ECE 329 | Fields and Waves I | Spring 2024

University of Illinois | Chen, Shao, Waldrop

Exam 2

Monday, March 25, 2024 — 7:00-8:15 PM

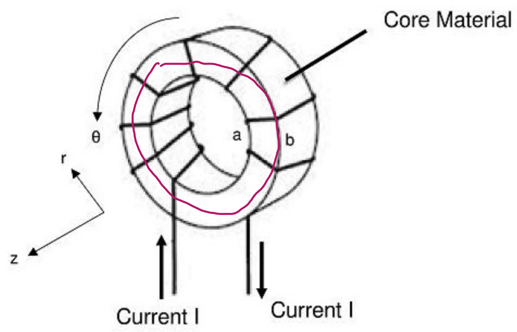
Please clearly PRINT your name in CAPITAL LETTERS and circle your section in the boxes below.

Name:	Solutions		
Section:	12 PM	1 PM	2 PM

This is a closed-book and closed-notes exam and calculators are not allowed. You may bring a single 4"x6" index card of handwritten notes. Please show all your work and make sure to include your reasoning for each answer. All answers should include units wherever appropriate. You may use the back of the exam as scratch paper, but no credit will be given for work done on scratch pages.

Problem 1 (25 points)	
Problem 2 (20 points)	
Problem 3 (30 points)	
Problem 4 (25 points)	
TOTAL (100 points)	

1. (25 points) The figure below depicts an ideal torodial coil, where a conducting wire is wrapped around a core torus in N total loops. The torus has an inner radius of a (i.e., its inner boundary is at $r = a$) and outer radius of b (at $r = b$). The conducting wire carries a steady current I in the direction shown, which generates a static magnetic flux density $\mathbf{B}$.



a) (5 points) What is the direction of the magnetic flux density field inside the core material, $\mathbf{B}(a < r < b)$? Choose the answer below according to the cylindrical coordinate definition shown in the figure.

i. in the $+\hat{z}$ direction along the center axis of the torus

ii. in the $-\hat{z}$ direction along the center axis of the torus

iii. in the radial $+\hat{r}$ direction

iv. in the radial $-\hat{r}$ direction

v. in the azimuthal $+\hat{\theta}$ direction

vi. in the azimuthal $-\hat{\theta}$ direction

b) (4 points) On the figure, draw a circulation loop (Amperian Loop) that can be used with Ampere's law to determine the magnetic flux density $\mathbf{B}$ inside the core material (i.e, for $a < r < b$).

c) (8 points) Assuming that the core material is not magnetic (i.e., its permeability is the same as that of free space), use the loop depicted in part (b) to find an expression for the magnetic flux density $\mathbf{B}$ inside the core material (i.e, for $a < r < b$).

Applying Ampere's law :

$$\oint B \cdot dl = \mu_0 I_{enc}.$$

$$B \cdot 2\pi\rho = \mu_0 N I.$$

$$B = \frac{\mu_0 N I}{2\pi\rho}.$$ (inside the coil)

Your Answer (include appropriate units):

$\mathbf{B}(a < r < b) = -\frac{\mu_0 N I}{2\pi\rho}\ \hat{\phi}\ \ Wb/m^2.$

d) (4 points) Which ONE of the following statements best describes the magnetic flux density in the region outside of the core material?

i. $|\mathbf{B}|$ decreases with increasing r for $r > b$ and decreases with decreasing r for $r < a$.

ii. $|\mathbf{B}|$ increases with increasing r for $r > b$ and increases with decreasing r for $r < a$.

iii. $|\mathbf{B}|$ decreases with increasing r for $r > b$ and increases with decreasing r for $r < a$.

iv. $|\mathbf{B}|$ is zero for $r > b$ and for $r < a$.

v. $|\mathbf{B}|$ is zero for $r < a$ but decreases with increasing r for $r > b$

e) (4 points) If the non-magnetic material comprising the torus core were replaced with one which is paramagnetic, how would the magnetic flux density change?

Your Answer:

$B = \mu_0 (1 + \chi_m) H.$

positive for paramagnets.

Eg. Al : $\chi_m = 2.2 \times 10^{-5}$.

The magnetic field is enhanced.

2. (20 points) A long solenoid is wound on a cylinder core made of iron ($\mu_r = 5000$). The solenoid has radius 2 cm and length $l = 1$ m and is wound with 100 loops. The axis of the solenoid is on the $z-$axis and a current of $I = 1$ A is flowing in the wire in the $\hat{\theta}-$direction (counter-clockwise when viewed from above). Ignore all fringing fields at the ends of the solenoid.

a) (10 points) What is the inductance L of the solenoid?

$$L = N^2 \mu \frac{A}{\ell} = 100^2 (5000 \mu_0) \frac{\pi (0.02)^2}{1}$$

$$= 4\pi (5000 \mu_0)$$

$$= 20000\pi \mu_0 \; [H]$$

Your Answer (include appropriate units):

$L =$ $20000\pi \mu_0 \; [H]$

b) (10 points) Now the iron core is hollowed up by drilling a hole of radius $r = 1$ cm through its center axis. What is the new inductance L of the solenoid?

$$L = 100^2 \left[(5000 \mu_0) \left(\frac{\pi (0.02)^2 - \pi (0.01)^2}{1} \right) + \mu_0 \frac{\pi (0.01)^2}{1} \right]$$

$$= 15001 \pi \mu_0 \; [H]$$

Your Answer (include appropriate units):

$L =$ $15001 \pi \mu_0 \; [H]$

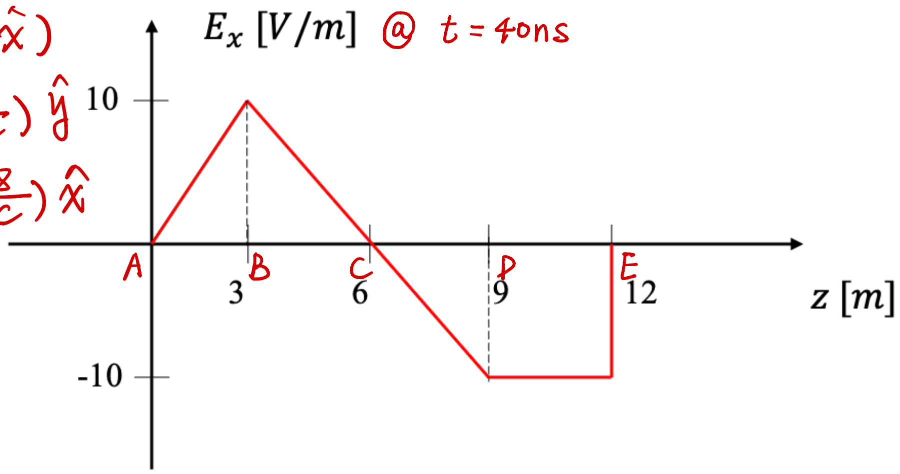
3. (30 points) An infinite current sheet with surface current $\mathbf{J_s}$ is located at $z = 0$ and generates outgoing transverse electromagnetic waves in the free space regions on $+\hat{z}$ and $-\hat{z}$ directions. The electric field of the electromagnetic wave that propagates in the $\boxed{+z \text{ direction at } t = 40\,\text{ns}}$ is shown below. Note that the free-space wave impedance $\eta_0 = 120\pi\,\Omega$, and the wave propagates with a speed of $c \approx 3 \times 10^8\, m/s$.

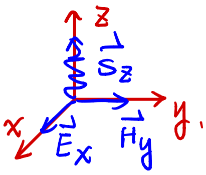
$c \approx 0.3\,m/ns$.

$\vec{J} = J_{s0} f(t)(-\hat{x})$

$\vec{H} = \pm\frac{J_{s0}}{2} f(t \mp \frac{z}{c})\,\hat{y}$

$\vec{E} = \eta_0 \frac{J_{s0}}{2} f(t \mp \frac{z}{c})\,\hat{x}$





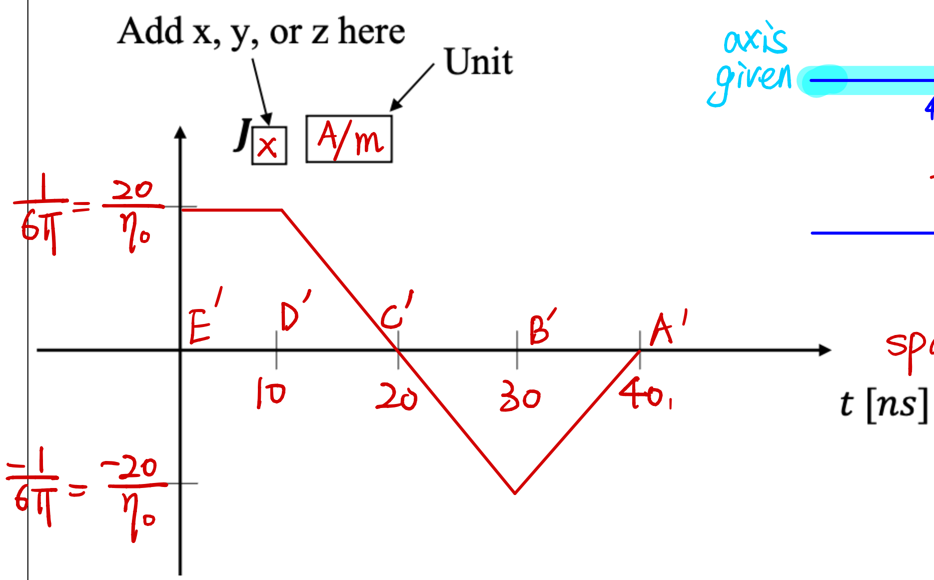
a) (10 points) Sketch the non-zero component of current source $\mathbf{J_s}$ as a function of time. Please make sure to label the direction of the surface current density $\mathbf{J}$ and put appropriate **numerical** labels on both axes.

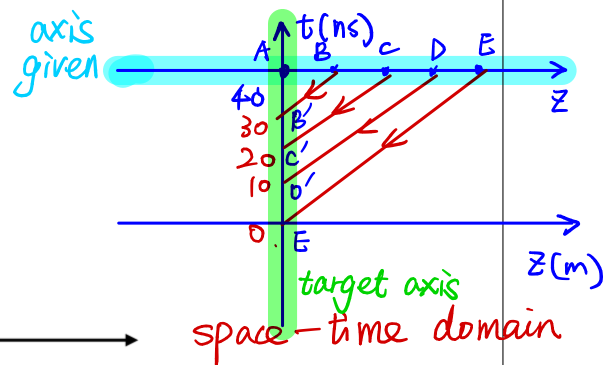
Method 1: $\vec{E}(z, t = 40\text{ns}) = \frac{\eta_0}{2} J_{s0} f(40\times10^{-9} - \frac{z}{3\times10^8})\hat{x} = \frac{\eta_0}{2} J_{s0} f(\frac{12 - z}{3\times10^8})\,\hat{x}$.

$\vec{J} = J_{s0} f(t)\,(-\hat{x}) = \frac{2}{\eta_0} E(t, z=0)(-\hat{x}) = \frac{\eta_0}{2} J_{s0} f(t)(-\hat{x})$ flip vertically

Your Answer (sketch and label all coordinates, axis, including amplitudes and units, and indicate direction):

Method 2:



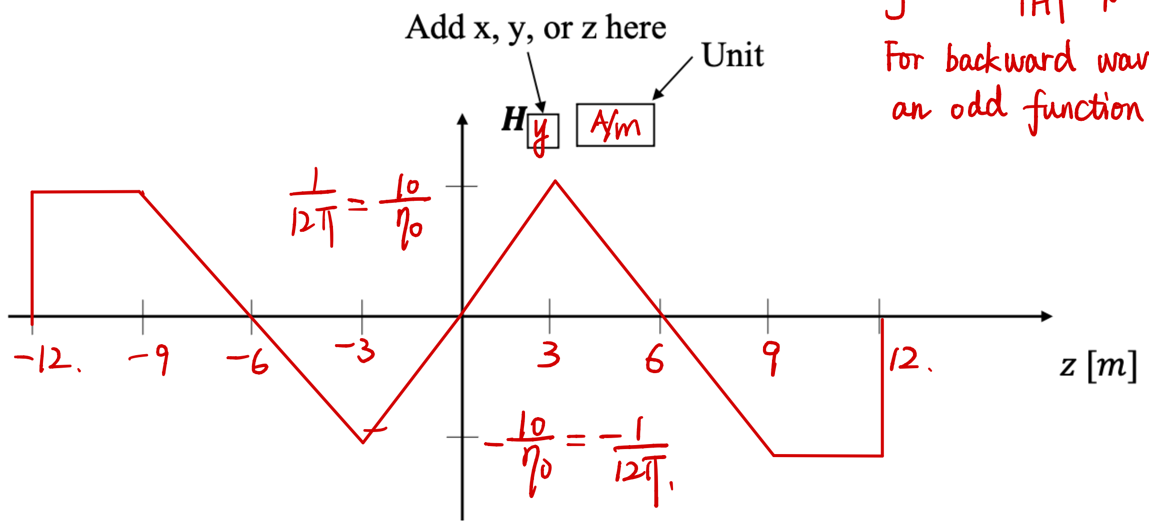


b) (10 points) Sketch the magnetic field at $t = 40\,\text{ns}$ as a function of location z. Please make sure to label the direction of the magnetic field and put appropriate numerical labels on both axes.

$$\left.\begin{array}{l} \text{forward } \vec{E} = f(t-\frac{z}{c})\,\hat{x} \\ \text{forward } \vec{H} = \frac{1}{\eta_0} f(t-\frac{z}{c})\,\hat{y} \end{array}\right\}$$ Both observe at $t=40\text{ns}$, so forward wave only have magnitude $\frac{|E|}{|H|}=\eta_0$ difference.

For backward wave, H is an odd function.

Your Answer (sketch and label all coordinates, axis, including amplitudes and units, and indicate direction):



c) (10 points) Sketch the electric field as a function of time at $z = 9\,\text{m}$. Please make sure to label the direction of the magnetic field and put appropriate numerical labels on both axes.

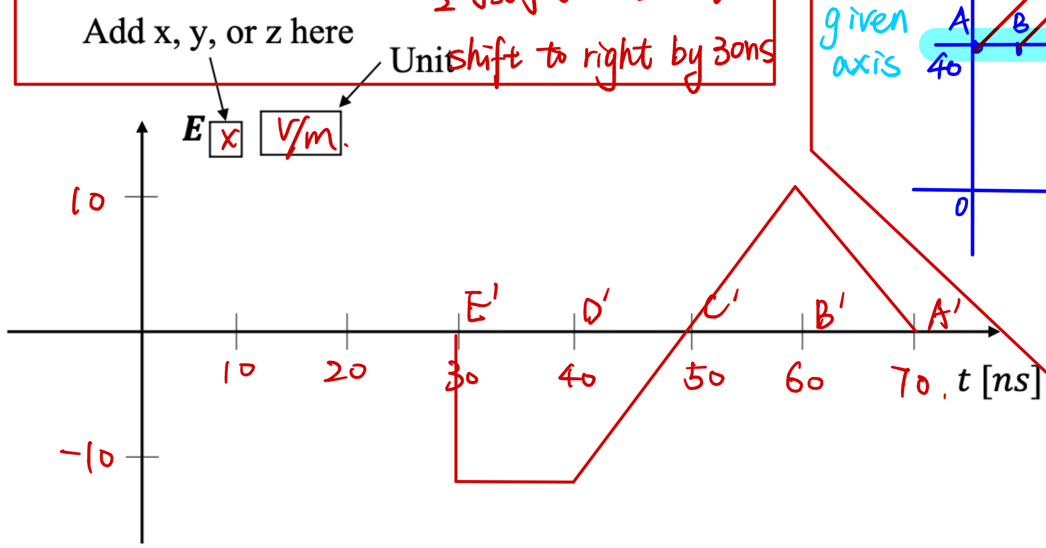
Method 1: Given $\vec{E} = \frac{\eta_0}{2} J_{s0} f(\frac{-12+z}{c})\,\hat{x}$

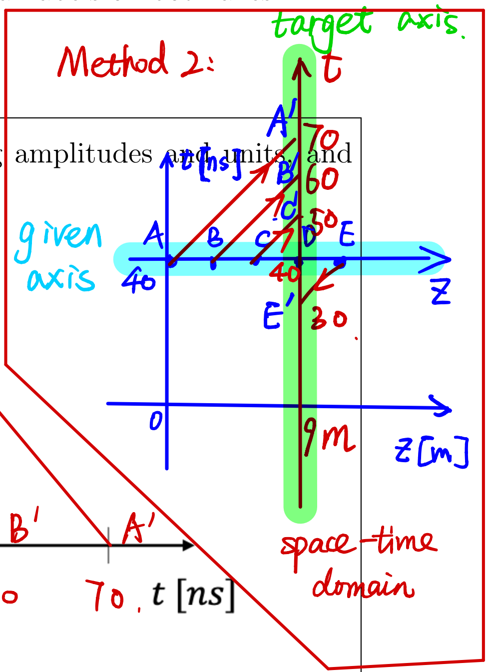
Target $\vec{E} = \frac{\eta_0}{2} J_{s0} f(t-\frac{9}{3\times10^8})\,\hat{x}$

$= \frac{\eta_0}{2} J_{s0} f(t-30\text{ns})\,\hat{x}$

shift to right by 30ns

Your Answer (sketch and label all coordinates, axis, including amplitudes and units, and indicate direction):





4. (25 points) The three parts of this problem are independent.

a) (9 points) Consider the Poynting Theorem $\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E}+\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H}\right)+\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{J}\cdot\mathbf{E}=0$. Identify the physical meaning of each term by choosing the appropriate letters below:

$$A. \quad \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E}+\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H}\right)$$
$$B. \quad \nabla\cdot(\mathbf{E}\times\mathbf{H})$$
$$C. \quad \mathbf{J}\cdot\mathbf{E}$$

i. (3 points) Energy transport

Your Answer (circle the appropriate letter):
A $\quad$ B $\quad$ C

ii. (3 points) Power absorbed per unit volume

Your Answer (circle the appropriate letter):
A $\quad$ B $\quad$ C

iii. (3 points) Time rate of change of field energy density

Your Answer (circle the appropriate letter):
A $\quad$ B $\quad$ C

b) (8 points) There is an electric field $\mathbf{E}=\hat{z}E_0e^{-2t}$ V/m in free space. Consider a circular loop in the $xy-$plane with radius $r=2$ m and resistance $R=1\,\Omega$ centered on the $z-$axis. What is the displacement current through the surface defined by the loop? Be sure to indicate the direction.

$\bar{D}=\varepsilon_0\bar{E}=\hat{z}\,\varepsilon_0E_0e^{-2t}$

Displacement current density $=\frac{\partial\bar{D}}{\partial t}=\hat{z}(-2)\varepsilon_0E_0e^{-2t}\left[\frac{A}{m^2}\right]$

Total displacement current $=\int_S\frac{d\bar{D}}{dt}\cdot d\bar{s}$

Area $=4\pi$ [m^2] $\qquad = 8\pi\varepsilon_0E_0e^{-2t}$ [A]

Your Answer (include appropriate units):
Displacement current = $8\pi\varepsilon_0E_0e^{-2t}$ [A]

Direction: $-\hat{z}$

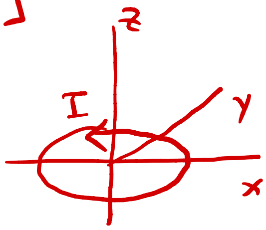
c) (8 points) There is a magnetic field $\mathbf{H} = \hat{z} H_0 e^{-2t}$ A/m in free space. Consider a circular loop in the $xy-$plane with radius $r = 2\,\text{m}$ and resistance $R = 1\,\Omega$ centered on the $z-$axis. What is the induced current from EMF on the loop? Be sure to indicate the direction.

$$\bar{B} = \mu \bar{H} = \hat{z}\mu_o H_o e^{-2t} \quad [\tfrac{Wb}{m^2}]$$

$$\Phi = \int_S \bar{B} \cdot d\bar{S} = 4\pi \mu_o H_o e^{-2t} \quad [Wb]$$

$$\mathcal{E} = \frac{-d\Phi}{dt} = +8\pi\mu_o H_o e^{-2t} \quad [V]$$

$$I = \frac{\mathcal{E}}{R} = +8\pi\mu_o H_o e^{-2t} \quad [A]$$



Your Answer (include appropriate units):
Induced current = $8\pi\mu_o H_o e^{-2t}$ [A]

Direction: counter-clockwise

(Scratch Page – No credit given for work done on this page.)