**ECE 329** **Introduction to Electromagnetic Fields** **Fall 12**

**University of Illinois** **Kim, Heil, Waldrop, Wasserman**

# Exam 1

**Thursday, Sept 27, 2012 — 7:00-8:15 PM**

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|---|---|
| Name: | |
| Section: | 8 AM 10 AM 12 Noon 1 PM |

Please clearly PRINT your name in CAPITAL LETTERS and CIRCLE YOUR SECTION in the above boxes.

This is a closed book exam and calculators are not allowed. Please show all your work and make sure to include your reasoning for each answer. All answers should include units wherever appropriate.

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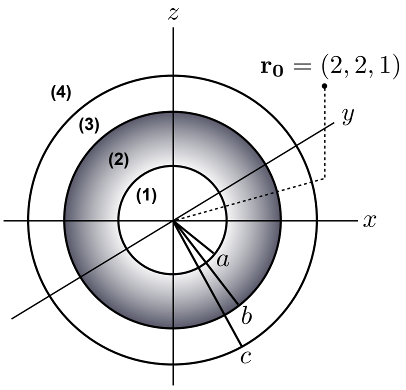
|---|---|
| Problem 1 (25 points) | |
| Problem 2 (25 points) | |
| Problem 3 (25 points) | |
| Problem 4 (25 points) | |
| TOTAL (100 points) | |

1. Consider the following spherically symmetric configuration of composite materials in steady state equilibrium: a metal sphere holding a total charge $Q = +4$ C is surrounded by a perfect silicone dielectric (i.e., glass) shell, which is in turn surrounded by a spherical shell of metal holding a total charge $Q = -2$ C. The entire configuation is embedded in a vacuum. The figure below illustrates the configuration of the materials and lists the material parameters for each region.



Region (1): $r < a$
$\sigma = 10^4$ $\epsilon = 3\epsilon_0$

Region (2): $a < r < b$
$\sigma = 0$ $\epsilon = 4\epsilon_0$

Region (3): $b < r < c$
$\sigma = 10^4$ $\epsilon = 3\epsilon_0$

Region (4): $r > c$
$\sigma = 0$ $\epsilon = \epsilon_0$

a) (2 pts) What is the displacement field $\mathbf{D}(\mathbf{r})$ in region (1)?

b) (5 pts) What is the polarization field $\mathbf{P}(\mathbf{r})$ in region (2)?

c) (4 pts) What is the charge density on the spherical surface at $r = b$?

d) (4 pts) What is the charge density on the spherical surface at $r = c$?

e) (6 pts) What is the electric field $\mathbf{E}$ at the position $\mathbf{r_0} = (2, 2, 1)$ m located in region (4)?

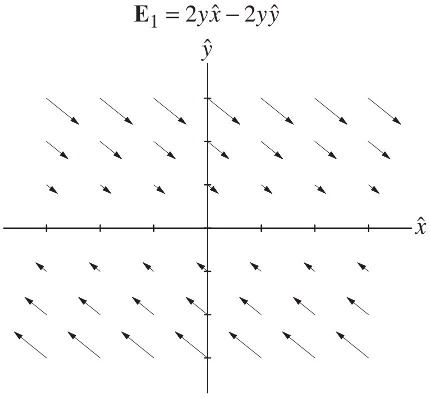
f) (1 pt) **TRUE** or **FALSE**: $\mathbf{E} = -\nabla V$ in region (3).

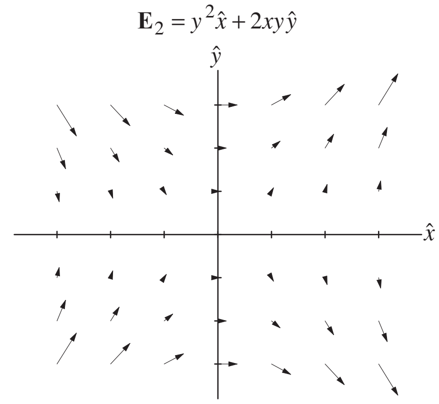
g) (1 pt) **TRUE** or **FALSE**: $\nabla^2 V = 0$ in region (4).

h) (1 pt) **TRUE** or **FALSE**: The potential drop from $r = b$ to $r = c$ is non-zero.

i) (1 pt) **TRUE** or **FALSE**: Before reaching steady-state equilibrium, conducting electrons inside the metal shell (region 3) experience Lorentz force acceleration in the $+\hat{r}$ direction.

2. Consider the following two electric field vector functions in free space and their corresponding graphical representations shown on the $\hat{x} - \hat{y}$ plane.





a) (5 pts) Which one of the fields above is an *electrostatic* field? Justify your answer.

b) (5 pts) Write the expression for the charge density $\rho$ in the region corresponding to the static electric field in part (a).

c) (7 pts) Write the expression for the electrostatic potential function, $V(x, y, z)$, corresponding to field $\mathbf{E}_2$. Assume that the potential at the origin is $V(0, 0, 0) = 0$.

d) (3 pts) Consider a charged particle $Q = 3$ C which is moved from position $(2, 1, 3)$ to position $(-1, 2, -1)$ in field $\mathbf{E}_2$. How much work is done on the particle?

e) (5 pts) Given an electrostatic potential field $V(x, y, z) = 3x(y - 2) + 2z^2$ V, determine the corresponding electric field $\mathbf{E}$.

3. Two infinite conducting copper plates lie in the $x = 1$ and $x = 4$ planes. Suppose the space between the plates is intially vacuum ($\epsilon = \epsilon_o$). The plates hold equal and oppositely signed charge densities $\pm\rho_S$, such that the electric field in the vacuum between the plates is initially $\mathbf{E} = -8\hat{x}$ V/m.

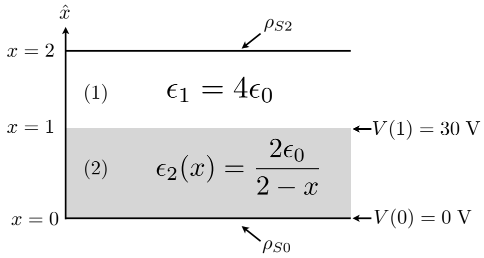
   a) (5 pts) What are the corresponding displacement vector **D** and polarization vector **P** in the vacuum between the plates?.

   b) (5 pts) What is the surface charge density of the copper plate at $x = 1$?

   c) (10 pts) If the vacuum between the plates is replaced with a crystalline perfect dielectric with $\epsilon = 13\epsilon_o$ , what would the new values for **E**, **D**, and **P** be, assuming the charge on the plates remains unchanged?

   d) (5 pts) The perfect dielectric is now doped with atoms which can donate (mobile) free carriers to the crystal lattice, giving the dielectric a non-zero conductivity $\sigma = 2$ S/m. What are the new values of **E**, **D**, and **P** once steady-state equilibrium is reached? Breifly explain the thinking behind your answer.

4. Consider a parallel-plate capacitor with unknown surface charge densities on two perfectly conducting plates, each having area A, that are placed in the $x = 0$ and $x = 2$ m planes. Two perfect dielectric slabs are placed between the plates, each having thickness of 1 m but each having different electric permittivities given in the figure below. Also as shown in the figure, the bottom plate is grounded, and the boundary between the two materials is at a potiential of 30 V. Assuming that fringe effects can be neglected, the electric field in region (1) between the plates is known to be $\mathbf{E_1} = -10\hat{x}$ V/m.



a) (4 pts) Determine the displacement fields $\mathbf{D_1}$ and $\mathbf{D_2}$ in region (1) and region (2), respectively. **Hint**: there is no free charge inside perfect dielectrics.

b) (3 pts) Determine the electric field intensity $|\mathbf{E_2}|$ in region (2).

c) (6 pts) Could Laplace's equation ($\nabla^2 V = 0$) be used to solve for the potential function in region (1)? If so, determine $V_1(x)$ using Laplace's equation. If not, explain why it is not possible, then write down the appropriate equation and solve it.

d) (8 pts) Could Laplace's equation be used to solve for the potential function in region (2)? If so, determine $V_2(x)$ using Laplace's equation. If not, explain why it is not possible, then write down the appropriate equation and solve it.

e) (4 pts) Determine the surface charge densities $\rho_{S0}$ and $\rho_{S2}$ on the bottom and top plates, respectively, and the capacitance per unit area, $C/A$, of the capacitor.

# #1:

(a) $\vec{D}_1(\vec{r}) = 0$ since all fields → 0 inside conductors in steady-state equilibrium!

(b) $\vec{P}_2(\vec{r}) = \vec{D}_2(\vec{r}) - \varepsilon_0 \vec{E}_2(\vec{r}) = \varepsilon_2 \vec{E}_2(\vec{r}) - \varepsilon_0 \vec{E}_2(\vec{r}) = (\varepsilon - \varepsilon_0)\vec{E}_2(\vec{r})$

where $\oint \varepsilon_2 \vec{E}_2 \cdot \vec{ds} = (4\varepsilon_0) E_r(\vec{r}) 4\pi r^2 = Q_{encl} = 4$.

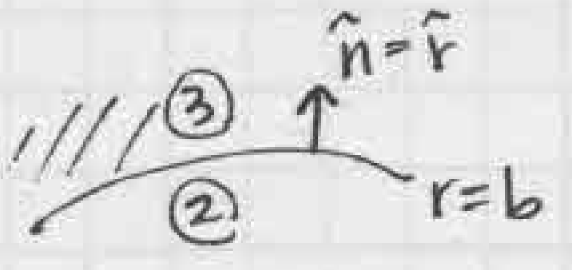
so $E_r(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0 r^2}$ $(a < r < b)$

$\therefore \vec{P}_2(\vec{r}) = (4\varepsilon_0 - \varepsilon_0) \cdot \dfrac{1}{4\pi\varepsilon_0 r^2}\hat{r} = \dfrac{3}{4\pi r^2}\hat{r}$ $[C/m^2]$

(c) <u>approach #1</u>: since $\vec{D}_3 = 0$ (region 3 is a conductor!), the charge on $r=b$ surface must cancel charge on $r=a$ $(=+4C)$

so $\rho_s = \dfrac{-4}{4\pi b^2} = \dfrac{-1}{\pi b^2}$ $[C/m^2]$

<u>approach #2</u>: via boundary conditions $\hat{n}\cdot(\bar{D}^+ - \bar{D}^-) = \rho_s$



$\hat{r}\cdot(\vec{D}_3 \xrightarrow{} 0 - \vec{D}_2) = \rho_s$ and $\vec{D}_2 = \dfrac{Q_{encl}}{4\pi b^2}$ @ $r=b$.

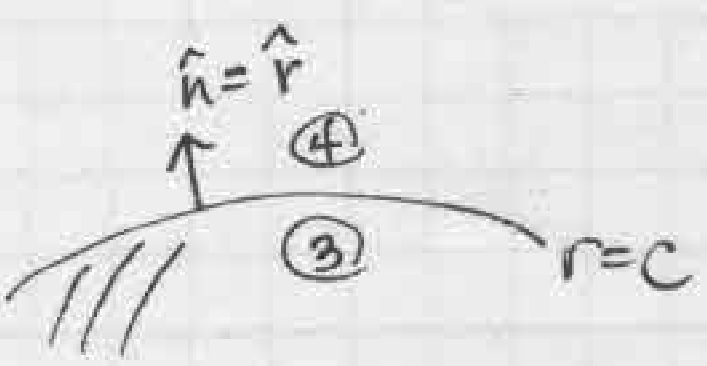
so $\rho_s = -D_{2r}\Big|_{r=b} = \dfrac{-4}{4\pi b^2} = -\dfrac{1}{\pi b^2}$ $[C/m^2]$

(d) <u>approach #1</u>: region 3 holds a net charge of $-2C$.

With $-4C$ on $r=b$ surface, $r=c$ must have $+2C$.

so $\rho_s = \dfrac{2}{4\pi c^2} = \dfrac{1}{2\pi c^2}$ $[C/m^2]$.

<u>approach #2</u>: via boundary conditions:



$\hat{r}\cdot(\vec{D}_4 - \vec{D}_3 \xrightarrow{} 0) = \rho_s = D_{4r}$ (@ $r=c$)

$= \dfrac{2}{4\pi c^2} = \dfrac{1}{2\pi c^2}$ $[C/m^2]$

(e) $\vec{E}_4(\vec{r}) = \dfrac{Q_{encl}}{4\pi\varepsilon_0 r^2}\hat{r}$ $[V/m]$. @ $\vec{r} = \vec{r}_0$: $|r_0| = \sqrt{2^2+2^2+1^2} = 3$.

so $\vec{E}_4(\vec{r}_0) = \dfrac{2}{4\pi\varepsilon_0 (3)^2}\left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3}\right)\left[\dfrac{V}{m}\right]$ and $\hat{r} = \left(\dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r}\right)$

#1 (con't)

(f) <u>TRUE</u>. $\vec{E}$ is static.

(g) <u>TRUE</u>. $\rho = 0$ & $\varepsilon$ is uniform in region (4)

(h) <u>FALSE</u>. V = constant inside conductors in steady state equil.

(g) <u>FALSE</u>. $\vec{F} = q\vec{E}_3 = -e\vec{E}_3 = -eE_{3r}\hat{r} \propto -\hat{r}$.

$E_{3r} > 0$ since Qencl ($b<r<c$) is $> 0$ before charges redistribute to cancel out the field.

#2:

(a) electrostatic $\Rightarrow$ $\nabla \times \bar{E} = 0$.

$$\nabla \times \bar{E}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & -2y & 0 \end{vmatrix} = -2\hat{z} \neq 0 \quad \therefore \text{ not electrostatic}$$

$$\nabla \times \bar{E}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & 2xy & 0 \end{vmatrix} = (2y-2y)\hat{z} = 0 \quad \therefore \text{ electrostatic}$$

(b) $$\rho = \nabla \cdot \vec{D}_2 = \nabla \cdot (\varepsilon_2 \bar{E}_2) = \nabla \cdot (\varepsilon_0 \bar{E}_2) = \frac{\partial}{\partial x}(\varepsilon_0 E_x) + \frac{\partial}{\partial y}(\varepsilon_0 E_y) + 0$$

$$= 0 + 2x\varepsilon_0 + 0 = 2x\varepsilon_0 \ [C/m^3]$$

(c) $$V(x,y,z) - \overset{0}{\cancel{V(0,0,0)}} = \int_{(0,0,0)}^{(x,y,z)} -\vec{E}_2 \cdot d\vec{l}$$

$$= -\int_0^x E_x\Big|_{\substack{y=0\\z=0}} dx - \int_0^y E_y\Big|_{\substack{x=x\\z=0}} dy - \int_0^z \overset{0}{\cancel{E_z}}\Big|_{\substack{x=x\\y=y}} dz$$

$$= -\int_0^x (0)\,dx - \int_0^y 2xy\,dy = -2x\left[\tfrac{1}{2}y^2\Big|_0^y\right] = -xy^2 \ [V]$$

#2 (con't)

(d) work $W = \int_A^B \vec{F}\cdot\vec{dl}$ . (let $(2,1,3) \equiv A$ ; $(-1,2,-1) \equiv B$.)

$$= \int_A^B Q\vec{E}_2\cdot\vec{dl} \quad \text{(via Lorentz force)}$$

$$= Q\left[\int_A^0 \vec{E}_2\cdot\vec{dl} + \int_0^B \vec{E}_2\cdot\vec{dl}\right] = Q\left[+V(A) - V(B)\right]$$

$V(A) = -(2)(1)^2 = -2$ V

$V(B) = -(-1)(2)^2 = 4$ V.

so $W = Q(-2-4) = -18$ [N·m] or [C·V]

(e). $\vec{E} = -\nabla V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$ [V/m].

$$= \hat{x}\left[-3(y-2)\right] + \hat{y}\left[-3x\right] + \hat{z}\left[-4z\right] \text{ V/m}$$

#3:

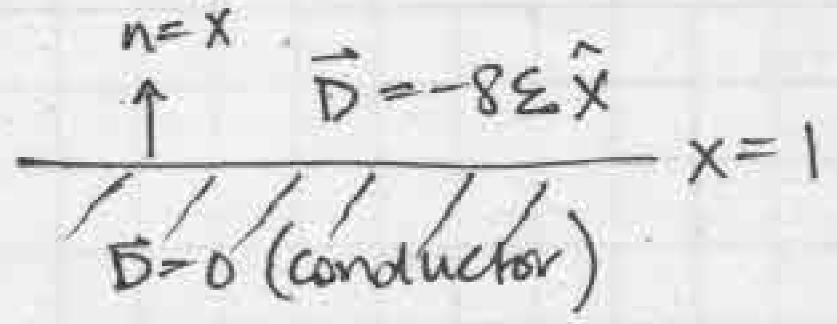
(a). $\vec{P} = 0$ in a vacuum.

$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\left[-8\hat{x}\right] = -8\varepsilon_0\hat{x}$ [C/m²]

(b) use boundary conditions:

$\hat{x}\cdot(\vec{D}^+ - \cancelto{0}{\vec{D}^-}) = \rho_s$

$\rho_s = -8\varepsilon_0$ [C/m²].



(c) charge on plates is unchanged → $\vec{D}$ is unchanged! (see part (b))

$\vec{D} = -8\varepsilon_0\hat{x}$ [C/m²].

$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{-8\varepsilon_0\hat{x}}{13\varepsilon_0} = -\frac{8}{13}\hat{x}$ [V/m].

$\vec{P} = \vec{D} - \varepsilon_0\vec{E} = (\varepsilon - \varepsilon_0)\vec{E} = (12\varepsilon_0)\left(\frac{8}{13}\right)(-\hat{x})$ [C/m²].

(d) steady state equilibrium + conductor = no fields!

$\vec{E} = 0$ ; $\vec{D} = 0$ ; $\vec{P} = 0$ . mobile charge moves to cancel external field.

#4 :

(a) $\vec{D}_1 = \varepsilon_1 \vec{E}_1 = 4\varepsilon_0(-10\hat{x}) = -40\varepsilon_0 \hat{x}\ [C/m^2]$

no other free charge around $(\nabla \cdot \vec{D} = 0)$ so

$\vec{D}_2 = \vec{D}_1 = -40\varepsilon_0 \hat{x}\ [C/m^2]$.

(b) $$|\vec{E}_2| = \frac{|\vec{D}_2|}{\varepsilon_2} = \frac{40\varepsilon_0}{2\varepsilon_0}(2-x) = 20(2-x) = 40-20x\ [V/m]$$

(c) YES : since $\rho = 0$ and $\varepsilon_1$ is uniform.

$\nabla^2 V_1 = 0 \Rightarrow V_1(x) = Ax + B$.

$$\vec{E}_1 = -\nabla V_1 = \frac{-\partial V_1}{\partial x}\hat{x} = -A\hat{x} = \underbrace{-10\hat{x}}_{\text{given}} \quad \text{so} \quad A = 10$$

boundary conditions :

$$V(x=1) = 10x + B = \underbrace{30\ V}_{\text{given}} \quad \text{so} \quad B = 20.$$

$\therefore V_1(x) = 10x + 20\ [V]$

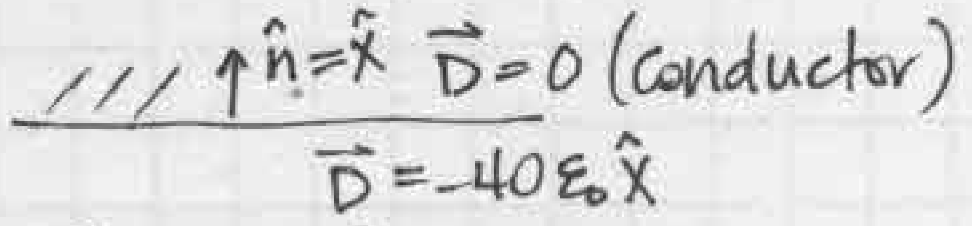
(d) No : $\rho$ is zero, but $\varepsilon_2$ is not uniform.

$$\rho = 0 = \nabla \cdot \big(\varepsilon_2 \underbrace{(-\nabla V_2)}_{\vec{E}_2}\big)$$

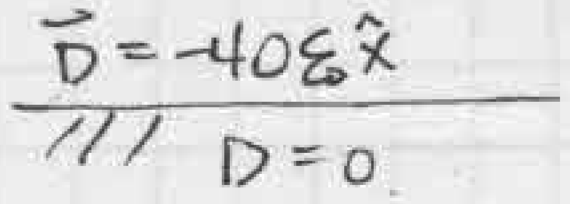
$$V_2(x) - \overset{0 \text{ given}}{\cancel{V_2(0)}} = \int_x^0 E_x\,dx = \int_x^0 (40-20x)\,dx = 40x\Big|_x^0 - 10x^2\Big|_x^0$$

$\therefore V_2(x) = 40x - 10x^2\ [V]$

check : is V continuous @ x=1? $V_2(x=1) = 40-10 = 30\ V$ ✓

(e) @ x=2 : 

$\hat{x} \cdot (0 - (-40\varepsilon_0 \hat{x})) = 40\varepsilon_0 = \rho_{s2}\ [C/m^2]$

@ x=0 : 

$\Rightarrow \rho_{s0} = -\rho_{s2} = -40\varepsilon_0\ [C/m^2]$.

check : $\vec{D}$ points from pos. to neg. charge? YES!

$$C/A = (Q/A)/V = |\rho_s|/V = 40\varepsilon_0 / V_2(2) = \frac{40\varepsilon_0}{40} = \varepsilon_0\ [F/m^2]$$