\quad [W].$$

$$(4):\quad \vec{S}_4 = \left[\eta_0\cos(\omega t-\beta z)\hat{x} - \eta_0 \overset{\sin}{\cancel{\cos}}(\omega t-\beta z-\pi/2)\hat{y}\right]\times\left[\sin(\omega t-\beta z-\pi/2)\hat{x} + \cos(\omega t-\beta z)\hat{y}\right]$$

$$= \eta_0\cos(\omega t-\beta z)\underbrace{\left[(\hat{x}+\hat{y})\times(-\hat{x}+\hat{y})\right]}_{\hat{z}+\hat{z}\ =\ 2\hat{z}} \quad (\text{check: both} = \hat{v}?\ \text{YES})$$

$$= 2\eta_0\cos^2(\omega t-\beta z)\quad [W].$$

(d) time avg power $\langle S\rangle = \frac{1}{2}\mathrm{Re}\{\tilde{E}\times\tilde{H}^*\}$ crossing surface: $d\vec{s} = \hat{z}\,dx\,dy$

$$(2):\quad \langle S_2\rangle = \frac{1}{2}\mathrm{Re}\{\tilde{E}_2\times\tilde{H}_2^*\} = \frac{1}{2}\mathrm{Re}\left\{E_0 e^{j\beta x}(-j\hat{y}-\hat{z})\times\frac{E_0}{\eta_0}e^{-j\beta x}(-j\hat{z}-\hat{y})\right\}$$

$$= \frac{1}{2}E_0^2\frac{1}{\eta_0}\mathrm{Re}\{\underbrace{-(\hat{y}\times\hat{z})}_{-\hat{x}} + \underbrace{(-\hat{z}\times-\hat{y})}_{-\hat{x}}\} \quad (\rightarrow \hat{v} = -\hat{x}\ \checkmark)$$

$$= -\frac{E_0^2}{\eta_0}\hat{x}\quad [W/m^2].$$

$(\hat{x}\cdot\hat{z}) = 0$

$\int\langle S_2\rangle\cdot d\vec{s} = 0$!

$$(3):\quad \langle S_3\rangle = \frac{1}{2}\mathrm{Re}\left\{\eta_0 e^{j\beta z}e^{j\pi/6}(\hat{y}-\hat{x})\times e^{-j\beta z}e^{-j\pi/6}(\hat{x}+\hat{y})\right\}$$

$$= \frac{\eta_0}{2}\left[\underbrace{(\hat{y}\times\hat{x})}_{-\hat{z}} + \underbrace{(-\hat{x}\times\hat{y})}_{-\hat{z}}\right] \quad (\rightarrow \hat{v} = -\hat{z}\ \checkmark)$$

$$= \eta_0\hat{z}\quad [W/m^2].\quad \text{so } \int\langle S_3\rangle\cdot d\vec{s} = \eta_0\ [W]$$