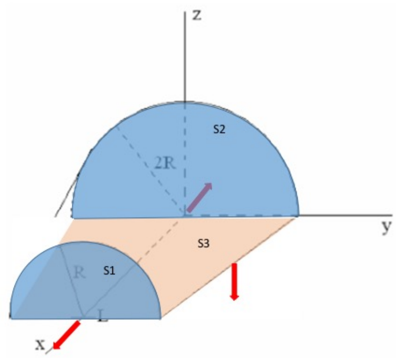
1. (10 points) Applying Gauss' law $\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$ and defining the closed surface S composed by the semicircle S_1 with radius R, the semicircle S_2 with radius $2R$, the trapezoid S_3 with bases $2R$ and R and height L, and partial cone surface S_4 as shown in the figure below, we can rewrite Gauss' law as,



$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_{S_1} \mathbf{B} \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{B} \cdot d\mathbf{S}_2 + \int_{S_3} \mathbf{B} \cdot d\mathbf{S}_3 + \int_{S_4} \mathbf{B} \cdot d\mathbf{S_4} = 0.$$

Now, the flux through each surface can be calculated as follows. On the semicircle S_1, the magnetic flux is given by

$$\int_{S_1} \mathbf{B} \cdot d\mathbf{S}_1 = \int_{S_1} (-3B_o\hat{x} + 2B_o\hat{y} + -1B_o\hat{z}) \cdot (-\hat{x})\, dS_1 = 6\pi R^2 B_o.$$

On the semicircle S_2, the magnetic flux is given by

$$\int_{S_2} \mathbf{B} \cdot d\mathbf{S}_2 = \int_{S_2} (-3B_o\hat{x} + 2B_o\hat{y} + -1B_o\hat{z}) \cdot \hat{x}\, dS_2 = -\frac{3}{2}\pi R^2 B_o$$

On the trapezoid S_3, the magnetic flux is given by

$$\int_{S_3} \mathbf{B} \cdot d\mathbf{S}_3 = \int_{S_3} (-3B_o\hat{x} + 2B_o\hat{y} + -1B_o\hat{z}) \cdot (-\hat{z})\ dS_3 = 3RLB_o.$$

Finally, the magnetic flux through S_4 is calculated as

$$\begin{aligned}\int_{S_4} \mathbf{B} \cdot d\mathbf{S}_4 &= -\left(\int_{S_1} \mathbf{B} \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{B} \cdot d\mathbf{S}_2 + \int_{S_3} \mathbf{B} \cdot d\mathbf{S_3}\right)\\ &= -RB_o\left(\frac{9}{2}\pi R + 3L\right) \text{ [Wb]}.\end{aligned}$$

2. An infinite current sheet with a uniform current density $J_s = J_s\hat{z}\,\mathrm{A/m}$ produces magnetostatic fields B with $\frac{\mu_o J_s}{2}$ magnitude on both sides of the sheet and with opposing direction.

   (a) (5 points) Magnetic field intensity at the origin due to sheet 1 ($\mathbf{J}_{s1} = 2\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$) is given by

$$\mathbf{H}_1 = \frac{1}{2}\mathbf{J}_{s1} \times \hat{x} = 1\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}},$$

   whereas the magnetic field intensity at the origin due to sheet 2 ($\mathbf{J}_{s2} = 2\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$) is found as

$$\mathbf{H}_2 = \frac{1}{2}\mathbf{J}_{s2} \times (-\hat{x}) = -1\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   Then, we find the resultant displacement vector due to both sheets as

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = \mathbf{0}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   (b) (5 points) Magnetic field intensity at the origin due to sheet 1 ($\mathbf{J}_{s1} = 2\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}$) is given by

$$\mathbf{H}_1 = \frac{1}{2}\mathbf{J}_{s1} \times \hat{x} = -1\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   whereas the magnetic field intensity at the origin due to sheet 2 ($\mathbf{J}_{s2} = -2\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}$) is found as

$$\mathbf{H}_2 = \frac{1}{2}\mathbf{J}_{s2} \times (-\hat{x}) = -1\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   Then, we find the resultant displacement vector due to both sheets as

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = -2\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   (c) (5 points) Magnetic field intensity at the origin due to sheet 1 ($\mathbf{J}_{s1} = -2\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$) is given by

$$\mathbf{H}_1 = \frac{1}{2}\mathbf{J}_{s1} \times \hat{x} = -1\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   whereas the magnetic field intensity at the origin due to sheet 2 ($\mathbf{J}_{s2} = 2\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}$) is found as

$$\mathbf{H}_2 = \frac{1}{2}\mathbf{J}_{s2} \times (-\hat{x}) = 1\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   Then, we find the resultant displacement vector due to both sheets as

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = -\hat{y} + \hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

3. (a) (5 points) Using integral form of Ampere's Law : $\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 \mathrm{I}_{\mathrm{enc}}$ where $\mathrm{I}_{\mathrm{enc}}$ is the current enclosed by the defined path. In this geometry setup, we will define our closed path as a circle with radius r. For $r < a$, since there is no current enclosed, $\mathbf{B} = 0$. For $a < r < b$:

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 \mathrm{I}_{\mathrm{enc}}$$

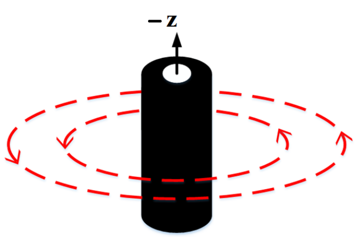
$$\begin{aligned} B \cdot 2\pi r &= -\mu_0[\pi(r^2 - a^2)J_0] \\ \mathbf{B} &= -\frac{\mu_0(r^2 - a^2)J_0}{2r}\hat{\phi} \end{aligned}$$

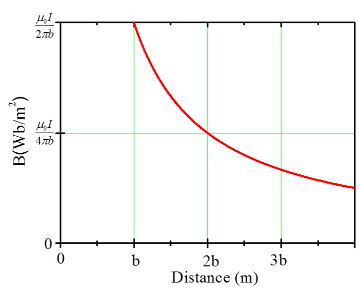
For $r > b$:

$$\begin{aligned} B \cdot 2\pi r &= -\mu_0[\pi(b^2 - a^2)J_0] \\ \mathbf{B} &= -\frac{\mu_0(b^2 - a^2)J_0}{2r}\hat{\phi} \end{aligned}$$

where $\hat{\phi}$ is the azimuthal unit vector in cylindrical coordinate system.

(b) (2 points) The direction of the field can be found using right-hand rule with thumb pointing in $-\hat{z}$, and the intensity of the field shows a $\frac{1}{r}$ decrease when moving away from the hollow cylinder. In general, for $r > b$, the magnetic field is the same as an infinite line with current I in $-\hat{z}$ direction.





(c) (5 points) For $r < a$, since there is no current enclosed, $\mathbf{B} = 0$. For $a < r < b$:

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 \mathrm{I}_{\mathrm{enc}}$$

$$\begin{aligned} B \cdot 2\pi r &= -\mu_0 \int_a^r \int_0^{2\pi} \frac{J_0}{r'} r' dr' d\phi \\ \mathbf{B} &= -\frac{\mu_0 J_0 (r - a)}{r}\hat{\phi} \end{aligned}$$

For $r > b$:

$$\begin{aligned} B \cdot 2\pi r &= -\ \mu_0 \int_a^b \int_0^{2\pi} \frac{J_0}{r'} r' d\phi dr' \\ \mathbf{B} &= -\frac{\mu_0 J_0 (b - a)}{r}\hat{\phi} \end{aligned}$$

(d) (3 points) The differential form of Gauss's Law for magnetic field states that $\nabla \cdot \mathbf{B} = 0$. For $r < a$, since the magnetic field is 0, the divergence of it is also 0. For $a < r < b$:

$$\nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\partial}{\partial \phi}[\frac{-\mu_0 J_0 (r-a)}{r}] = 0$$

since B_ϕ does not have any ϕ dependence, the divergence is 0. For $r > b$, this is the same case, so divergence of magnetic field is also 0.

4. (a) (2 points) Biot-Savart law tells us that the direction of the magnetic field generated by an infinitesimal current element is parallel to the cross product between the direction of the current and the vector joining the current element and the point under consideration. In our case, the current is parallel to $\hat{x}$. Then, using the right-hand rule, we can easily verify that, above the slab ($z > 1$), $\mathbf{B}$ should be along $\hat{x}$,

$$\hat{b} = 2\hat{y} \times \hat{z} \Longrightarrow \hat{b} = 2\hat{x},$$

while, below the slab ($z < -3$), $\mathbf{B}$ should be along $-\hat{x}$,

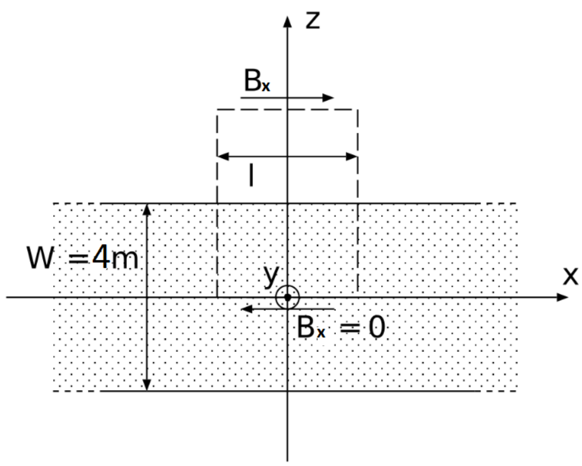
$$\hat{b} = 2\hat{y} \times -\hat{z} \Longrightarrow \hat{b} = -2\hat{x}.$$

Since the current distribution is symmetric with respect to the plane at $z = -1$, the fields at the same distance from above and below the slab should have equal amplitudes.

(b) (1 point) The magnetic fields slightly above and below $z = -1$ should have equal amplitudes but point in opposite directions. Exactly at $z = -1$ the fields must cancel each other such that

$$\mathbf{B}(z = -1) = \mathbf{0}\,\mathrm{Wb/m^2}.$$

(c) (3 points) To compute $\mathbf{B}$ above the slab using Ampere's law, let us first define a rectangle R placed on the x-z plane as it is shown in the figure below.



Integrating $\mathbf{B}$ along the perimeter of the rectangle and noting that only the segment above the slab contributes to the integration, we have

$$\frac{1}{\mu_o}\oint_R \mathbf{B}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{S}$$

$$\frac{1}{\mu_o}(B_x \times l) = 2 \times \frac{W}{2} \times l$$

$$B_x = \mu_o W = 4\mu_0\,\mathrm{Wb/m^2}.$$

Similarly, we can compute $\mathbf{B}$ below the slab and obtain

$$\mathbf{B} = \begin{cases} 4\mu_o\,\hat{x}\,\mathrm{Wb/m^2} & \text{for } z > 1\,\mathrm{m}, \\ -4\mu_o\,\hat{x}\,\mathrm{Wb/m^2} & \text{for } z < -3\,\mathrm{m}. \end{cases}$$

(d) (2 points) Applying the same technique, but considering a rectangle of height shorter than $W/2$ (2 m), we can compute $\mathbf{B}$ inside the slab (i.e., for $-3 \le z \le 1$ m) as follows

$$\frac{1}{\mu_o}\oint_R \mathbf{B}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{S}$$
$$\frac{1}{\mu_o}(B_x l) = 2\times(z+1)\times l$$
$$B_x = 2\mu_o(z+1).$$

Finally, extending this result to negative z, we obtain

$$\mathbf{B} = 2\mu_o(z+1)\,\hat{x}\,\mathrm{Wb/m^2} \quad \text{for } -3\,\mathrm{m} \le z \le 1\,\mathrm{m}.$$

(e) (2 points) Plotting B_x as function of z.