1. (a) (1 point) The vector wave field $\mathbf{E}(x,t)$ is given by

$$\mathbf{E}(y,t) = 80\Delta(\frac{t+y/c}{\tau})\,\hat{z}\,[\frac{\text{V}}{\text{m}}].$$

(b) (1 point) The associated wave field $\mathbf{H}(x,t)$ is

$$\mathbf{H}(y,t) = -\frac{80}{\eta_o}\Delta(\frac{t+y/c}{\tau})\,\hat{x}\,[\frac{\text{A}}{\text{m}}].$$

(c) (2 points) The Poynting vector is

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = -\frac{6400}{\eta_o}\Delta^2(\frac{t+y/c}{\tau})\,\hat{y}\,[\frac{\text{W}}{\text{m}^2}],$$

and its maximum value is

$$\max\left(|\mathbf{E}\times\mathbf{H}|\right) = \frac{6400}{\eta_o} \approx \frac{160}{3\pi}\,[\frac{\text{W}}{\text{m}^2}].$$

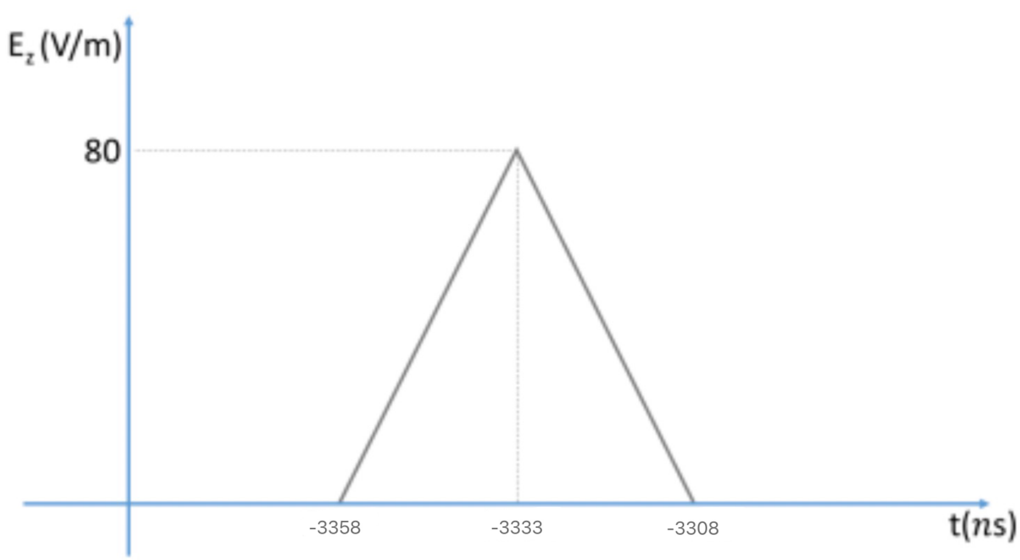
(d) (2 points) The location of the peak of $\mathbf{E}\times\mathbf{H}$ evolves according to

$$\frac{t+y/c}{\tau} = 0 \quad \rightarrow \quad y = -ct.$$

(e) (2 points) The field E_z given by

$$E_z(y,t)|_{y=1000m} = 80\triangle\left(\frac{t+3.33\times10^{-6}\,\text{s}}{50\,ns}\right)\,[\frac{\text{V}}{\text{m}}],$$

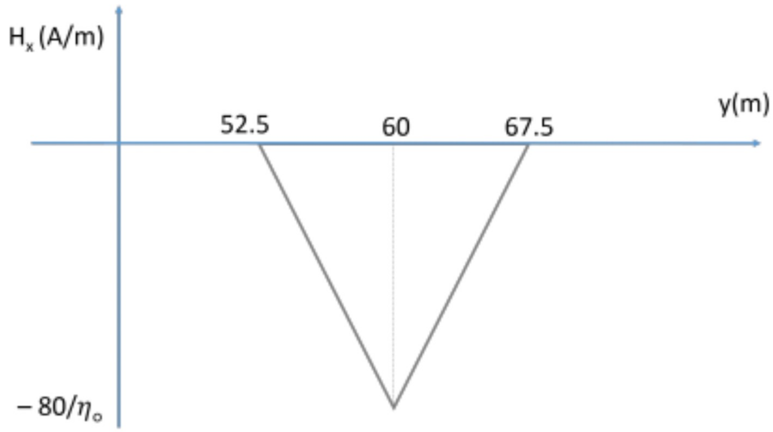
It is plotted in the following figure:



(f) (2 points) The field H_y given by

$$H_x(y,t)|_{t=-200\,ns} = -\frac{80}{\eta_o}\Delta(\frac{-60+y}{15})\,[\frac{\text{A}}{\text{m}}],$$

It is plotted in the following figure:



2. (a) (2 points) The propagation velocity is given by

$$v = \frac{c}{4} = 7.5 \times 10^7\,[\frac{\text{m}}{\text{s}}].$$

(b) (4 points) If $\epsilon = \epsilon_r\epsilon_o$ and $\mu = \mu_r\mu_o$, then the intrinsic impedance η can take the following form

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_r}{\epsilon_r}}\eta_o$$

while the wave propagation velocity v can be written as

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{\sqrt{\epsilon_r\mu_r}}.$$

Using the results of parts (a) and (b) we find that

$$\epsilon_r = 16, \mu_r = 1.$$

(c) (2 points) Since $E_x = \eta H_y$we have that

$$u(t - \frac{z}{c/4}) = \eta H_{yo} u(t - \frac{4z}{c})$$

with $\eta = \frac{1}{4}\eta_0$, then $H_{yo} = \frac{1}{30\pi}$

(d) (2 points) Finally, since $E_y = -\eta H_x$ (recall that the propagation direction is $\hat{z} = -\hat{y} \times \hat{x}$, hence the minus sign), we have that

$$g(t - \frac{z}{c/4}) = -10 \times (\frac{c}{4}t - z)rect(\frac{t - \frac{4z}{c} + 2ns}{4ns}),$$

then

$$g(t) = -\frac{5c}{2}t\,rect(\frac{t+2ns}{4ns}).$$

3. The pulse of sheet current $\mathbf{J}_s(t) = \hat{y}8t\text{rect}(\frac{t}{\tau})\,\frac{\text{A}}{\text{m}}$ will produce magnetic and electric fields.The magnetic field is: (it's direction can be verify using the right-hand-rule for Ampere's law $\oint_C \mathbf{H}\cdot d\mathbf{l} = I_C$)

$$\begin{aligned}\mathbf{H}^{\pm}(z,t) &= \pm\frac{1}{2}J_{so}\left(t \pm \frac{z}{c}\right)\hat{x}\,[\frac{\text{A}}{\text{m}}] \\ &= \pm 4\left(t \mp \frac{z}{c}\right)\text{rect}\left(\frac{t \mp \frac{z}{c}}{\tau}\right)\hat{x}\,[\frac{\text{A}}{\text{m}}] \quad \text{for } z \gtrless 0,\end{aligned}$$

whereas the electric field is given by

$$\mathbf{E}^{\pm}(x,t) = -\frac{\eta_o}{2}J_{so}\left(t \mp \frac{z}{c}\right)\hat{y} = -4\eta_o\left(t \mp \frac{z}{c}\right)\text{rect}\left(\frac{t \mp \frac{z}{c}}{\tau}\right)\hat{y}\,[\frac{\text{V}}{\text{m}}] \quad \text{for } z \gtrless 0,$$

where $\eta_o = 120\pi\,\Omega$ is the intrinsic impedance of free-space.

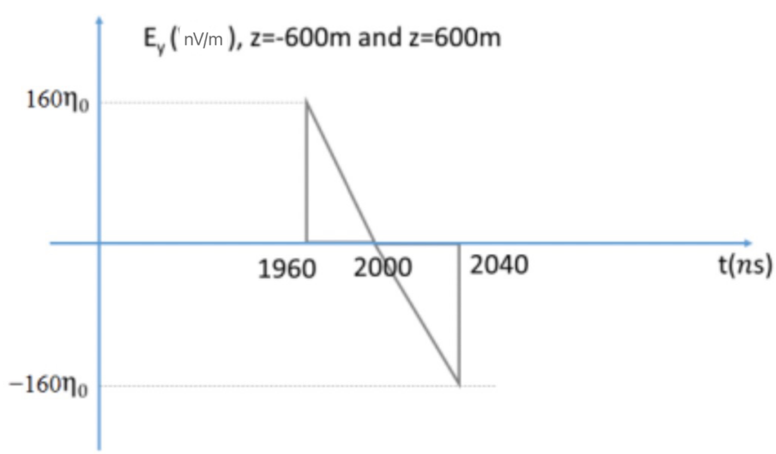
(a) (8 points) The fields are given by

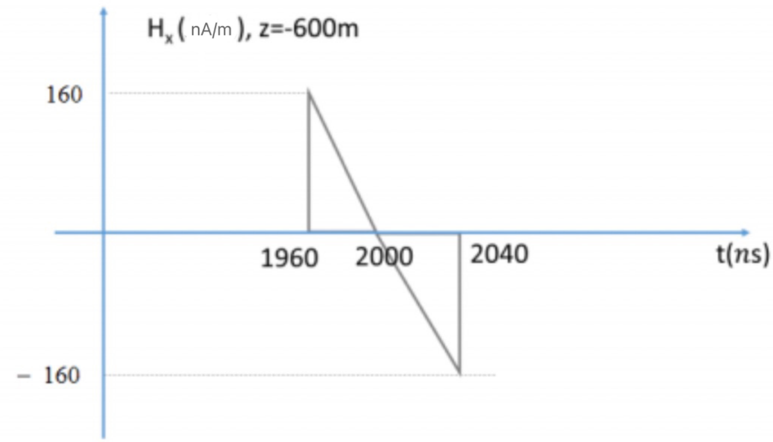
$$E_y(z,t)|_{z=-600\,m} = -4\eta_o\,(t - 2000\,ns)\ \text{rect}\left(\frac{t-2000\,ns}{80\,ns}\right)\,[\frac{\text{V}}{\text{m}}]$$

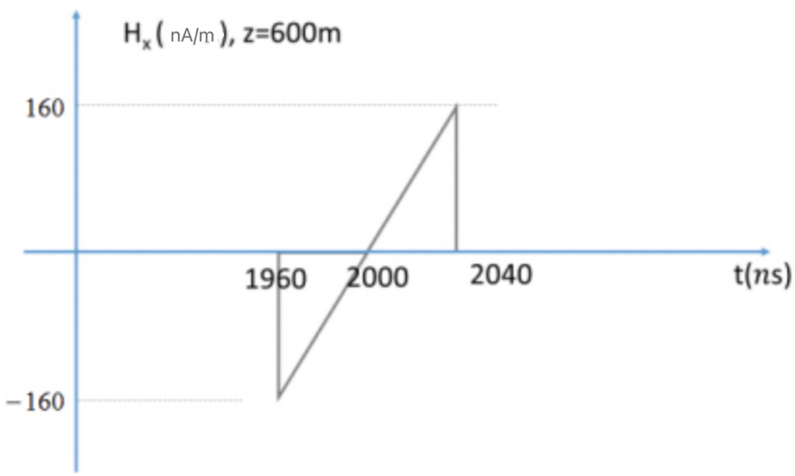
$$E_y(z,t)|_{z=600\,m} = -4\eta_o\,(t - 2000\,ns)\ \text{rect}\left(\frac{t-2000\,ns}{80\,ns}\right)\,[\frac{\text{V}}{\text{m}}]$$

$$H_x(z,t)|_{z=-600\,m} = -4\,(t - 2000\,ns)\ \text{rect}\left(\frac{t-2000\,ns}{80\,ns}\right)\,[\frac{\text{A}}{\text{m}}]$$

$$H_x(z,t)|_{z=600\,m} = 4\,(t - 2000\,ns)\ \text{rect}\left(\frac{t-2000\,ns}{80\,ns}\right)\,[\frac{\text{A}}{\text{m}}]$$



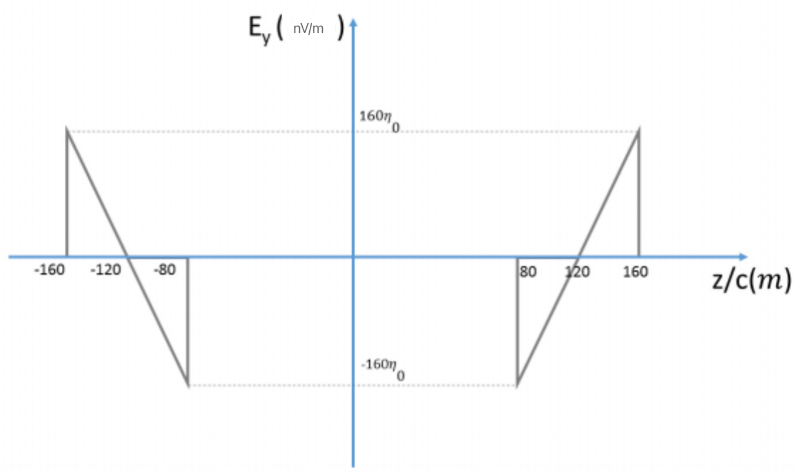


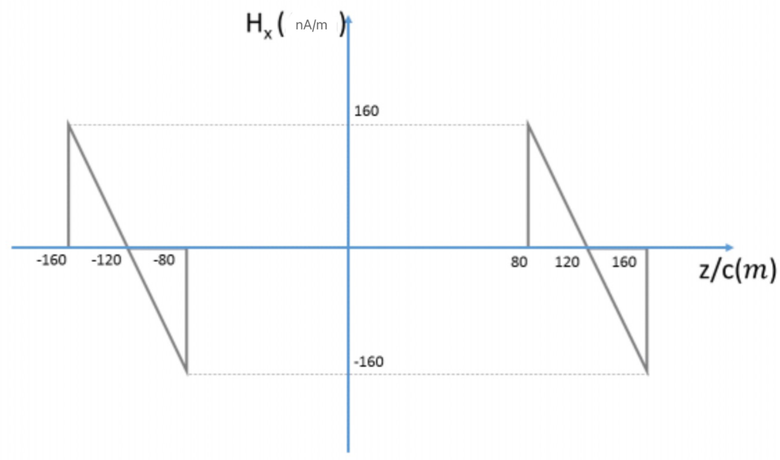


(b) (4 points) The fields are given by

$$E_y(z,t)|_{t=120\,n\text{S}} = -4\eta_0\left(120\,ns \mp \frac{z}{c}\right)\,\text{rect}\left(\frac{120ns \mp \frac{z}{c}}{80\,ns}\right)\,[\frac{\text{V}}{\text{m}}]$$

$$H_x(z,t)|_{t=120\,n\text{S}} = \pm 4\left(120\,ns \mp \frac{z}{c}\right)\,\text{rect}\left(\frac{120ns \mp \frac{z}{c}}{80\,ns}\right)\,[\frac{\text{A}}{\text{m}}]$$





(c) (2 points) Following the hint given in the problem, we can write

$$-\mathbf{J}_s \cdot \mathbf{E} = -\left(\hat{y}\, 8t\,\mathrm{rect}\left(\frac{t}{\tau}\right)\right) \cdot \left(-4\eta_0 t\,\mathrm{rect}\left(\frac{t}{\tau}\right)\,\hat{y}\right)$$
$$= 32\eta_o t^2\,\mathrm{rect}^2\left(\frac{t}{\tau}\right)\,[\frac{\mathrm{W}}{\mathrm{m}^2}].$$

Then, the TEM wave density energy is

$$\int -\mathbf{J}_s \cdot \mathbf{E}\, dt = \int 32\eta_o t^2\,\mathrm{rect}^2\left(\frac{t}{\tau}\right)\, dt$$
$$= \int_{-\tau/2}^{\tau/2} 32\eta_o t^2\, dt \; = \; \frac{32}{3}\eta_o\,\left[t^3\right]\Big|_{-\tau/2}^{\tau/2}$$
$$= \frac{8}{3}\eta_o\tau^3 \approx 5.14 \times 10^{-19}\,[\frac{\mathrm{J}}{\mathrm{m}^2}].$$

4. (a) (3 points) For the plane wave described by $\mathbf{E}_1 = 3\cos(\omega t - \beta z)\hat{y}\,[\frac{\text{V}}{\text{m}}]$:

   i. The magnetic field $\mathbf{H}$ should satisfy $\mathbf{H} = -\frac{\mathbf{E}\times\hat{\beta}}{\eta}$, where $\hat{\beta}$ is the unit vector parallel to the propagation direction. Then, we can find the expressions for $\mathbf{H}$ field of the given plane wave as

$$\mathbf{H_1} = -\frac{3}{\eta_o}\cos(\omega t - \beta z)\hat{x}\,[\frac{\text{A}}{\text{m}}].$$

   ii. The instantaneous power flow density is given by the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$. Therefore, the instantaneous power that crosses some surface A is given by $P = \int_A \mathbf{S}\cdot d\mathbf{A}$, where $d\mathbf{A} \equiv \hat{n}\cdot A$. Therefore, the Poynting vector is found as

$$\mathbf{S}_1 = \mathbf{E}_1 \times \mathbf{H}_1 = \frac{9}{\eta_o}\cos^2(\omega t - \beta z)\hat{z}\,[\frac{\text{W}}{\text{m}^2}],$$

   and expression for instantaneous power that crosses a $1\,\text{m}^2$ area (i.e. $A = 1\,\text{m}^2$) in the xy-plane from $-z$ to $+z$ may be written as

$$P_1 = \frac{9}{\eta_o}\cos^2(\omega t - \beta z)\,[\text{W}].$$

   iii. We can calculate the time-average of the Poynting vector using the trigonometric identity: $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$. Based on the fact that the time average of the cosine wave is zero ($\frac{1}{T}\int_T \cos(\omega t)dt = 0$), we can write

$$\langle\mathbf{S}_1\rangle = \left\langle \frac{9}{\eta_o}\cos^2(\omega t - \beta z)\hat{z}\,\frac{\text{W}}{\text{m}^2}\right\rangle = \frac{9}{2\eta_o}\hat{z}\,[\frac{\text{W}}{\text{m}^2}].$$

   Therefore, the average power that crosses some surface A is given by

$$\langle P_1\rangle = \langle\mathbf{S}_1\rangle\cdot\hat{n}\,A = \frac{9}{2\eta_o}\,[\text{W}].$$

(b) (3 points) For the plane wave described by $\mathbf{E}_2 = E_o\left(\cos(\omega t - \beta z)\hat{x} + \sin(\omega t - \beta z)\hat{y}\right)\,[\frac{\text{V}}{\text{m}}]$:

   i. The propagation direction is $\hat{\beta}_2 = \hat{z}$. Thus, the magnetic field $\mathbf{H}_2$ is given by

$$\mathbf{H}_2 = \frac{E_o}{\eta_o}\left(\cos(\omega t - \beta z)\hat{y} - \sin(\omega t - \beta z)\hat{x}\right)\,[\frac{\text{A}}{\text{m}}].$$

   ii. The Poynting vector is given by

$$\begin{aligned}\mathbf{S}_2 &= \mathbf{E}_2 \times \mathbf{H}_2\\ &= E_o\left(\cos(\omega t - \beta z)\hat{x} + \sin(\omega t - \beta z)\hat{y}\right) \times \frac{E_o}{\eta_o}\left(\cos(\omega t - \beta z)\hat{y} - \sin(\omega t - \beta z)\hat{x}\right)\\ &= \frac{E_o^2}{\eta_o}\left(\cos^2(\omega t - \beta z)\hat{z} + \sin^2(\omega t - \beta z)\hat{z}\right) = \frac{E_o^2}{\eta_o}\hat{z}\,[\frac{\text{W}}{\text{m}^2}].\end{aligned}$$

   Therefore, the instantaneous power crossing the area $A = 1\,\text{m}^2$ is

$$P_2 = \frac{E_o^2}{\eta_o}\,[\text{W}].$$

iii. The Poynting vector is constant in time, thus the time-average power is

$$\langle P_2 \rangle = \frac{E_o^2}{\eta_o} \,[\mathrm{W}].$$

(c) (3 points) For the plane wave described by $\mathbf{H}_3 = \cos(\omega t + \beta z + \frac{\pi}{3})\hat{x} - \sin(\omega t + \beta z - \frac{\pi}{6})\hat{y} \,[\frac{\mathrm{A}}{\mathrm{m}}]$:

i. The electric field $\mathbf{E}$ should satisfy $\mathbf{E} = \eta(\mathbf{H} \times \hat{\beta})$ where $\hat{\beta} = -\hat{z}$ in this case. Then, we can find the expressions for the $\mathbf{E}$ field of the given plane wave as

$$\mathbf{E}_3 = \eta_o \left(\cos(\omega t + \beta z + \frac{\pi}{3})\hat{y} + \sin(\omega t + \beta z - \frac{\pi}{6})\hat{x} \right) \,[\frac{\mathrm{V}}{\mathrm{m}}].$$

ii. The Poynting vector is given by

$$\begin{aligned}
\mathbf{S}_3 &= \mathbf{E}_3 \times \mathbf{H}_3 \\
&= -\eta_o \left(\cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} + \sin^2(\omega t + \beta z - \frac{\pi}{6})\hat{z} \right) \\
&= -\eta_o \left(\cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} + \cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} \right) \\
&= -2\eta_o \cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} \,[\frac{\mathrm{W}}{\mathrm{m}^2}].
\end{aligned}$$

Therefore, the instantaneous power crossing the area $A = 1\,\mathrm{m}^2$ is

$$P_3 = -2\eta_o \cos^2(\omega t + \beta z + \frac{\pi}{3}) \,[\mathrm{W}].$$

iii. The time-average power crossing a $1\,\mathrm{m}^2$ area is

$$\langle P_3 \rangle = -\eta_o \,[\mathrm{W}].$$

(d) (3 points) For the plane wave described by $\mathbf{H}_4 = \cos(\omega t - \beta x)\hat{z} + \sin(\omega t - \beta x)\hat{y} \,[\frac{\mathrm{A}}{\mathrm{m}}]$:

i. The propagation direction is $\hat{\beta}_4 = \hat{x}$. Thus, the electric field $\mathbf{E}_4$ is given by

$$\mathbf{E}_4 = \eta_o \left(\cos(\omega t - \beta x)\hat{y} - \sin(\omega t - \beta x)\hat{z} \right) \,[\frac{\mathrm{V}}{\mathrm{m}}].$$

ii. The wave is propagating in the $+x$ direction, therefore there is no flux of energy flowing into the z direction. Therefore, the instantaneous power crossing a $1\,\mathrm{m}^2$ area in the xy-plane from $-z$ to z is

$$P_4 = 0 \,[\mathrm{W}].$$

iii. The time-average power crossing a $1\,\mathrm{m}^2$ area in the xy-plane from $-z$ to z is also

$$\langle P_4 \rangle = 0 \,[\mathrm{W}].$$

5. (10 points) The current produced is proportional to $\mid \mathbf{E}_1 + \mathbf{E}_2 \mid^2$, hence it can be described by

$$I = A \mid \mathbf{E}_1 + \mathbf{E}_2 \mid^2 = AE_o(cos(\omega_1 t - \beta_1 z) + cos(\omega_2 t - \beta_2 z))^2.$$

Since we are not interested in the z dependence, we may very well just ignore it by setting z=0. Also remember:

$$\cos(a)^2 = \frac{\cos(2a) + 1}{2}$$
$$\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b)$$

Therefore:

$$\begin{aligned}
I|_{z=0} &= AE_o[\cos(\omega_1 t - \beta_1 z) + \cos(\omega_2 t - \beta_2 z)]^2|_{z=0} \\
&= AE_o[\cos(\omega_1 t) + \cos(\omega_2 t)]^2 \\
&= AE_o[\cos^2(\omega_1 t) + \cos^2(\omega_2 t) + 2\cos(\omega_1 t)\cos(\omega_2 t)] \\
&= AE_o[\frac{2 + \cos(2\omega_1 t) + \cos(2\omega_2 t)}{2}] \\
&+ AE_o[\cos((\omega_1 - \omega_2)t) + \cos((\omega_1 + \omega_2)t)]
\end{aligned}$$

Hence, the frequencies produced are $2\omega_1$, $2\omega_2, \omega_1 + \omega_2$ and $|\omega_1 - \omega_2|$.