1. (10 points) Verifying vector identity

$$\mathbf{H}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{H}=\nabla\cdot(\mathbf{E}\times\mathbf{H})$$

for $\mathbf{E}=5\hat{z}e^{-\alpha y}$ and $\mathbf{H}=10\hat{x}e^{-\alpha y}$. The left-hand side of the identity gives

$$\begin{aligned}\mathbf{H}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{H} &= \left(10\hat{x}e^{-\alpha y}\right)\cdot\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 0 & 5e^{-\alpha y}\end{vmatrix}-\left(5\hat{z}e^{-\alpha y}\right)\cdot\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 10e^{-\alpha y} & 0 & 0\end{vmatrix}\\ &= \left(10\hat{x}e^{-\alpha y}\right)\cdot\left(-5\alpha\hat{x}e^{-\alpha y}\right)-\left(5\hat{z}e^{-\alpha y}\right)\cdot\left(10\alpha\hat{z}e^{-\alpha y}\right)\\ &= -50\alpha e^{-2\alpha y}-50\alpha e^{-2\alpha y}\\ &= -100\alpha e^{-2\alpha y},\end{aligned}$$

and the right-hand side gives

$$\begin{aligned}\nabla\cdot(\mathbf{E}\times\mathbf{H}) &= \nabla\cdot\left(5\hat{z}e^{-\alpha y}\times 10\hat{x}e^{-\alpha y}\right)\\ &= \nabla\cdot\left(50\hat{y}e^{-2\alpha y}\right)\\ &= \frac{\partial}{\partial y}\left(50e^{-2\alpha y}\right)\\ &= -100\alpha e^{-2\alpha y}.\end{aligned}$$

Thus, the identity is verified.

2. (a) (6 points) The closed surface S can be decomposed into 6 surfaces:

$$S=S_1+S_2+S_3+S_4+S_5+S_6,$$

where

$$\begin{aligned}S_1 &: \quad x=-1.5\\ S_2 &: \quad x=1.5\\ S_3 &: \quad y=-1.5\\ S_4 &: \quad y=1.5\\ S_5 &: \quad z=-1.5\\ S_6 &: \quad z=1.5\end{aligned}$$

Therefore,

$$\oint_S \mathbf{J}\cdot d\mathbf{S}=\int_{S_1+S_2+S_3+S_4+S_5+S_6}\mathbf{J}\cdot d\mathbf{S},$$

where

$$\begin{aligned}\int_{S_1+S_2}\mathbf{J}\cdot d\mathbf{S} &= \int_{-1.5}^{1.5}\int_{-1.5}^{1.5}\left[4x(y-1)^2\hat{x}+6y\hat{y}+8x^2y^2\hat{z}\right]|_{x=-1.5}\cdot(-\hat{x})\,dydz\\ &\quad+\int_{-1.5}^{1.5}\int_{-1.5}^{1.5}\left[4x(y-1)^2\hat{x}+6y\hat{y}+8x^2y^2\hat{z}\right]|_{x=1.5}\cdot\hat{x}dydz\\ &= \int_{-1.5}^{1.5}\int_{-1.5}^{1.5}6(y-1)^2dydz\;+\int_{-1.5}^{1.5}\int_{-1.5}^{1.5}6(y-1)^2dydz\\ &= 189\ [\mathrm{A}].\end{aligned}$$

$$\begin{aligned}\int_{S_3+S_4}\mathbf{J}\cdot d\mathbf{S} &= \int_{-1.5}^{1.5}\int_{-1.5}^{1.5}\left[4x(y-1)^2\hat{x}+6y\hat{y}+8x^2y^2\hat{z}\right]|_{y=-1.5}\cdot(-\hat{y})\,dxdz\\ &\quad+\int_{-1.5}^{1.5}\int_{-1.5}^{1.5}\left[4x(y-1)^2\hat{x}+6y\hat{y}+8x^2y^2\hat{z}\right]|_{y=1.5}\cdot\hat{y}dxdz\\ &= \int_{-1.5}^{1.5}\int_{-1.5}^{1.5}9dxdz\ +\ \int_{-1.5}^{1.5}\int_{-1.5}^{1.5}9dxdz\\ &= 162\ [\mathrm{A}]\,.\end{aligned}$$

$$\begin{aligned}\int_{S_5+S_6}\mathbf{J}\cdot d\mathbf{S} &= \int_{-1.5}^{1.5}\int_{-1.5}^{1.5}\left[4x(y-1)^2\hat{x}+6y\hat{y}+8x^2y^2\hat{z}\right]|_{z=-1.5}\cdot(-\hat{z})\,dxdy\\ &\quad+\int_{-1.5}^{1.5}\int_{-1.5}^{1.5}\left[4x(y-1)^2\hat{x}+6y\hat{y}+8x^2y^2\hat{z}\right]|_{z=1.5}\cdot\hat{z}dxdy\\ &= \int_{-1.5}^{1.5}\int_{-1.5}^{1.5}(-8x^2y^2)dxdy\ +\ \int_{-1.5}^{1.5}\int_{-1.5}^{1.5}(8x^2y^2)dxdy\\ &= 0\ [\mathrm{A}]\,.\end{aligned}$$

Therefore, $\oint_S \mathbf{J}\cdot d\mathbf{S} = 351$ [A].

(b) (3 points) We know that the unit of J_x, J_y, J_z is [A/m^2], and x, y, z has unit [m], thus

$$\begin{aligned}[J_x] &= [4x(y-1)^2] = [\mathrm{A/m^2}] &\rightarrow\quad [5] &= [\mathrm{A/m^5}],\\ [J_y] &= [6y] = [\mathrm{A/m^2}] &\rightarrow\quad [6] &= [\mathrm{A/m^3}],\\ [J_z] &= [8x^2y^2] = [\mathrm{A/m^2}] &\rightarrow\quad [8] &= [\mathrm{A/m^6}].\end{aligned}$$

(c) (2 points) Based on the integral form of charge conservation

$$\frac{dQ}{dt} = \frac{d}{dt}\int_V \rho dV = -\oint_S \mathbf{J}\cdot d\mathbf{S} = -351\ [\mathrm{A}]\,,$$

which means the total charge contained in the cube is decreasing.

(d) (4 points) Taking the divergence of $\mathbf{J} = 4x(y-1)^2\hat{x}+6y\hat{y}+8x^2y^2\hat{z}$ [A/m^2], we get

$$\nabla\cdot\mathbf{J} = \frac{\partial}{\partial x}\left(4x(y-1)^2\right)+\frac{\partial}{\partial y}(6y)+\frac{\partial}{\partial z}\left(8xy^2\right) = 4(y-1)^2+6.$$

According to the continuity equation $\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0$, we have

$$\rho(0,0,0,t)-\rho(0,0,0,0) = \int_0^t \frac{\partial\rho}{\partial t'}\,|_{(0,0,0)}\,dt' = -\int_0^t \nabla\cdot\mathbf{J}\,|_{(0,0,0)}\,dt' = -\int_0^t 10dt' = -10t\ [\mathrm{C/m^3}]\,.$$

Since $\rho(0,0,0,0)=0$, we have

$$\rho(0,0,0,t) = -10t\ [\mathrm{C/m^3}]\,.$$

3. (a) (2 points)

$$\because \qquad \nabla \cdot \mathbf{J} = \nabla \cdot (\sigma \mathbf{E}) = \sigma \nabla \cdot \mathbf{E} = \sigma \nabla \cdot \left(\frac{\mathbf{D}}{\epsilon_0} \right) = \frac{\sigma}{\epsilon_0} \rho, \quad \text{Gauss' Law}$$

$$\textit{and} \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad \text{Continuity Equation}$$

$$\therefore \qquad \frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_0} \rho = 0.$$

(b) (3 points) From $\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_0} \rho = 0$ we have $\frac{d\rho}{\rho} = -\frac{\sigma}{\epsilon_0} dt$, or

$$d\left(\ln \rho\right) = -\frac{\sigma}{\epsilon_0} dt.$$

By integrating both sides, we obtain $\ln \rho \mid_0^t = -\frac{\sigma}{\epsilon_0} t$, or

$$\ln \frac{\rho\left(x, y, z, t\right)}{\rho\left(x, y, z, 0\right)} = -\frac{\sigma}{\epsilon_0} t.$$

Therefore,

$$\rho\left(x, y, z, t\right) = \rho\left(x, y, z, 0\right) e^{-\frac{\sigma}{\epsilon_0} t} = 2\rho_0 \sin\left(100z\right) e^{-\frac{\sigma}{\epsilon_0} t} \ \left(\mathrm{C/m^3}\right).$$

(c) (2 points) From $2\rho_0 \sin\left(100z\right) e^{-\frac{\sigma}{\epsilon_0} t} = \frac{2\rho_0}{e} \sin\left(100z\right)$, we obtain

$$e^{-\frac{\sigma}{\epsilon_0} t} = \frac{1}{e},$$

or

$$t = \frac{\epsilon_0}{\sigma} = \frac{8.854 \times 10^{-12}}{24\pi \times 10^5} \approx 1.18 \times 10^{-18} \ \left(\mathrm{s}\right).$$

(d) (3 points) Because ρ is a function of z and t only, we have

$$\nabla \cdot \mathbf{E} = \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0} = \frac{2\rho_0}{\epsilon_0} \sin\left(100z\right) e^{-\frac{\sigma}{\epsilon_0} t}.$$

Therefore,

$$E_z = -\frac{\rho_0}{50\epsilon_0} \cos\left(100z\right) e^{-\frac{\sigma}{\epsilon_0} t}.$$

Furthermore, we get

$$\mathbf{J}\left(x, y, z, t\right) = \sigma \mathbf{E} = -\hat{z} \frac{\sigma \rho_0}{50\epsilon_0} \cos\left(100z\right) e^{-\frac{\sigma}{\epsilon_0} t}.$$

4. (a) (1 point) Gauss' Law: $\nabla \cdot \mathbf{D} = \rho$ (= 0 for this problem). Since $\mathbf{E} = \hat{x}E_x(z,t)$, which is independent of x and y coordinates, we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_0 \mathbf{E}) = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = 0.$$

(b) (2 points) From Faraday's Law (Equation (2) in the problem), we have

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = -\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\left(\hat{x} \cdot 0 + \hat{y}\frac{\partial E_x}{\partial z} + \hat{z} \cdot 0 \right) = -\hat{y}\frac{\partial E_x}{\partial z}.$$

(c) (2 points) Take one more curl of the result of Part (b)

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{\partial E_x}{\partial z} & 0 \end{vmatrix} = -\hat{x}\frac{\partial^2 E_x}{\partial z^2} + \hat{y} \cdot 0 + \hat{z} \cdot 0 = -\hat{x}\frac{\partial^2 E_x}{\partial z^2}.$$

(d) (3 points) Again, from Faraday's Law (Equation (2) in the problem), after taking curl of both sides, we have

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}).$$

By plugging in Ampere's Law (Equation (4) in the problem), we obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\hat{x}\mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}.$$

Combing the above equation with the result of Part (c) yields

$$-\hat{x}\frac{\partial^2 E_x}{\partial z^2} = -\hat{x}\mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}, \text{ or } \frac{\partial^2 E_x}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0.$$

(e) (2 points) The solution to the 1D scalar wave equation (propagating in the $\hat{z}$ direction) can be written as

$$E_x(z,t) = \cos\left[\omega\left(t - \sqrt{\mu_0 \epsilon_0} z\right)\right] = \cos\left[\omega\left(t - \frac{z}{v}\right)\right].$$

Thus, the propagation speed is

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

5. (a) (2 points)

$$\frac{\partial^2 E_x}{\partial z^2} = -\left(\mp\beta\right)^2 E_0 \cos\left(\omega t \mp \beta z + \phi\right) = -\beta^2 E_0 \cos\left(\omega t \mp \beta z + \phi\right).$$

Since $\beta = \omega\sqrt{\mu_0\epsilon_0}$, the above equation becomes

$$\frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu_0 \epsilon_0 E_0 \cos\left(\omega t \mp \beta z + \phi\right).$$

On the other hand,

$$\mu_0\epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = \mu_0\epsilon_0 \cdot \left(-\omega^2\right) E_0 \cos\left(\omega t \mp \beta z + \phi\right) = -\omega^2 \mu_0 \epsilon_0 E_0 \cos\left(\omega t \mp \beta z + \phi\right).$$

Therefore,

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0\epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0.$$

(b) (6 points) Integrating

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

from 0 to t, we have

$$\begin{aligned}
\mathbf{B}\left(z,t\right) - \mathbf{B}\left(z,0\right) &= \pm\frac{E_0\beta}{\omega}\hat{y}\left[\cos\left(\omega t \mp \beta z + \phi\right) - \cos\left(\pm\beta z + \phi\right)\right] \\
&= \pm\hat{y}\left[\frac{E_x}{c} - \frac{E_0}{c}\cos\left(\pm\beta z + \phi\right)\right] \\
&= \pm\hat{y}\left[\frac{E_x\left(z,t\right)}{c} - \frac{E_x\left(z,0\right)}{c}\right].
\end{aligned}$$

If evaluated at $z = 0$, the above equation becomes

$$\mathbf{B}\left(0,t\right) - \mathbf{B}\left(0,0\right) = \pm\hat{y}\left[\frac{E_x\left(0,t\right)}{c} - \frac{E_x\left(0,0\right)}{c}\right].$$

Therefore,

$$\begin{aligned}
\mathbf{B}\left(0,t\right) &= \pm\frac{E_x\left(0,t\right)}{c}\hat{y} \mp \frac{E_x\left(0,0\right)}{c}\hat{y} + \mathbf{B}\left(0,0\right) \\
&= \pm\frac{E_x\left(0,t\right)}{c}\hat{y},
\end{aligned}$$

which means **B** and **E** line up at $z = 0$ for all time t. If consider arbitrary z, we obtain

$$\begin{aligned}
\mathbf{B}\left(z,t\right) &= f\left(t \mp \frac{z}{v_p}\right) = \mathbf{B}\left(0, t \mp \frac{z}{v_p}\right) \\
&= \pm\frac{E_x\left(0, t \mp \frac{z}{v_p}\right)}{c}\hat{y} = \pm\frac{E_x\left(z,t\right)}{c}\hat{y},
\end{aligned}$$

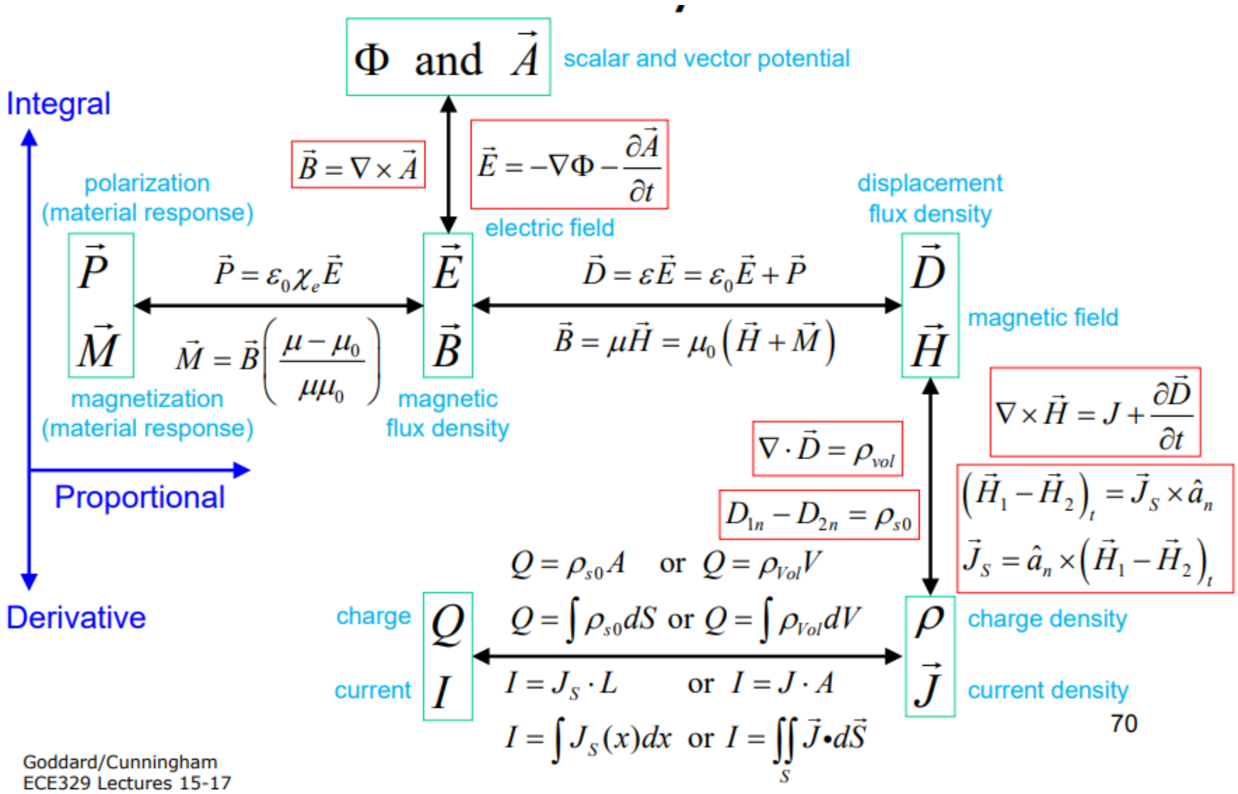
which means **B** and **E** line up for all z and t. As the phase shift, ϕ could be arbitrary value.

(c) (2 points) By comparing $\mathbf{B} = \pm\frac{E_x}{c}\hat{y}$ and $\mathbf{B} = \mu_0\mathbf{H}$, we obtain

$$\mathbf{H} = \pm\frac{E_x}{\mu_0 c}\hat{y} = \pm\frac{E_x}{\mu_0}\sqrt{\mu_0\epsilon_0}\hat{y} = \pm\frac{E_x}{\sqrt{\mu_0/\epsilon_0}}\hat{y}.$$

If we define $\eta_0 = \sqrt{\mu_0/\epsilon_0}$, then

$$\mathbf{H} = \pm\frac{E_x}{\eta_0}\hat{y}.$$



6. (5 points) Please refer to the above picture: