ECE 329 Fields and Waves I
Homework 4

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Homework Policy:

- Write your name and NetID on top of every page. This habit will help you in exams in the event of having loose page(s).
- Tag all the questions in Gradescope. Failure to do so results in a 5 points deduction.
- Cheating results in ZERO and 50% reduction in HW average on first offense. A 100% reduction in HW average on second offense.
- Please show detailed process for each problem instead of just an answer. No partial credits would be given otherwise. All answers should include units wherever appropriate.
- No late HW is accepted.
- Regrade requests are available one week following grade release.

You are allowed to work with anyone else, but the work you submit should only belong to you. Note that if you have knowledge of a violation of the Honor Code, then you are obligated to report it. By submitting this homework, you are agreeing to the Honor Code: "I have neither given nor received unauthorized aid on this homework, nor have I concealed any violations of the Honor Code."

Question	Points	Score
1	10	
2	10	
3	15	
4	15	
5	16	
6	10	
Total:	76	

1. Coulomb's field of a charge Q stationed at the origin of a right-handed Cartesian coordinate system can be expressed as:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o r^2}\hat{r} = \frac{Q}{4\pi\epsilon_o r^2}\frac{(x,y,z)}{r},$$

where $r^2 \equiv x^2 + y^2 + z^2$ and $r \geq 0$.

   (a) (5 points) Verify that $\nabla \times \mathbf{E} = 0$ by showing that when $\nabla \times \mathbf{E}$ is expanded as usual, all of its Cartesian components cancel out exactly.

   (b) (5 points) Assuming that the electrostatic potential V associated with $\mathbf{E}$ is zero at $r = \infty$, show that $V = \frac{Q}{4\pi\epsilon_o r}$. **Hint:** In this spherically symmetric situation, $\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r}$

2. Consider a simplified model of a *vacuum diode* consisting of a cathode in the $z = 0$ plane and an anode in the $z = d$ plane. Electrons are released from the heated cathode, which is held at potential zero, and attracted by the anode, which is held at a positive potential $V_0 = 4$ V, resulting in a net flow of electrons across the gap. In steady state, the distribution of charge in the region between the cathode and anode plates is a constant function of position z, such that the associated *static* electric field (in V/m) is given by

$$\mathbf{E} = -\hat{z}\frac{4V_0}{3d}(\frac{z}{d})^{1/3}$$

Given that the distance d between the plates is much smaller than their spatial dimensions in the xy-plane, determine the following:

   (a) (4 points) The distance d (in m) beween the cathode and anode if the electrostatic potential V at $z = 1$ is 0.25 V.

   (b) (4 points) Volumetric charge density ρ at $z = 5d/8$ between the cathode and anode,

   (c) (2 points) The surface charge density ρ_s on the anode (assume zero field outside the gap region).

3. Two infinite, plane parallel, perfectly conducting plates in the $x = 0$ and $x = 10$ m planes hold equal and opposite signed surface charge density of $\rho_s = \pm 5\epsilon_0$ [C/m^2]. The region between the plates is filled with two slabs of perfect dielectric materials having constant electric permittivities ϵ_1 for $0 < x < 2$ m (region 1) and ϵ_2 for $2 < x < 10$ m (region 2). The plate at $x = 10$ is held at constant potential $V(10) = 0$ and the electrostatic field is known to be $\mathbf{E}_1 = -2\hat{x}$ V/m in region 1, and $\mathbf{E}_2 = -\frac{1}{2}\hat{x}$ V/m in region 2.

   (a) (2 points) Which plate is holding the negative charge density?

   (b) (4 points) What are the dielectric material permittivities ϵ_1 and ϵ_2?

   (c) (2 points) Verify that the above field satisfies Maxwell's boundary condition regarding $\mathbf{D}$ at the boundary between the two dielectric slabs (i.e., at $x = 2$).

   (d) (4 points) What is the electrostatic potential $V(x)$ in each region between the plates?

   (e) (1 point) Does the potential determined in part (d) above satisfy Laplace's equation in the region $0 < x < 10$ m? Explain.

(f) (2 points) What is the capacitance per unit area $\frac{C}{A}$?

4. The gap between a pair of parallel, infinite copper plates extends from $z = 0$ to $z = 2$ m and is occupied by a vacuum. The plates carry equal and oppositely signed surface charge densities of ± 2 C/m^2 that give rise to a constant electric field pointing along the $-\hat{z}$ direction in the gap. The plate at $z = 0$ is grounded $V(0) = 0$ V.
   (a) (7 points) What are the corresponding electric field $\mathbf{E}$, displacement vector $\mathbf{D}$, polarization vector $\mathbf{P}$, and potential $V(z)$ in the gap region? What is the capacitance per unit area $\frac{C}{A}$?
   (b) (5 points) What would be the new values of $\mathbf{E}$, $\mathbf{D}$, $\mathbf{P}$, and $V(z)$ if the gap is instead filled with pure water (a non-conducting fluid of permittivity $\epsilon = 80\epsilon_0$) while the surface charge densities on the copper plates remained the same? What is the new capacitance per unit area $\frac{C}{A}$?
   (c) (3 points) Now consider that a sufficient quantity of salt is dissolved into the water in the gap region (see part b), giving the fluid a conductivity of $\sigma = 4$ S/m (the conductivity of seawater). What are the new values of $\mathbf{E}$, $\mathbf{D}$, $\mathbf{P}$, and $V(z)$ after a steady state is reached within the salt water fluid? What is the new capacitance per unit area $\frac{C}{A}$?

5. (16 points) Consider the following spherically symmetric configuration of composite materials in steady-state equilibrium, centered on the origin:

   (i) In the region $r \leq a$ (where $r = \sqrt{x^2 + y^2 + z^2}$ is the radial distance from the origin), a perfect conductor holds a net charge of $Q = -2$ C.

   (ii) The region $a < r < b$ is filled with a conducting fluid characterized by $\epsilon = 2\epsilon_0$ and $\sigma = 10^4$ S/m.

   (iii) The region $b \leq r \leq c$ contains a shell of perfect dielectric material having $\epsilon = 4\epsilon_0$.

   (iv) The region $r > c$ is occupied by free space (a vacuum).

   Determine, in all four regions, the equilibrium (steady-state) values for (a) the displacement vector $\mathbf{D}$, (b) the electric field $\mathbf{E}$, (c) the polarization vector $\mathbf{P}$, and also determine (d) the surface charge densities on each of the surfaces at $r = a, b$ and c between the different material media.

   Hint: Make use of Gauss' Law in integral form with $\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$, along with a crucial fact about steady-state fields within conducting materials.

6. **Bonus Question**: In an anisotropic material, the permittivity is described by a tensor (matrix) rather than a scalar. Suppose $\epsilon = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \epsilon_0$.
   (a) (3 points) For $\mathbf{E} = E_0\hat{y}$, find $\mathbf{D}$ and $\mathbf{P}$ and give a brief physical explain of what it means for $\mathbf{D}$ and $\mathbf{E}$ to point in the same direction.
   (b) (3 points) For $\mathbf{E} = E_0\hat{x}$, find $\mathbf{D}$ and $\mathbf{P}$ and give a brief physical explain of what it means for $\mathbf{D}$ and $\mathbf{E}$ to not point in the same direction.
   (c) (4 points) Find three directions such that $\mathbf{D}$ and $\mathbf{E}$ point in the same direction.