1. (a) (5 points) We can write $\mathbf{E}$ as

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\left(\hat{x}\frac{x}{r^3} + \hat{y}\frac{y}{r^3} + \hat{z}\frac{z}{r^3}\right).$$

Therefore,

$$\begin{aligned}\nabla \times \mathbf{E} &= \frac{Q}{4\pi\epsilon_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3}\end{vmatrix} \\ &= \frac{Q}{4\pi\epsilon_0}\hat{x}\left[\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right] + \\ &\quad \frac{Q}{4\pi\epsilon_0}\hat{y}\left[\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) - \frac{\partial}{\partial x}\left(\frac{z}{r^3}\right)\right] + \\ &\quad \frac{Q}{4\pi\epsilon_0}\hat{z}\left[\frac{\partial}{\partial x}\left(\frac{y}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right].\end{aligned}$$

Take the calculation of $\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right)$ for example:

$$\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) = \frac{\partial}{\partial y}\left[\frac{z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right] = -\frac{3zy}{\left(\sqrt{x^2+y^2+z^2}\right)^5}.$$

Similarly, we can obtain the other partial differentials. Finally,

$$\begin{aligned}\nabla \times \mathbf{E} &= \hat{x}\frac{Q}{4\pi\epsilon_0}\left[-\frac{3zy}{\left(\sqrt{x^2+y^2+z^2}\right)^5} + \frac{3yz}{\left(\sqrt{x^2+y^2+z^2}\right)^5}\right] \\ &\quad +\hat{y}\frac{Q}{4\pi\epsilon_0}\left[-\frac{3xz}{\left(\sqrt{x^2+y^2+z^2}\right)^5} + \frac{3zx}{\left(\sqrt{x^2+y^2+z^2}\right)^5}\right] \\ &\quad +\hat{z}\frac{Q}{4\pi\epsilon_0}\left[-\frac{3yx}{\left(\sqrt{x^2+y^2+z^2}\right)^5} + \frac{3xy}{\left(\sqrt{x^2+y^2+z^2}\right)^5}\right] \\ &= \mathbf{0}.\end{aligned}$$

(b) (5 points) In this spherically symmetric situation,

$$\nabla V = \frac{\partial V}{\partial r}\hat{r}.$$

Therefore,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r} = -\left(-\frac{Q}{4\pi\epsilon_0 r^2}\right)\hat{r} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}.$$

We know that

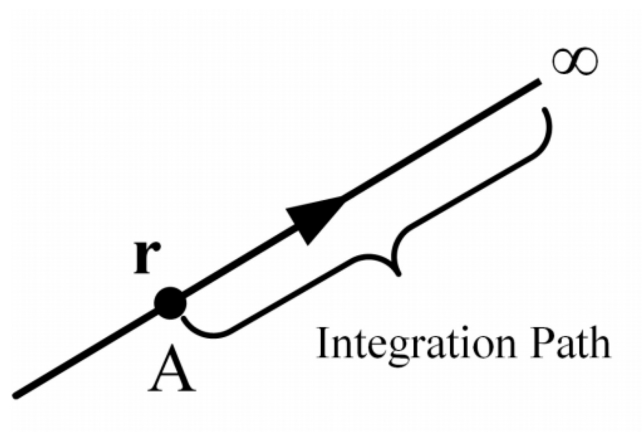
$$V(\mathbf{r}) - V(\infty) = \int_r^{\infty} \mathbf{E} \cdot d\mathbf{l}$$

From part a), we know $\nabla \times \mathbf{E} = \mathbf{0}$, which means that the above integration is path-independent. Thus, we can simply take the path to be a straight line from point A to $r = \infty$:

$$\begin{aligned} V(\mathbf{r}) - V(\infty) &= \int_r^{\infty} \mathbf{E} \cdot d\mathbf{r} \\ &= \int_r^{\infty} \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \cdot \hat{r} dr \\ &= \int_r^{\infty} \frac{Q}{4\pi\epsilon_0 r^2} dr \\ &= 0 - \left(-\frac{Q}{4\pi\epsilon_0 r}\right) \\ &= \frac{Q}{4\pi\epsilon_0 r}. \end{aligned}$$

By plugging in $V(\infty) = 0$, we have

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r}.$$



2. A vacuum diode consists of a cathode in the z = 0 plane and an anode in the z = d plane. The electric field between the the plates is given by $\mathbf{E} = -\hat{z}\frac{4V_0}{3d}(\frac{z}{d})^{1/3}$ and the field outside the of the gap region is zero.

   (a) (4 points) Electric field is the negative gradient of V(z),

$$-\nabla V = \mathbf{E} = -\frac{4V_0}{3d}(\frac{z}{d})^{1/3}\hat{z}$$

$$\begin{array}{rcl} V(z) & = & \int_0^z -\mathbf{E}(z')dz' + V_b \\ & = & V_0(\frac{z}{d})^{4/3} + V_b \end{array}$$

   By applying the given boundary conditions V(d) = 4 and V(0) = 0,

$$V(z) = 4(\frac{z}{d})^{4/3}$$

   Solve V(1) = 0.25, we can get that,

$$d = 8\,[m]$$

   (b) (4 points) For volumetric charge density,

$$\rho = \nabla \cdot \mathbf{D} = \epsilon_o \nabla \cdot \mathbf{E} = -\frac{4\epsilon_o V_0}{9d^2}(\frac{z}{d})^{-2/3}$$

   Evaluating at z=5d/8 ,

$$\rho(5d/8) = -\frac{5^{-\frac{2}{3}}\epsilon_o}{9}\,[C/m^3]$$

   (c) (2 points) The surface charge density on the anode (we choose $\hat{n} = -\hat{z}$) is

$$\rho_s(d) = \hat{n} \cdot \mathbf{D}(d) = -\hat{z} \cdot \epsilon_o(-\frac{4V_0}{3d}(1)^{1/3}\hat{z})$$

   Plug in the constants and we have

$$\rho_s(d) = \frac{2}{3}\epsilon_o\,[\mathrm{C/m^2}]$$

3. (a) (2 points) Since the electric field is in the $-\hat{x}$ direction, we can tell that the plate at x = 10 m is holding positive charge density, and the plate at x = 0 m is holding negative charge density.

   (b) (4 points) By superposing the electric fields generated by the two plates, we know that outside of the region between the two plates, the electric field is **0**. Now, since we already know the charge density of the plate at x = 0 m, by applying boundary condition at x = 0 m, we can find ϵ_1

$$\rho_s = \hat{n} \cdot (\mathbf{D}_+ - \mathbf{D}_-) = \hat{x} \cdot (-2\epsilon_1\hat{x} - 0) = -2\epsilon_1 = -5\epsilon_o$$

$$\epsilon_1 = \frac{5}{2}\epsilon_o$$

and similarly applying the boundary condition at x = 10 m ,

$$\rho_s = \hat{n} \cdot (\mathbf{D}_+ - \mathbf{D}_-) = -\hat{x} \cdot (-\frac{\epsilon_2}{2}\hat{x} - 0) = \frac{\epsilon_2}{2} = 5\epsilon_o$$

$$\epsilon_2 = 10\epsilon_o$$

Notice that $\mathbf{D} = -5\epsilon_0$ everywhere between the plates and thus is continuous at x = 2 m

   (c) (2 points) Since both of the slabs are perfect dielectrics, there should be no charge at the interface at x = 2,

$$\begin{aligned} \rho &= \hat{n} \cdot (\mathbf{D}_+ - \mathbf{D}_-) \\ &= \hat{x} \cdot [(-\frac{1}{2}\hat{x}10\epsilon_o - (-2\hat{x}\tfrac{5}{2}\epsilon_o)] \\ &= 0 \end{aligned}$$

   (d) (4 points) From $\mathbf{E} = -\nabla V$, we know that the potential in the two regions should have the form of $V(x) = mx + k$. For $0 \le x \le 2$,

$$\mathbf{E} = -\nabla V = -2\hat{x}$$

The potential should have the form of

$$V(x) = 2x + A$$

And for $2 \le x \le 10$,

$$V(x) = \frac{x}{2} + B$$

Apply boundary condition at x =10 m, V(10) = 0, we can get B = -5. Then, the potential should be continuous at x=2, so we can have

$$\begin{aligned} 4 + A &= -4 \\ A &= -8 \end{aligned}$$

The potential is

$$\begin{aligned} V(x) &= 2x - 8, \quad 0 \le x \le 2 \\ V(x) &= \frac{x}{2} - 5, \quad 2 \le x \le 10 \end{aligned}$$

(e) (1 point) Inside the two regions, we have $\nabla^2 V(x) = \frac{\partial^2}{\partial z^2} V(x) = 0$, which means that Laplace's equation is satisfied inside the two regions. However, at the interface x=2 m, the derivative of $V(x)$ is not differentiable, so Laplace's equation is not satisfied.

(f) (2 points) From (d) , we can know the the voltage drop between the 2 conducting plates is 8 [V], and $\frac{Capacitance}{Area} = \frac{Q/Area}{V} = \frac{\rho}{V}$

$$\frac{C}{A} = \frac{5\epsilon_0}{8}\,[\mathrm{F/m^2}]$$

4. Since the field is in $-\hat{z}$ direction, we can know that the plate at z = 0 m is holding a surface charge density of $-2\,[\mathrm{C/m^2}]$. The electric field associated with surface charge density of ρ_s is

$$\mathbf{E} = \hat{z}\frac{\rho_s}{2\epsilon_o} sgn(z)$$

The field between the two plates is $\mathbf{E} = -\frac{2}{\epsilon_o}\hat{z}\,[\mathrm{V/m}]$ and $\mathbf{E} = 0\,[\mathrm{V/m}]$ elsewhere

(a) (7 points) The displacement in the gap

$$\mathbf{D} = \epsilon_o \frac{-2}{\epsilon_0}\hat{z} = -2\hat{z}\,[C/m^2]$$

By definition,

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_o\mathbf{E} + \mathbf{P}$$

The gap is occupied by vacuum, $\epsilon = \epsilon_o$

$$\mathbf{P} = 0\,[\mathrm{C/m^2}]$$

From $\mathbf{E} = -\nabla V$, we can know $V(z) = \frac{2z}{\epsilon_0} + C$. Given that $V(0) = 0$,

$$V(z) = \frac{2z}{\epsilon_0}\,[\mathrm{V}]$$

Voltage drop between the two copper plates is

$$V(2) - V(0) = \frac{4}{\epsilon_0}$$

so capacitance per unit area is

$$\frac{C}{A} = \frac{\epsilon_0}{2}\,[\mathrm{F/m^2}]$$

(b) (5 points) The displacement does not change even if vacuum is replaced by pure water because the charge densities on the plates are the same. So

$$\mathbf{E} = \frac{-2}{\epsilon} = -\frac{2}{80\epsilon_o} = -\frac{1}{40\epsilon_o}\hat{z}$$

$$\begin{aligned} \mathbf{D} &= \epsilon_o\mathbf{E} + \mathbf{P} \\ -2\hat{z} &= -\frac{1}{40}\hat{z} + \mathbf{P} \\ \mathbf{P} &= -\frac{79}{40}\hat{z}\,[\mathrm{C/m^2}] \end{aligned}$$

From $\mathbf{E} = -\nabla V$, we can know $V(z) = \frac{z}{40\epsilon_0} + C$. Given that $V(0) = 0$,

$$V(z) = \frac{z}{40\epsilon_0}\,[\mathrm{V}]$$

Voltage drop between the two copper plates is

$$V(2) - V(0) = \frac{1}{20\epsilon_0}\,[\mathrm{V}]$$

so capacitance per unit area is

$$\frac{C}{A} = 40\epsilon_0\,[\mathrm{F/m^2}]$$

(c) (3 points) If the conductivity of the material between the parallel plates becomes different from zero then $\mathbf{E} = \mathbf{D} = 0$ and $\mathbf{P} = 0$ as the steady-state equilibrium is reached. Also, the whole conducting system is at the same potential, so $V = 0$. This happens because in a conducting medium all equilibrium fields vanish after the rearrangements of the net charge on the bounding surface. In this particular case, the salt water shorts out the original field between the plates. Since there is no potential difference between the two plates, and the two plates do not carry any charge, the capacitance is undefined.

5. We will look at each region at a time.

   (a) (4 points) For $r \leq a$, it is a perfect conductor. For a conductor, there is no electric field within it when equilibrium is reached, and all the charges should be located on the surface. However, since the region $a < r < b$ is also made of conductor, the electric field should also be 0 within it. According to Gauss' Law, $\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{enclosed}$, the charge enclosed in a spherical Gausian surface with radius R, which $a < R < b$, is 0. So we can conclude for $r \leq a$,

$$\mathbf{D_1} = 0\,[\frac{C}{m^2}] \quad \mathbf{E_1} = 0\,[\frac{V}{m}] \quad \mathbf{P_1} = 0\,[\frac{C}{m^2}] \quad \rho_s|_{r=a} = 0\,[\frac{\mathrm{C}}{\mathrm{m}^2}]$$

   (b) (4 points) As explained above, for $a < r < b$

$$\mathbf{D_2} = 0\,[\frac{C}{m^2}] \quad \mathbf{E_2} = 0\,[\frac{V}{m}] \quad \mathbf{P_2} = 0\,[\frac{\mathrm{C}}{\mathrm{m}^2}]$$

   This region is surround by a shell of perfect dielectric material, so the charge, $Q = -2\,[\mathrm{C}]$, is sitting on the interface at r=b. The charge density at the interface is

$$\rho_s|_{r=b} = \frac{Q}{A} = \frac{-2}{4\pi b^2} = \frac{-1}{2\pi b^2}\,[\frac{\mathrm{C}}{\mathrm{m}^2}]$$

   (c) (4 points) For $b < r < c$, this shell is made of perfect dielectric material with $\epsilon = 4\epsilon_o$. Apply Gauss' Law, $\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{enclosed}$, with the Gaussian surface being a sphere with radius R and $b < R < c$

$$\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{s} &= Q_{enclosed} \\ \mathbf{D_3} \cdot 4\pi R^2 &= -2 \end{aligned}$$

   We have

$$\mathbf{D_3} = \frac{-1}{2\pi R^2}\hat{r}\,[\frac{\mathrm{C}}{\mathrm{m}^2}]$$

$$\mathbf{E_3} = \frac{\mathbf{D_3}}{\epsilon} = \frac{-1}{8\epsilon_o \pi R^2}\hat{r}\,[\frac{\mathrm{V}}{\mathrm{m}}]$$

$$\mathbf{P_3} = \mathbf{D_3} - \epsilon_o \mathbf{E_3} = \frac{-3}{8\pi R^2}\hat{r}\,[\frac{\mathrm{C}}{m^2}]$$

   With $\hat{r}$ being the radial direction unit vector. Since this region is made of perfect dielectric material,

$$\rho_s|_{r=c} = 0\,[\frac{C}{m^2}]$$

   (d) (4 points) For $r > c$, again, apply Gauss' Law, $\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{enclosed}$, with the Gaussian surface to be a sphere with radius R and $R > c$

$$\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{s} &= Q_{enclosed} \\ \mathbf{D_4} 4\pi R^2 &= -2 \end{aligned}$$

   We get

$$\mathbf{D_4} = \frac{-1}{2\pi R^2}\hat{r}\,[\frac{C}{m^2}]$$

$$\mathbf{E_4} = \frac{\mathbf{D_4}}{\epsilon} = \frac{-1}{2\epsilon_o \pi R^2}\hat{r}\,[\frac{V}{m}]$$

And $\epsilon = \epsilon_o$, so

$$\mathbf{P_4} = 0\hat{r}\,[\frac{C}{m^2}]$$

We can also calculate the surface charge density at $r = c$ using the boundary condition. At $r = c$, $\mathbf{D_3} = \mathbf{D_4}$, so $\rho_s|_{r=c} = \hat{r} \cdot (\mathbf{D_4} - \mathbf{D_3}) = 0\,[\frac{C}{m^2}]$

6. **Bonus Question:**
   (a) (3 points) Use the constitutive equation for the electric field and flux:

$$\mathbf{D} = \epsilon \mathbf{E}$$

if the permittivity is a tensor, we can treat the electric field and flux as vectors:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \epsilon_0 \times \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

By letting $\mathbf{E} = E_0 \hat{y} = (0, E_0, 0)$, then $\mathbf{D} = (0, 2\epsilon_0 E_0, 0)$. To find the polarization $\mathbf{P}$,we use

$$\mathbf{D} = \epsilon_\mathbf{0} \mathbf{E_0} + \mathbf{P}$$

$$\mathbf{P} = \epsilon_0 E_0 \hat{y}$$

In this case, the electric field and flux are pointing to the same direction. The polariztion $\mathbf{P}$ is the same the free space flux $\epsilon \mathbf{E_0}$. The reason is because the entities in the second column of the tensor are zeros for x and z directions. It means that this material polarized in y direction under electric field in y direction. Therefore, the electric dipoles are orientated in the same direction with the applied electric field.

   (b) (3 points) Using the same equation in part (a), but letting $\mathbf{E} = E_0 \hat{x} = (E_0, 0, 0)$, we find:

$$\mathbf{D} = 2\epsilon_0 E_0 \hat{x} - \epsilon_0 E_0 \hat{z}$$

$$\mathbf{P} = \epsilon_0 E_0 \hat{x} - \epsilon_0 E_0 \hat{z}$$

In this case, the electric field and flux are pointing to different directions. The polarization due to the electric dipoles is then pointing to a different direction under the applied electric field.

   (c) (4 points) This problem is equivalent to finding the eigenvectors of the given matrix (tensor). If $\mathbf{D}$ and $\mathbf{E}$ are pointing to the same direction, we can write $\mathbf{D} = \lambda \mathbf{E}$, where λ is a scalar number. Then:

$$\epsilon \mathbf{E} = \lambda \mathbf{E}$$

This is an eigenvalue problem: λ is the eigenvalue and $\mathbf{E}$ is the eigenvector. We can only find three eigenvectors for this problem since ϵ is a 3×3 matrix and it is non-singular. We can calcuate the eigenvectors of ϵ:

$$\mathbf{E_1} = -\frac{\sqrt{2}}{2}\hat{\mathbf{x}} - \frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\hat{\mathbf{z}}[a.u.]$$

$$\mathbf{E_2} = \hat{\mathbf{y}}[a.u.]$$

$$\mathbf{E_3} = -\frac{\sqrt{2}}{2}\hat{\mathbf{x}} + \frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\hat{\mathbf{z}}[a.u.]$$

Then if $\mathbf{E}$ in one of the above direction, $\mathbf{D}$ and $\mathbf{E}$ will be in the same direction.