5 min.

1. (18 points) The three parts of this problem are independent.

a) (6 points) The electric field in a given region of free space ($\varepsilon = \varepsilon_0$) is known to be $\mathbf{E} = 2xy\hat{x} + x^2\hat{y} - 2z^2\hat{z}$ V/m. What is the volumetric charge density $\rho(x, y, z)$ in this region?

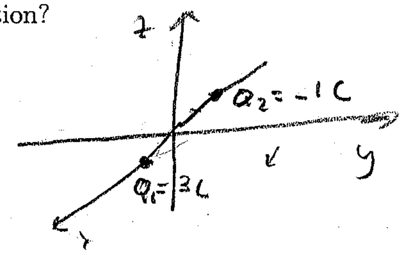
$$\rho = \varepsilon_0 \nabla \cdot \vec{E} = \varepsilon_0(2y - 4z)\ C/m^3$$

Your Answer (include appropriate units):

$\rho(x, y, z) =$ $\varepsilon_0(2y - 4z)$ C/m^3

b) (6 points) Two point charges $Q_1 = 3$ C and $Q_2 = -1$ C are located along the $\hat{x}$ axis at $(x, y, z) = (1, 0, 0)$ and at $(x, y, z) = (-2, 0, 0)$, respectively. What is the displacement flux $\int \mathbf{D} \cdot d\mathbf{S}$ through the entire $x = 0$ plane in the $+\hat{x}$ direction?

$$-\frac{3}{2} + \frac{-1}{2} = -2$$



Your Answer (include appropriate units):

$\int_{x=0} \mathbf{D} \cdot d\mathbf{S} =$ -2 C

c) (6 points) The volumetric free current density in a given region of space is known to be $\mathbf{J} = 2\sin(z)\hat{z}$ A/m^2. At point $(x, y, z) = (0, 0, 0)$ within this region, is the volumetric charge density ρ increasing with time, decreasing with time, or constant? Explain your reasoning.

$$-\nabla \cdot \vec{J} = -2\cos z = -2 \quad \text{at } (0,0,0)$$

$$= \frac{\partial \rho}{\partial t}$$

Your Answer (circle correct answer):

Increasing　　　Decreasing　　　Constant

since $\frac{\partial \rho}{\partial t} < 0$ by continuity

5 min.

2. (16 points) A infintesimally thin spherical shell (having radius a) holds a total charge of 4 C. The shell is centered on the origin and embedded within free space (permittivity $\epsilon = \epsilon_0$).

a) (4 points) What is the surface charge density, ρ_s, on the shell? Be sure to include units with your answer.

$$\frac{4C}{4\pi a^2 \; m^2}$$

Your Answer (include appropriate units):
$\rho_s = \frac{1}{\pi a^2} \frac{C}{m^2}$

b) (4 points) What is the vector electric field in the region inside the shell, $\mathbf{E}(r < a)$?

Your Answer (include appropriate units):
$\mathbf{E}(r < a) =$ 0 V/m

c) (4 points What is the vector electric field in the region outside the shell, $\mathbf{E}(r > a)$?

$$\varepsilon_0 |E| \, 4\pi r^2 = 4$$

Your Answer (include appropriate units):
$\mathbf{E}(r > a) = \frac{1}{\pi \varepsilon_0 r^2} \hat{r} \; \frac{V}{m}$

d) (4 points) Is the divergence of the electric field, $\nabla \cdot \mathbf{E}$, evaluated at any point outside the shell ($r > a$), less than, greater than, or equal to zero? Explain your answer.

$$\nabla \cdot \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r)$$

$\hat{r}$ divergence term in spherical, $r^2 E_r = $ const.

Your Answer (circle correct answer):

Less than zero | Greater than zero | Equal to zero

3. (32 points) The two parts of this problem are independent.

a) (16 points) In free space, there is a constant line of charge with density $\lambda\,[\frac{\mathrm{C}}{\mathrm{m}}]$ along the z-axis. An electron with charge $-q\,[\mathrm{C}]$ is moved from a position $r = 1\,[\mathrm{m}]$ to $r = 2\,[\mathrm{m}]$ away from the z-axis. What is the change in potential energy of the electron? (Note: if there is a net loss of potential energy, then the answer should be negative.)

$$\bar{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}\ \left[\frac{V}{m}\right]$$

$$V(2) - V(1) = -\int_1^2 \bar{E}\cdot dr\hat{r} = \frac{-\lambda}{2\pi\varepsilon_0}\int_1^2 \frac{1}{r}\,dr$$

$$= \frac{-\lambda}{2\pi\varepsilon_0}\ln(2)\ [V]$$

$$W = -qV = \frac{q\lambda}{2\pi\varepsilon_0}\ln(2)\ [J]$$

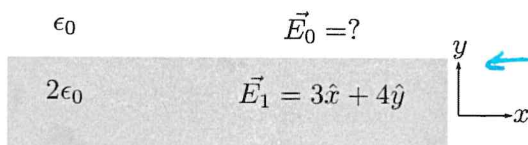
Your Answer (include appropriate units):

$W = \dfrac{q\lambda}{2\pi\varepsilon_0}\ln(2)$ or $\dfrac{-q\lambda}{2\pi\varepsilon_0}\ln\left(\frac{1}{2}\right)$ $[J]$

b) (16 points) There is a slab of perfect dielectric with $\epsilon = 2\epsilon_0$ laying in free space, with the surface parallel to the xz-plane as shown in the figure below. Inside the dielectric, the electric field is measured as $\mathbf{E}_1 = 3\hat{x} + 4\hat{y}\,[\frac{\mathrm{V}}{\mathrm{m}}]$. Find the electric field $\mathbf{E}_0$ in the free space above the slab.

$D_n^+ - D_n^- = 0$

$E_t^+ - E_t^- = 0$



$\leftarrow \rho_s = 0$ Perfect dielectric

$$\bar{D}_1 = \underbrace{6\varepsilon_0\hat{x}}_{\text{tangential}} + \underbrace{8\varepsilon_0\hat{y}}_{\text{normal}}$$

$D_{0y} = 8\varepsilon_0 \rightarrow E_{0y} = 8\ \left[\frac{V}{m}\right]$

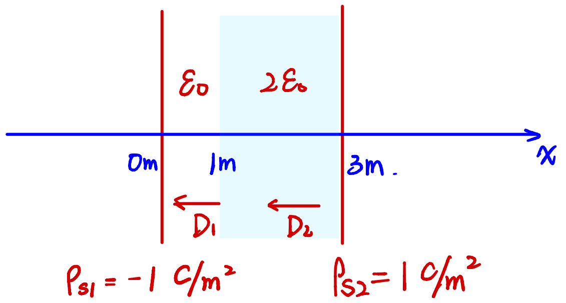
$E_{0x} = 3\ \left[\frac{V}{m}\right]$

Your Answer (include appropriate units):

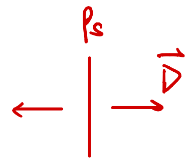
$\mathbf{E}_0 = 3\hat{x} + 8\hat{y}\ \left[\frac{V}{m}\right]$

4. (34 points) A pair of infinite conducting plates at $x = 0$ and $x = 3\,[\mathrm{m}]$ carry equal and opposite surface charge densities of $-1\,\frac{\mathrm{C}}{\mathrm{m}^2}$ and $+1\,\frac{\mathrm{C}}{\mathrm{m}^2}$, respectively. Region $0 < x < 1\,[\mathrm{m}]$ is free space, and the region $1 < x < 3\,[\mathrm{m}]$ is occupied by a perfect dielectric with permittivity $2\epsilon_0$. There are no background fields.

a) (9 points) Find **D**, **E**, **P** in the region $0 < x < 1\,[\mathrm{m}]$.



Since



$\vec{D} = \frac{\rho_s}{2}\,\mathrm{sgn}(x)\,\hat{x}$

Use superposition $\vec{D} = -\frac{\rho_s}{2} - \frac{\rho_s}{2} = -\rho_s \hat{x}\ C/m^2$

$(0 < x < 3m)$.

Between $0 < x < 1m$

$\vec{E}_1 = \frac{\vec{D}}{\varepsilon_0} = \frac{-1}{\varepsilon_0}\hat{x}\ V/m.$

$\vec{D}_1 = \vec{D} = \varepsilon_0\vec{E} + \vec{P} \Rightarrow \vec{P}_1 = 0\ \ C/m^2$

Your Answer (include appropriate units):

D $= -1\ \hat{x}\ \ C/m^2$

E $= -\frac{1}{\varepsilon_0}\ \hat{x}\ \ V/m$

P $= 0\ \ C/m^2$

b) (9 points) Find **D**, **E**, **P** in the region $1 < x < 3\,[\mathrm{m}]$.

$\vec{D}_2 = -\rho_s = -1\,\hat{x}\ \ C/m^2$

$\vec{E}_2 = \frac{\vec{D}}{2\varepsilon_0} = -\frac{1}{2\varepsilon_0}\hat{x}\ \ V/m$

$\vec{P} = \vec{D}_2 - \varepsilon_0\vec{E}_2 = -1\,\hat{x} - \varepsilon_0 \cdot \left(-\frac{1}{2\varepsilon_0}\hat{x}\right) = -\frac{1}{2\varepsilon_0}\hat{x}\ \ C/m^2$

Your Answer (include appropriate units):

D $= -1\,\hat{x}\ \ C/m^2$

E $= -\frac{1}{2\varepsilon_0}\ \hat{x}\ \ V/m$

P $= -\frac{1}{2}\ \hat{x}\ \ C/m^2$

c) (12 points) The two-plate system constitutes a capacitor. Find the capacitance per unit area C of the system.

$$\Delta V = -\int_0^3 \vec{E}\cdot d\vec{\ell} = -\int_0^1 \vec{E_1}\cdot d\vec{\ell} - \int_1^3 \vec{E_2}\cdot d\vec{\ell}$$

$$= -\int_0^1 \left(-\frac{1}{\varepsilon_0}\hat{x}\right)\cdot(dx\,\hat{x}) - \int_1^3 \left(-\frac{1}{2\varepsilon_0}\hat{x}\right)\cdot(dx\,\hat{x})$$

$$= \frac{1}{\varepsilon_0} + \frac{1}{2\varepsilon_0}\cdot 2 = \frac{2}{\varepsilon_0}\ [V].$$

Or, since $\vec{E}$ is in 1-D, $\Delta V = |E_1|d_1 + |E_2|d_2 = \frac{1}{\varepsilon_0}\cdot 1 + \frac{1}{2\varepsilon_0}\cdot 2 = \frac{2}{\varepsilon_0}$

per unit area $\quad C = \frac{\rho_s}{\Delta V} = \frac{1}{2/\varepsilon_0} = \frac{\varepsilon_0}{2}\ [F/m^2]$

Your Answer (include appropriate units):

$C = \frac{\varepsilon_0}{2}\ [F/m^2]$

d) (2 points) Suppose the surface charge densities are fixed, but we remove the slab of perfect dielectric from $1 < x < 3$ [m], will the per unit area capacitance of the two-plate system increase, decrease, or remain the same?

Easy way is to think $C = \varepsilon\frac{A}{d}$. $\varepsilon_{eff}\downarrow$ so $C\downarrow$.

ECE 329 way: $V\uparrow$ $V = E\cdot d\uparrow$

$2\varepsilon_0 \to \varepsilon_0$ $E = \frac{D}{\varepsilon\downarrow}\uparrow$ $E \uparrow \leftarrow D$ fixed

$|D| = \frac{\rho_s}{2}$ ρ_s fixed

$C_{\text{per unit area}} = \frac{\rho_s}{V\uparrow}\downarrow$

Your Answer (circle correct answer):

Increase | Decrease | Remain the same

e) (2 points) Suppose the potentials on the plates are fixed, but we remove the slab of perfect dielectric from $1 < x < 3$ [m], will the per unit area capacitance of the two-plate system increase, decrease, or remain the same?

ECE 329 way: V fixed

fixed $E = \frac{V}{d}$ $E \to D$ $D = \varepsilon E\downarrow$.

ρ_s $\rho_s = 2|D|\downarrow$.

$C_{\text{per unit area}} = \frac{\rho_s\downarrow}{V}\downarrow$.

Your Answer (circle correct answer):

Increase | Decrease | Remain the same