1. (a) (1 point) The vector wave field $\mathbf{E}(x, t)$ is given by

$$
\mathbf{E}(y, t) = 80\Delta \left( \frac{t + y/c}{\tau} \right) \hat{z} \text{[V/m]},
$$

(b) (1 point) The associated wave field $\mathbf{H}(x, t)$ is

$$
\mathbf{H}(y, t) = -\frac{80}{\eta_o} \Delta \left( \frac{t + y/c}{\tau} \right) \hat{x} \text{[A/m]},
$$

(c) (2 points) The Poynting vector is

$$
\mathbf{S} = \mathbf{E} \times \mathbf{H} = -\frac{6400}{\eta_o} \Delta^2 \left( \frac{t + y/c}{\tau} \right) \hat{y} \text{[W/m$^2$]},
$$

and its maximum value is

$$
\max(|\mathbf{E} \times \mathbf{H}|) = \frac{6400}{\eta_o} \approx \frac{160}{3\pi} \text{[W/m$^2$]}.
$$

(d) (2 points) The location of the peak of $\mathbf{E} \times \mathbf{H}$ evolves according to

$$
\frac{t + y/c}{\tau} = 0 \rightarrow y = -ct.
$$

(e) (2 points) The field $E_z$ given by

$$
E_z(y, t)|_{y=1000m} = 80\Delta \left( \frac{t + 3.33 \times 10^{-6} \text{s}}{50 \text{ns}} \right) \text{[V/m]},
$$

It is plotted in the following figure:
(f) (2 points) The field $H_y$ given by

$$H_x(y, t)\big|_{t=-200\text{ ns}} = -\frac{80}{\eta_o} \Delta\left(\frac{-60 + y}{15}\right) \left[\frac{A}{m}\right].$$

It is plotted in the following figure:
2. (a) (2 points) The propagation velocity is given by
\[ v = \frac{c}{4} = 7.5 \times 10^7 \text{ [m/s]} \].

(b) (4 points) If \( \epsilon = \epsilon_r \epsilon_o \) and \( \mu = \mu_r \mu_o \), then the intrinsic impedance \( \eta \) can take the following form
\[ \eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_r}{\epsilon_r}} \eta_o \]
while the wave propagation velocity \( v \) can be written as
\[ v = \frac{1}{\sqrt{\epsilon \mu}} = \frac{c}{\sqrt{\epsilon_r \mu_r}}. \]

Using the results of parts (a) and (b) we find that
\[ \epsilon_r = 16, \mu_r = 1. \]

(c) (2 points) Since \( E_x = \eta H_y \), we have that
\[ u(t - \frac{z}{c/4}) = \eta H_{yo}u(t - \frac{4z}{c}) \]
with \( \eta = \frac{1}{4} \eta_o \), then \( H_{yo} = \frac{1}{30\pi} \).

(d) (2 points) Finally, since \( E_y = -\eta H_x \) (recall that the propagation direction is \( \hat{z} = -\hat{y} \times \hat{x} \), hence the minus sign), we have that
\[ g(t - \frac{z}{c/4}) = -10 \times \left( \frac{c}{4} t - z \right) \text{rect}\left( \frac{t - \frac{4z}{c} + 2ns}{4ns} \right), \]
then
\[ g(t) = -\frac{5c}{2} t \text{rect}\left( \frac{t + 2ns}{4ns} \right). \]
3. The pulse of sheet current \( \mathbf{J}_s(t) = \hat{y}8\text{rect}(\frac{t}{\tau}) \frac{A}{m} \) will produce magnetic and electric fields. The magnetic field is: (it’s direction can be verify using the right-hand-rule for Ampere’s law \( \oint \mathbf{H} \cdot d\mathbf{l} = I_C \))

\[
\mathbf{H}^{\pm}(z, t) = \pm \frac{1}{2} J_{so} \left( t \mp \frac{z}{c} \right) \hat{x} \left[ \frac{A}{m} \right] \\
= \pm 4 \left( t \mp \frac{z}{c} \right) \text{rect} \left( \frac{t \mp \frac{z}{c}}{\tau} \right) \hat{x} \left[ \frac{A}{m} \right] \quad \text{for} \quad z \geq 0,
\]

whereas the electric field is given by

\[
\mathbf{E}^{\pm}(x, t) = -\frac{\eta_o}{2} J_{so} \left( t \mp \frac{z}{c} \right) \hat{y} = -4\eta_o \left( t \mp \frac{z}{c} \right) \text{rect} \left( \frac{t \mp \frac{z}{c}}{\tau} \right) \hat{y} \left[ \frac{V}{m} \right] \quad \text{for} \quad z \geq 0,
\]

where \( \eta_o = 120\pi \Omega \) is the intrinsic impedance of free-space.

(a) (8 points) The fields are given by

\[
E_y(z, t)\bigg|_{z=-600\,m} = -4\eta_o \left( t - 2000\,ns \right) \text{rect} \left( \frac{t - 2000\,ns}{80\,ns} \right) \left[ \frac{V}{m} \right]
\]

\[
E_y(z, t)\bigg|_{z=600\,m} = -4\eta_o \left( t - 2000\,ns \right) \text{rect} \left( \frac{t - 2000\,ns}{80\,ns} \right) \left[ \frac{V}{m} \right]
\]

\[
H_x(z, t)\bigg|_{z=-600\,m} = -4 \left( t - 2000\,ns \right) \text{rect} \left( \frac{t - 2000\,ns}{80\,ns} \right) \left[ \frac{A}{m} \right]
\]

\[
H_x(z, t)\bigg|_{z=600\,m} = 4 \left( t - 2000\,ns \right) \text{rect} \left( \frac{t - 2000\,ns}{80\,ns} \right) \left[ \frac{A}{m} \right]
\]
(b) (4 points) The fields are given by

\[ E_y(z, t) \big|_{t=120\,\text{ns}} = -4\eta_0 \left( 120\,\text{ns} \mp \frac{z}{c} \right) \text{rect} \left( \frac{120\,\text{ns} \mp \frac{z}{c}}{80\,\text{ns}} \right) \left[ \frac{\text{V}}{\text{m}} \right] \]

\[ H_x(z, t) \big|_{t=120\,\text{ns}} = \pm 4 \left( 120\,\text{ns} \mp \frac{z}{c} \right) \text{rect} \left( \frac{120\,\text{ns} \mp \frac{z}{c}}{80\,\text{ns}} \right) \left[ \frac{\text{A}}{\text{m}} \right] \]
(c) (2 points) Following the hint given in the problem, we can write

\[- \mathbf{J}_s \cdot \mathbf{E} = - \left( \hat{y} 8t \operatorname{rect} \left( \frac{t}{\tau} \right) \right) \cdot \left( -4\eta_0 t \operatorname{rect} \left( \frac{t}{\tau} \right) \hat{y} \right) \]

\[= 32\eta_0 t^2 \operatorname{rect}^2 \left( \frac{t}{\tau} \right) \left[ \frac{W}{m^2} \right].\]

Then, the TEM wave density energy is

\[\int - \mathbf{J}_s \cdot \mathbf{E} \, dt = \int 32\eta_0 t^2 \operatorname{rect}^2 \left( \frac{t}{\tau} \right) \, dt\]

\[= \int_{-\tau/2}^{\tau/2} 32\eta_0 t^2 \, dt = \frac{32}{3} \eta_0 \left[ t^3 \right]_{-\tau/2}^{\tau/2}\]

\[= \frac{8}{3} \eta_0 \tau^3 \approx 5.14 \times 10^{-19} \left[ \frac{J}{m^2} \right].\]
4. (a) (3 points) For the plane wave described by \( \mathbf{E}_1 = 3 \cos(\omega t - \beta z) \hat{y} \text{[V]} \):

i. The magnetic field \( \mathbf{H} \) should satisfy \( \mathbf{H} = -\frac{\mathbf{E} \times \hat{p}}{\eta_0} \), where \( \hat{p} \) is the unit vector parallel to the propagation direction. Then, we can find the expressions for \( \mathbf{H} \) field of the given plane wave as

\[
\mathbf{H}_1 = -\frac{3}{\eta_0} \cos(\omega t - \beta z) \hat{x} \text{[A]}.
\]

ii. The instantaneous power flow density is given by the Poynting vector \( \mathbf{S} = \mathbf{E} \times \mathbf{H} \). Therefore, the instantaneous power that crosses some surface \( A \) is given by

\[
P = \int_A \mathbf{S} \cdot d\mathbf{A},
\]

where \( d\mathbf{A} \equiv \hat{n} \cdot d\mathbf{A} \). Therefore, the Poynting vector is found as

\[
\mathbf{S}_1 = \mathbf{E}_1 \times \mathbf{H}_1 = \frac{9}{\eta_0} \cos^2(\omega t - \beta z) \hat{z} \text{[W/m}\ ^2],
\]

and expression for instantaneous power that crosses a 1 m\(^2\) area (i.e. \( A = 1 \text{ m}^2 \)) in the \( xy \)-plane from \(-z\) to \(+z\) may be written as

\[
P_1 = \frac{9}{\eta_0} \cos^2(\omega t - \beta z) \text{[W]}.
\]

iii. We can calculate the time-average of the Poynting vector using the trigonometric identity: \( \cos^2 \theta = \frac{1}{2} (1 + \cos(2\theta)) \). Based on the fact that the time average of the cosine wave is zero (\( \frac{1}{T} \int_T \cos(\omega t) dt = 0 \)), we can write

\[
\langle \mathbf{S}_1 \rangle = \left\langle \frac{9}{\eta_0} \cos^2(\omega t - \beta z) \hat{z} \right\rangle \frac{W}{\text{m}^2} = \frac{9}{2\eta_0} \hat{z} \frac{W}{\text{m}^2}.
\]

Therefore, the average power that crosses some surface \( A \) is given by

\[
\langle P_1 \rangle = \langle \mathbf{S}_1 \rangle \cdot \hat{n} A = \frac{9}{2\eta_0} \text{[W]}.
\]

(b) (3 points) For the plane wave described by \( \mathbf{E}_2 = \mathbf{E}_o (\cos(\omega t - \beta z) \hat{x} + \sin(\omega t - \beta z) \hat{y}) \text{[V]} \):

i. The propagation direction is \( \hat{p}_2 = \hat{z} \). Thus, the magnetic field \( \mathbf{H}_2 \) is given by

\[
\mathbf{H}_2 = \frac{\mathbf{E}_o}{\eta_0} (\cos(\omega t - \beta z) \hat{y} - \sin(\omega t - \beta z) \hat{x}) \text{[A/m]},
\]

ii. The Poynting vector is given by

\[
\mathbf{S}_2 = \mathbf{E}_2 \times \mathbf{H}_2
\]

\[
= \mathbf{E}_o (\cos(\omega t - \beta z) \hat{x} + \sin(\omega t - \beta z) \hat{y}) \times \frac{\mathbf{E}_o}{\eta_0} (\cos(\omega t - \beta z) \hat{y} - \sin(\omega t - \beta z) \hat{x})
\]

\[
= \frac{E_o^2}{\eta_0} (\cos^2(\omega t - \beta z) \hat{z} + \sin^2(\omega t - \beta z) \hat{z}) = \frac{E_o^2}{\eta_0} \hat{z} \text{[W/m}^2].
\]

Therefore, the instantaneous power crossing the area \( A = 1 \text{ m}^2 \) is

\[
P_2 = \frac{E_o^2}{\eta_0} \text{[W]}.
\]
iii. The Poynting vector is constant in time, thus the time-average power is
\[ \langle P_2 \rangle = \frac{E_2^2}{\eta_o} \text{[W]} . \]

(c) (3 points) For the plane wave described by \( \mathbf{H}_3 = \cos(\omega t + \beta z + \frac{\pi}{3})\hat{x} - \sin(\omega t + \beta z - \frac{\pi}{6})\hat{y} \text{[A/m]} \):

i. The electric field \( \mathbf{E} \) should satisfy \( \mathbf{E} = \eta(\mathbf{H} \times \hat{\beta}) \) where \( \hat{\beta} = -\hat{z} \) in this case. Then, we can find the expressions for the \( \mathbf{E} \) field of the given plane wave as
\[ \mathbf{E}_3 = \eta_o \left( \cos(\omega t + \beta z + \frac{\pi}{3})\hat{y} + \sin(\omega t + \beta z - \frac{\pi}{6})\hat{x} \right) \text{[V/m]} . \]

ii. The Poynting vector is given by
\[ \mathbf{S}_3 = \mathbf{E}_3 \times \mathbf{H}_3 \]
\[ = -\eta_o \left( \cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} + \sin^2(\omega t + \beta z - \frac{\pi}{6})\hat{z} \right) \]
\[ = -\eta_o \left( \cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} + \cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} \right) \]
\[ = -2\eta_o \cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} \text{[W/m}^2]. \]

Therefore, the instantaneous power crossing the area \( A = 1 \text{ m}^2 \) is
\[ P_3 = -2\eta_o \cos^2(\omega t + \beta z + \frac{\pi}{3}) \text{[W]} . \]

iii. The time-average power crossing a \( 1 \text{ m}^2 \) area is
\[ \langle P_3 \rangle = -\eta_o \text{[W]} . \]

(d) (3 points) For the plane wave described by \( \mathbf{H}_4 = \cos(\omega t - \beta x)\hat{z} + \sin(\omega t - \beta x)\hat{y} \text{[A/m]} \):

i. The propagation direction is \( \hat{\beta}_4 = \hat{x} \). Thus, the electric field \( \mathbf{E}_4 \) is given by
\[ \mathbf{E}_4 = \eta_o \left( \cos(\omega t - \beta x)\hat{y} - \sin(\omega t - \beta x)\hat{z} \right) \text{[V/m]} . \]

ii. The wave is propagating in the \( +x \) direction, therefore there is no flux of energy flowing into the \( z \) direction. Therefore, the instantaneous power crossing a \( 1 \text{ m}^2 \) area in the \( xy \)-plane from \(-z\) to \( z\) is
\[ P_4 = 0 \text{ [W]} . \]

iii. The time-average power crossing a \( 1 \text{ m}^2 \) area in the \( xy \)-plane from \(-z\) to \( z \) is also
\[ \langle P_4 \rangle = 0 \text{ [W]} . \]
5. (10 points) The current produced is proportional to $|E_1 + E_2|^2$, hence it can be described by

$$I = A |E_1 + E_2|^2 = AE_o (\cos(\omega_1 t - \beta_1 z) + \cos(\omega_2 t - \beta_2 z))^2.$$

Since we are not interested in the $z$ dependence, we may very well just ignore it by setting $z=0$. Also remember:

$$\cos(a)^2 = \frac{\cos(2a) + 1}{2}$$

$$\cos(a + b) + \cos(a - b) = 2 \cos(a) \cos(b)$$

Therefore:

$$I|_{z=0} = AE_o [\cos(\omega_1 t - \beta_1 z) + \cos(\omega_2 t - \beta_2 z)]^2|_{z=0}$$

$$= AE_o [\cos(\omega_1 t) + \cos(\omega_2 t)]^2$$

$$= AE_o [\cos^2(\omega_1 t) + \cos^2(\omega_2 t) + 2 \cos(\omega_1 t) \cos(\omega_2 t)]$$

$$= AE_o \left[ \frac{1 + \cos(2\omega_1 t) + \cos(2\omega_2 t)}{2} \right]$$

$$+ AE_o [\cos((\omega_1 - \omega_2)t) + \cos((\omega_1 + \omega_2)t)]$$

Hence, the frequencies produced are $2\omega_1$, $2\omega_2$, $\omega_1 + \omega_2$ and $|\omega_1 - \omega_2|$. 