ECE 329 Fields and Waves I
Homework 7

Instructors: Chen, Goddard, Shao

Due March 9, 2023, 11:59 PM

Homework Policy:

- Write your name and NetID on top of every page. This habit will help you in exams in the event of having loose page(s).
- Tag all the questions in Gradescope. Failure to do so results in a 5 points deduction.
- Cheating results in ZERO and 50% reduction in HW average on first offense. A 100% reduction in HW average on second offense.
- Please show detailed process for each problem instead of just an answer. No partial credits would be given otherwise. All answers should include units wherever appropriate.
- No late HW is accepted.
- Regrade requests are available one week following grade release.

You are allowed to work with anyone else, but the work you submit should only belong to you. Note that if you have knowledge of a violation of the Honor Code, then you are obligated to report it. By submitting this homework, you are agreeing to the Honor Code: "I have neither given nor received unauthorized aid on this homework, nor have I concealed any violations of the Honor Code."

Question	Points	Score
1	10	
2	15	
3	10	
4	10	
5	10	
6	5	
Total:	60	

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1. (10 points) Verify that vector identity

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = \nabla \cdot (\mathbf{E} \times \mathbf{H})$$

   holds for $\mathbf{E} = 5\hat{z}e^{-\alpha y}$ and $\mathbf{H} = 10\hat{x}e^{-\alpha y}$ by expanding both sides of the identity. Treat α as a real constant.

   You should download the table of vector identities from the ECE 329 web site and examine the list to familiarize yourself with the listed identities — they are widely employed in electromagnetics as well as in other branches of engineering such as fluid dynamics.

2. Charge conservation states that the net outward flux of current density from a volume V through its bounding surface S equals the time rate of decrease of net charge contained within the volume:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho dV$$

   Note that applying the Divergence theorem yields the differential form:

$$\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}$$

   (a) (6 points) Calculate $\oint_S \mathbf{J} \cdot d\mathbf{S}$ for the surface of a cubic volume $V = 27\,\text{m}^3$ centered at the origin if the current density is $\mathbf{J} = 4x(y-1)^2\hat{x} + 6y\hat{y} + 8x^2y^2\hat{z}\,\text{A/m}^2$.

   (b) (3 points) Use dimensional analysis to determine the physical units of the coefficients 4, 6, and 8 used in defining $\mathbf{J}$.

   (c) (2 points) Is the total charge contained in the cube increasing, decreasing, or neither?

   (d) (4 points) Find the charge density at the origin, $\rho(0,0,0,t)$ as a function of time, if $\rho(0,0,0,0) = 0$.

3. Consider a homogeneous conductor where $\mathbf{J} = \sigma\mathbf{E}$ and $\sigma = 24\pi \times 10^5$ S/m.

   (a) (2 points) Use Gauss's law $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$ and the continuity equation $\nabla \cdot \mathbf{J} = -\frac{\partial\rho}{\partial t}$ to derive the differential equation:

$$\frac{\partial\rho}{\partial t} + \frac{\sigma}{\epsilon_o}\rho = 0$$

   for the charge density ρ.

   (b) (3 points) Find the solution of the differential equation above for $t > 0$ if at $t = 0$ the charge density is $\rho(x, y, z, 0) = 2\rho_0 sin(100z)$ C/m^3 over all space, where ρ_0 is positive.

   (c) (2 points) Find, using the properties of the conductor, the general expression and numerical value of the characteristic time over which the charge density ρ decreases in time by a factor of 1/e.

   (d) (3 points) Find the current density $\mathbf{J}(x, y, z, t)$ by using the continuity equation, knowing that ρ is a function of z and t only and assuming that there are no external applied $\mathbf{E}$ fields.

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4. In this problem, we will study the propagation of waves in free space. In a source-free region where $\mathbf{J} = 0$ and $\rho = 0$, Maxwell's equations reduce to:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{1}$$

For $\mathbf{E} = \hat{x} E_x(z,t)$, where $E_x(z,t)$ is an arbitrary function of coordinates z and t only,

(a) (1 point) Show that Gauss' Law is satisfied.

(b) (2 points) Determine $\frac{\partial \mathbf{B}}{\partial t}$ in terms of $E_x(z,t)$.

(c) (2 points) Determine $\nabla \times (\nabla \times \mathbf{E})$ in terms of $E_x(z,t)$.

(d) (3 points) Use your results for (a)-(c) to derive the **wave equation**:

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0$$

(e) (2 points) Using dimensional (unit) analysis, relate $\mu_0 \epsilon_0$ appearing in the wave equation in (d) to the speed of propagation of $E_x(z,t)$ in the $\hat{z}$- direction.

5. Let $\mathbf{E} = \hat{x} E_x(z,t)$ as in problem 4 above, where $E_x(z,t) = E_0 cos(\omega t \mp \beta z + \phi)$ and E_0 (amplitude), ω (radian frequency), β (wavenumber), and ϕ (phase shift) are positive real scalars. These fields describe waves polarized in the $\hat{x}$ direction and propagating in the $\pm \hat{z}$ directions, respectively.

(a) (2 points) Show that the specified E_x satisfies the wave equation derived above if:

$$\beta = \omega \sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c}$$

where $c \approx 3 \times 10^8$ m/s is the **speed of light** in free space.

(b) (6 points) Substitute the specified E_x into Faraday's law and solve for $\mathbf{B}(z,t)$ assuming that $\mathbf{B}(0,0) = \pm \frac{E_x(0,0)}{c} \hat{y}$. Show that your answer can be expressed as $\mathbf{B} = \pm \frac{E_x}{c} \hat{y}$ (be careful about maintaining the correct order for the upper and lower signs). Is there any restriction on the value for ϕ?

(c) (2 points) Use $\mathbf{B} = \mu_0 \mathbf{H}$ to show that $\mathbf{H} = \pm \frac{E_x}{\eta_0} \hat{y}$ in terms of an appropriately defined constant η_0. What is η_0 in terms of μ_0 and ϵ_0?

6. (5 points) Create a concept map that connects all the magnetostatic concepts: current I, current density $\mathbf{J}$, magnetic field $\mathbf{H}$, magnetic flux density $\mathbf{B}$, magnetization $\mathbf{M}$, and vector potential $\mathbf{A}$. For example, the concept map for charge Q, charge density ρ, and displacement field $\mathbf{D}$ is:

$$Q \xleftrightarrow{Q=\int \rho_{vol} dV} \rho \xleftrightarrow{\nabla \cdot \mathbf{D} = \rho_{vol}} \mathbf{D}$$

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