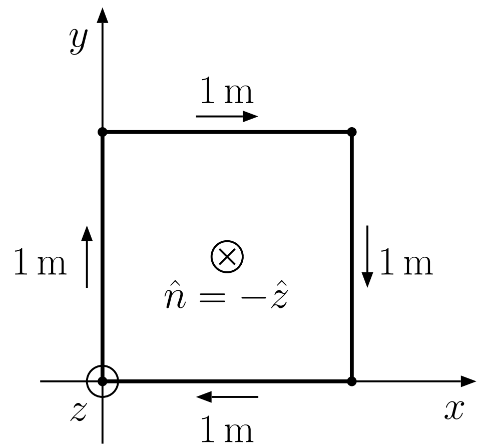
1. Since the closed paths are not varying in time and the magnetic field **B** is independent of position, we can rewrite Faraday's law as follows

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\mathbf{B}}{dt} \cdot \int_S d\mathbf{S} = \left(-\frac{d\mathbf{B}}{dt} \cdot \hat{n}\right) \cdot \text{Area},$$

where $\hat{n}$ is the unit vector normal to the surface, and

$$\frac{d\mathbf{B}}{dt} = B_0\left(\left(sin(\omega t) + t\omega cos(\omega t)\right)\hat{y} + \omega sin(\omega t)\,\hat{z}\right).$$

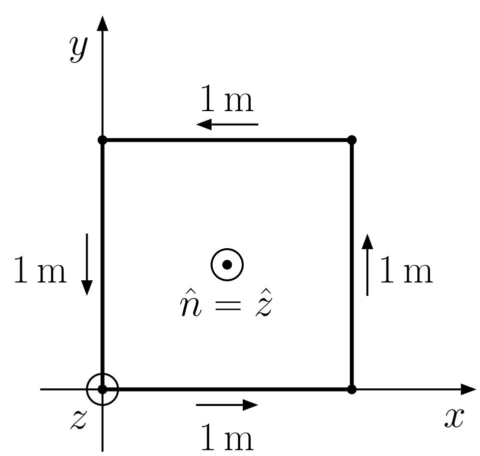
(a) (2 points) For the rectangular path shown in the figure below,



the area of the enclosed surface S is $1\,\mathrm{m}^2$ and the unit vector normal to S is $\hat{n} = -\hat{z}$. Therefore, the electromotive force is

$$\mathcal{E} = -\frac{d\mathbf{B}}{dt} \cdot \hat{z} = B_0\omega \sin(\omega t)\,\mathrm{V}.$$

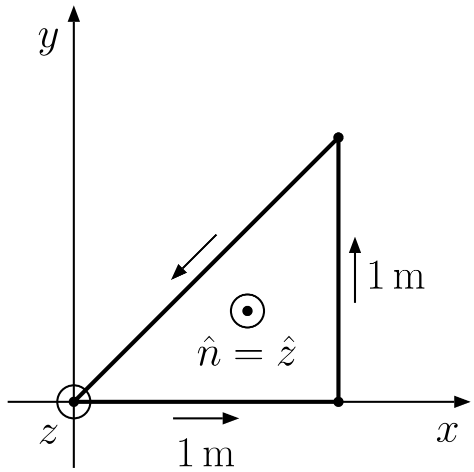
(b) (2 points) If we consider the same rectangle of part (a), but the direction of the path is reversed, we have that $\hat{n} = \hat{z}$.



As a result, the electromotive force is

$$\mathcal{E} = \frac{d\mathbf{B}}{dt} \cdot \hat{z} = -B_0\omega \sin(\omega t)\,[\mathrm{V}].$$

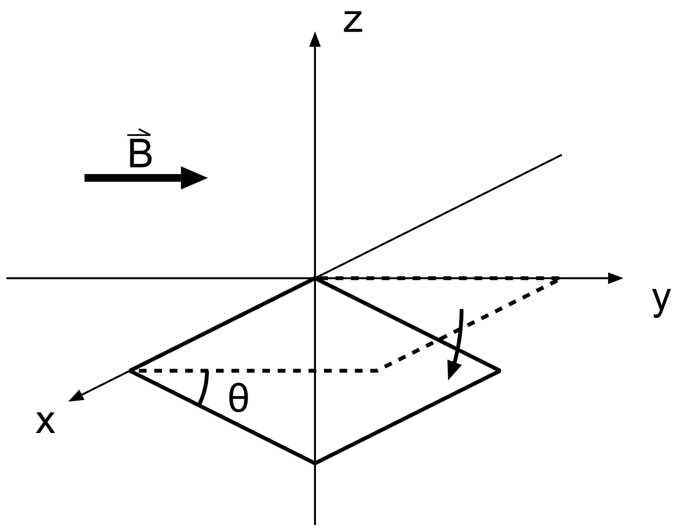
(c) (2 points) For the triangular path shown in the figure below,

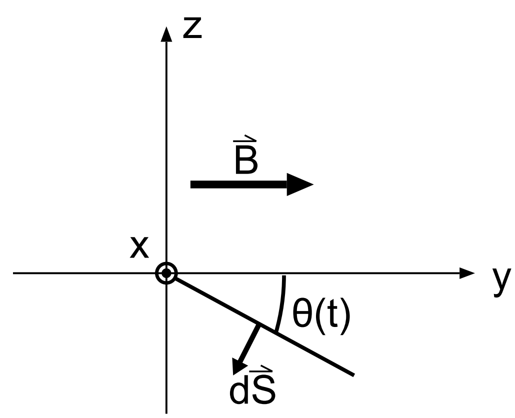


the area of the enclosed surface S is $\frac{1}{2}$ [m^2] and the unit vector normal to S is $\hat{n} = \hat{z}$. Then, the electromotive force is

$$\mathcal{E} = -\frac{d\mathbf{B}}{dt} \cdot \hat{z} \cdot \frac{1}{2} = -\frac{1}{2} B_0 \omega \sin(\omega t) \text{ [V]}.$$

2. Consider a square loop of wire of some finite resistance R with $4\,\text{cm}^2$ surface area that is located in a region of constant magnetic field $\mathbf{B} = 8\hat{y}\,\text{Wb/m}^2$. The loop can rotate around the x-axis as shown in the following figure.





(a) (2 points) If $\theta = 0°$, the area projection onto xz-plane is 0, and therefore, the magnetic flux is 0.
If $\theta = 90°$, the area projection onto xz-plane is $4\,[\text{cm}^2]$, and $d\mathbf{S} = -dS\hat{y}$, and therefore, the magnetic flux is

$$\Psi = \int_S \mathbf{B}\cdot d\mathbf{S} = \mathbf{B}\cdot\int_S d\mathbf{S} = 8\hat{y}\,\frac{\text{Wb}}{\text{m}^2}\cdot(-4\times10^{-4}\,\text{m}^2\hat{y}) = -32\times10^{-4}\,[\text{Wb}].$$

(b) (2 points) For any angle θ, the differential area is $d\mathbf{S} = -dS\,(\cos\theta\hat{z} + \sin\theta\hat{y})$, and therefore, the magnetic flux is

$$\Psi = \int_S \mathbf{B}\cdot d\mathbf{S} = 4\hat{y}\cdot\left(-8\times10^{-4}\,(\cos\theta\hat{z} + \sin\theta\hat{y})\right) = -32\times10^{-4}\sin\theta\,[\text{Wb}].$$

(c) (2 points) The induced emf is given by:

$$\mathcal{E} = -\frac{d\Psi}{dt} = 32\times10^{-4}\cos\theta\frac{d\theta}{dt}.$$

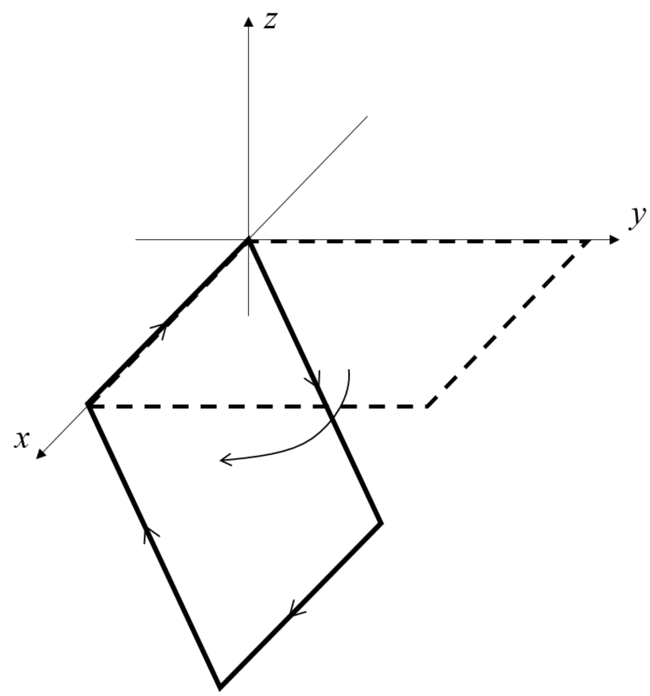
At t = 0.25 s,

$$\theta = \frac{1}{4}\pi$$

Considering $\theta = \frac{\pi}{4}$ rad and $\frac{d\theta}{dt} = \pi$ rad/s, we get

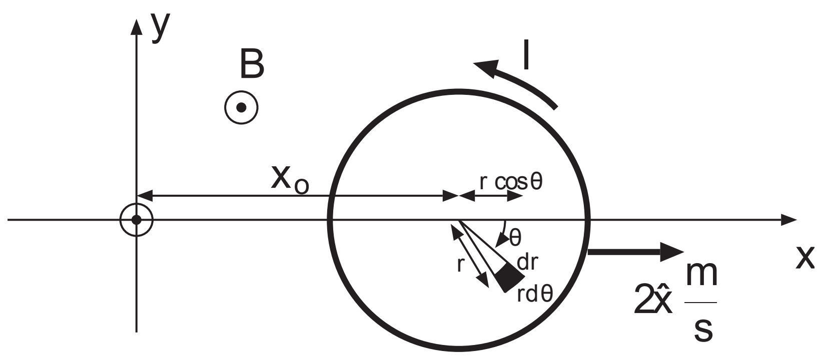
$$\mathcal{E}(\theta = \frac{\pi}{2}) = 32\times10^{-4}\left(\cos\frac{\pi}{4}\right)(\pi) = 0.0071\,[\text{V}].$$

(d) (2 points) The simple approach to determine the current direction is to remember: current is induced in such a way that the flux associated with the induced current opposes the change of the background flux. At this position, shown in the graph below, the loop rotates towards the xz-plane, which means the area projection onto xz-plane is increasing, which means the flux in the $+\hat{y}$ direction is increasing; as a result, the current is induced in such a way that the induced flux points towards the $-\hat{y}$ direction. Therefore, applying the right-hand rule, the current flow direction is specified in the graph as below.



(e) (2 points) In general you should know that $d\mathbf{S}$ can be assigned arbitrarily without affecting the physical result. Therefore, the answer is that the current flows in the same direction as in part (d). If you would like to think about this problem in terms of vector representations, here $d\mathbf{S}$ changes sign, which means the change of flux also changes sign, and then the induced current also changes "sign". The induced current is used to be in the opposite direction of $d\mathbf{S}$, but now it is in the same direction of $d\mathbf{S}$ (in the sense of right-hand rule, as always).

3. The geometry of the problem is shown in the figure below.



(a) (3 points) Looking at the described geometry in cylindrical coordinate, the magnetic flux is given by

$$\begin{aligned}
\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} &= \int_0^r \int_0^{2\pi} 25 \times 10^{-6} \left(1 - \frac{x_o + r' \cos\theta}{L}\right) r' d\theta dr' \\
&= 25 \times 10^{-6} \left(\int_0^r \int_0^{2\pi} r' d\theta dr' - \int_0^r \int_0^{2\pi} \frac{x_o}{L} r' d\theta dr' - \int_0^r \int_0^{2\pi} \frac{r' \cos\theta}{L} r' d\theta dr' \right) \\
&= 25 \times 10^{-6} \left(\pi r^2 - \frac{x_o}{L} \pi r^2 - \int_0^r r' dr' \int_0^{2\pi} \frac{\cos\theta}{L} d\theta \right) \\
&= 25 \times 10^{-6} \left(\pi r^2 - \frac{x_o}{L} \pi r^2 - 0 \right) \\
&= 25 \times 10^{-6} \pi r^2 \left(1 - \frac{x_o}{L} \right) \text{ [Wb]}.
\end{aligned}$$

Thus, the emf $\mathcal{E}$ is

$$\begin{aligned}
\mathcal{E} &= -\frac{d\Psi}{dt} = 25\pi r^2 \times 10^{-6} \times \frac{1}{L} \times \frac{dx_o}{dt} = 25\pi \times 10^{-6} \times \frac{1}{1000} \times 2 \\
&\approx 157.08 \text{ [nV]}.
\end{aligned}$$

(b) (1 point) The magnitude of the loop current with resistance $2\,\Omega$ is

$$I = \frac{|\mathcal{E}|}{R} = 78.54 \text{ [nA]}.$$

4. (a) (2 points) At time t, the area of the loop= $L(z_0 + v_0 t)$; for clockwise contour (following ABC), $d\mathbf{S} = dS(-\hat{x})$.

$$\begin{aligned}
\Psi &= \int_S \mathbf{B} \cdot d\mathbf{S} \\
&= B_0 L(z_0 + v_0 t) \\
\mathcal{E} &= -\frac{d\Psi}{dt} = -B_0 L v_0 \,[\text{V}]
\end{aligned}$$

(b) (2 points) $\mathcal{E} < 0$ so the current flows opposite to the original contour, i.e. along CBA (counterclockwise),

$$\begin{aligned}
I_0 &= \frac{\mathcal{E}}{R} = \frac{-B_0 v_0 L}{R} \,[\text{A}] \\
|I_0| &= \frac{B_0 v_0 L}{R} \,\text{A}
\end{aligned}$$

(c) (2 points) Lorentz force per unit length

$$\begin{aligned}
d\mathbf{F} &= I_0 d\mathbf{l} \times \mathbf{B} \\
&= I_0 (dy\hat{y}) \times B_0(-\hat{x}) \\
&= I_0 B_0 dy \hat{z}
\end{aligned}$$

Therefore the total $\mathbf{F}$

$$\mathbf{F} = \int d\mathbf{F} = \int_0^L I_0 B_0 dy \hat{z} = I_0 B_0 L \hat{z} \,[\text{N}]$$

For I_0 as above

$$\mathbf{F} = \frac{B_0^2 v_0 L^2}{R}(-\hat{z}) \,[\text{N}]$$

(d) (2 points) Since the current flows along the contour ABC, I' is in the $+\hat{y}$ direction in the armature. Similar to part (c), at the moment right when the mechanical force is removed:

$$\begin{aligned}
\mathbf{F} &= I' L \hat{y} \times B_0(-\hat{x}) \\
&= I' B_0 L \hat{z} \,\text{N} \\
\mathbf{a} &= \frac{I' B_0 L}{M} \hat{z} \,[\text{m/s}^2]
\end{aligned}$$

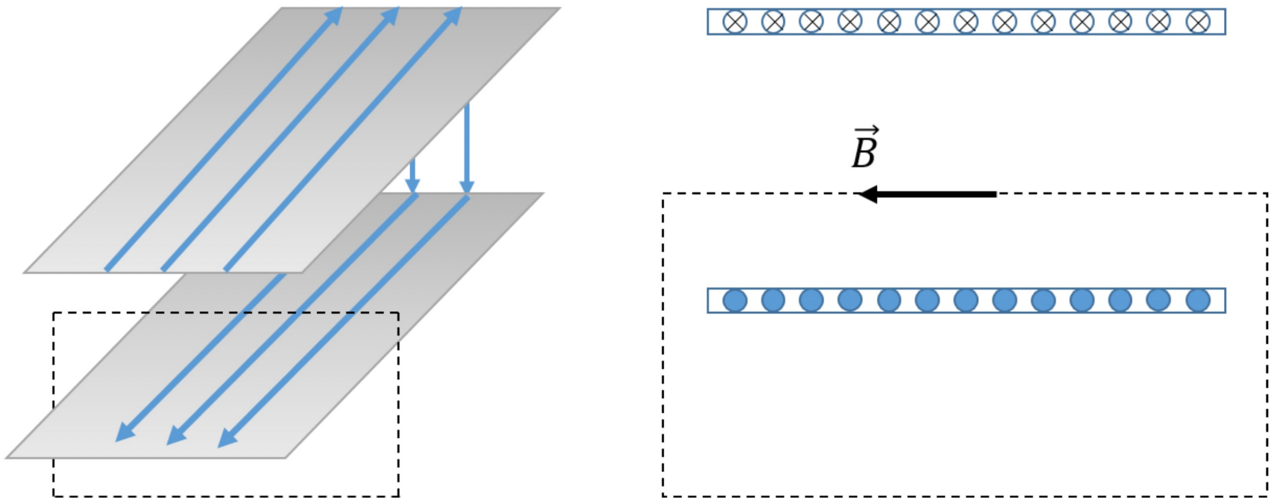
Alternatively, we could find the total force by superposing the Lorentz forces on each charge in the armature:

$$\mathbf{F} = NLAq\,(\mathbf{v} \times \mathbf{B})$$

where N is the number density of the charges in the armature, A is its cross-sectional area, and LA is its volume. Since $\mathbf{J} = qN\mathbf{v} = \frac{I'}{A}\hat{y}$, we can arrive at the same result above.

5. (5 points) Since $l \gg d$, an infinite parallel plate approximation can be invoked. Apply Ampere's law to a rectangular contour as shown in the figure gives

$$|B|\omega = \mu_0 I$$



The magnetic flux linked by the closed current path is

$$\Psi = \int_S B dS = |B| \int dS = |B| l \times d$$

Hence, the self-inductance is found as

$$L = \frac{\Psi}{I} = \frac{dl}{\omega}\mu_0.$$

6. (a) (2 points) The current in the outer loop can be calculated by $I_a = \frac{V}{R}$, where $R \equiv \frac{1}{G} = \frac{d}{A\sigma}$

$$I_a = \frac{V}{R} = \frac{V}{\frac{2\pi a}{\sigma \pi r^2}} = \frac{3}{\frac{2\times 0.08}{4\times 10^7 \times 0.001^2}} = 750\,[\mathrm{A}].$$

(b) (2 points) The magnetic flux density generated by the outer loop only has z component along the z axis

$$B_z = \frac{\mu_o I_a a^2}{2(a^2+z^2)^{\frac{3}{2}}}.$$

At the origin, $B_z(z=0) = \frac{\mu_o I_a}{2a}\,[\frac{Wb}{\mathrm{m}^2}]$. Since $b \ll a$, we assume that the B_z across the inner loop is constant. Thus,

$$\Psi_{a\to b} = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_S B_z(z=0)\hat{z}\cdot\hat{z}dS = \frac{\mu_o I_a}{2a}\pi b^2 = \frac{\mu_o \pi b^2}{2a} I_a\,[\mathrm{Wb}].$$

(c) (2 points) The numerical value of $L_{a\to b}$ is

$$L_{a\to b} = \frac{\Psi_{a\to b}}{I_a} = \frac{\mu_o \pi b^2}{2a}\,\mathrm{H} \approx 0.625\,[\mathrm{nH}].$$

(d) (2 points) Applying induced emf formula $\mathcal{E} = -\frac{d\Psi}{dt}$, we get

$$\mathcal{E}_{a\to b} = -\frac{d\Psi_{a\to b}}{dt} = -\frac{d}{dt}\int_S B\cdot dS = -\frac{d}{dt}(B_z A_b) = -A_b\frac{d}{dt}B_z.$$

Here, B_z is given by

$$B_z = \frac{\mu_o I_a a^2}{2(a^2+z^2)^{\frac{3}{2}}},$$

Therefore, we can write

$$\frac{d}{dt}B_z = -\frac{3}{2}\frac{\mu_o I_a a^2 z}{(a^2+z^2)^{\frac{5}{2}}}\frac{dz}{dt} = -\frac{3}{2}\frac{\mu_o I_a a^2 z}{(a^2+z^2)^{\frac{5}{2}}}v_z,$$

from which we obtain

$$\mathcal{E}_{a\to b} = \frac{3\pi}{2}\frac{\mu_o I_a a^2 b^2 z}{(a^2+z^2)^{\frac{5}{2}}}v_z\,\mathrm{V}.$$

If we consider, $z = v_z t$, we can write $\mathcal{E}_{a\to b}$ as a function of time t as

$$\mathcal{E}_{a\to b}(t) = \frac{3}{2}\frac{\mu_o I_a a^2 v_z^2 \pi b^2}{(a^2+v_z^2 t^2)^{\frac{5}{2}}}t\,[\mathrm{V}].$$

(e) (2 points) As the induced emf $\mathcal{E}$ is given by $\mathcal{E} = -\frac{d\Psi_t}{dt}$, also the current I by $I = \frac{\mathcal{E}}{R}$. Then, the induced current in the inner loop will be

$$I_b = -\frac{1}{R}\frac{d\Psi_t}{dt},$$

where $\Psi_t = \Psi_{a\to b} + \Psi_s$, $\Psi_{a\to b} = -\int \mathcal{E}_{a\to b}(t)\,dt$ and $\Psi_s = LI_b$ ($L = 0.2\,\mu\text{H}$). Thus, we can write

$$I_b = -\frac{1}{R}\frac{d\Psi_{a\to b}}{dt} - \frac{L}{R}\frac{dI_b}{dt}.$$

Rearranging the above equation, it will give us the differential equation as

$$\frac{dI_b}{dt} + \frac{R}{L}I_b = -\frac{1}{L}\frac{d\Psi_{a\to b}}{dt},$$

where right hand side is the emf $\mathcal{E}_{a\to b}(t)$ of part (d) divided by L.