1. (5 points) In electrostatics, we generate a curl-free vector field $\mathbf{E}(x, y, z)$ if we take the gradient of a scalar function $V(x, y, z)$. Therefore, $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = 0$.

$$\begin{aligned} \nabla \times \mathbf{E} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -y & x \end{vmatrix} \\ &= 0. \end{aligned}$$

2. (5 points) Following the logic from last question,

$$\begin{aligned} \mathbf{E} = -\nabla V &= -\nabla V \\ &= -\nabla(\sin(x)e^{-y}z^2) \\ &= (-\cos(x)e^{-y}z^2, \sin(x)e^{-y}z^2, -2\sin(x)e^{-y}z)\,\frac{V}{m} \end{aligned}$$

We use the differential form of Gauss's Law for calculating the static charge density,

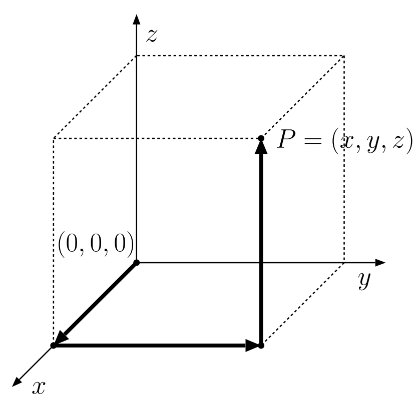
$$\rho = \nabla \cdot \mathbf{D} = -2\epsilon_o \sin(x)e^{-y}$$

The close loop line integral of a vector field $\mathbf{E}(x, y, z)$ generated from a scalar potential,

$$\int_P^P \mathbf{E} \cdot d\mathbf{l} = \int_P^P -\nabla V \cdot d\mathbf{l} = V(P) - V(P) = 0$$

Hence, it is conservative.

3. (10 points) The electrostatic potential V at any point $P = (x, y, z)$ can be calculated by performing a vector line integral by using the path shown in the below figure.

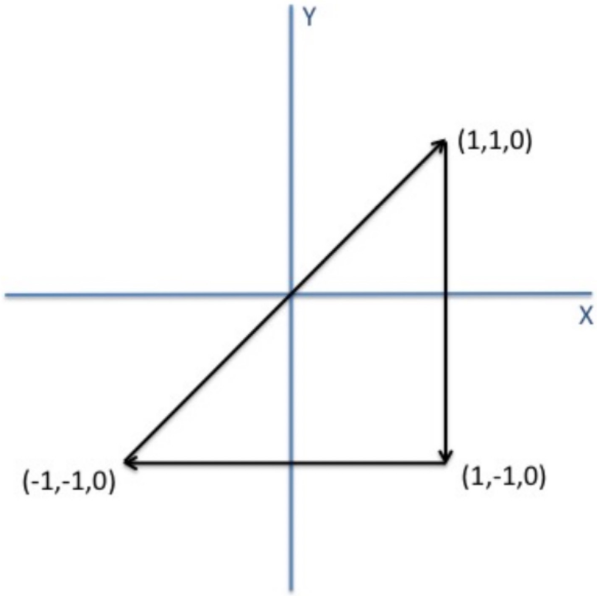


Therefore, we can write

$$\begin{aligned} V(P) - V(0) &= -\int_0^P \mathbf{E} \cdot d\mathbf{l} \\ &= -\int_0^x E_x(x, 0, 0)dx - \int_0^y E_y(x, y, 0)dy - \int_0^z E_z(x, y, z)dz \\ &= -1 - 2\sin(3)\,\mathrm{V}. \end{aligned}$$

Given that $V(0) = 3\,\mathrm{V}$, the electrostatic potential at $P = (1, 2, 3)$ is $V(1, 2, 3) = 2 - 2\sin(3)$ [V].

4. (a) (6 points) The triangular path defined in the problem is sketched in the figure below.



Referring to the hint given in the problem, we can write

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_{l_1} \mathbf{E}(x,-1,0)\cdot d\mathbf{l}_1 + \int_{l_2} \mathbf{E}(x,x,0)\cdot d\mathbf{l}_2 + \int_{l_3} \mathbf{E}(1,y,0)\cdot d\mathbf{l}_3.$$

Evaluating this equation for the field $\mathbf{E}$ with the + sign, we obtain

$$\oint_C \mathbf{E_1}\cdot d\mathbf{l} = \int_1^{-1} (-\hat{x}+\hat{y}x)\cdot(\hat{x})\,dx + \int_{-1}^{1} (\hat{x}x+\hat{y}x)\cdot(\hat{x}+\hat{y})\,dx + \int_1^{-1} (\hat{x}y+\hat{y})\cdot(\hat{y})\,dy = 0\,[\text{V}].$$

Likewise, we have

$$\oint_C \mathbf{E_2}\cdot d\mathbf{l} = \int_1^{-1} (-\hat{x}-\hat{y}x)\cdot(\hat{x})\,dx + \int_{-1}^{1} (\hat{x}x-\hat{y}x)\cdot(\hat{x}+\hat{y})\,dx + \int_1^{-1} (\hat{x}y-\hat{y})\cdot(\hat{y})\,dy = 4\,[\text{V}].$$

for the field $\mathbf{E}$ with the − sign.

(b) (4 points) Evaluating $\nabla\times\mathbf{E}$ for both the fields, first $\mathbf{E}$ with the + sign,

$$\begin{aligned}\nabla\times\mathbf{E_1} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} \\ &= 0\end{aligned}$$

and the - sign $\mathbf{E}$,

$$\begin{aligned}\nabla\times\mathbf{E_2} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} \\ &= -2\hat{z}.\end{aligned}$$

Using Stokes' equation,

$$\int_S (\nabla \times \mathbf{E_1}) \cdot d\mathbf{S} = 0$$

and similarly,

$$\int_S (\nabla \times \mathbf{E_2}) \cdot d\mathbf{S} = 2 \times Area = 4.$$

The result confirms the calculations carried out in part (a).

5. (5 points) Using vector line integral equation from question 3, the voltage drop is given by,

$$\begin{aligned} V(2) - V(-2) &= -\int_{-2}^{2} \mathbf{E} \cdot d\mathbf{l} \\ &= 4\int_{-2}^{2} dz \\ &= 16\,[\mathrm{V}]. \end{aligned}$$

6. (a) (3 points) Using formula for static charge sheet from Lecture 3,

$$\mathbf{D} = \begin{cases} \hat{z}\frac{\rho_0}{2}, & \text{for } z > 0 \\ -\hat{z}\frac{\rho_0}{2}, & \text{for } z < 0 \end{cases}$$

(b) (2 points) With the substitutions,

$$\begin{aligned} \mathbf{D}_1 &= \hat{a}_n \frac{\rho_0}{2}\,[\frac{\mathrm{C}}{\mathrm{m}^2}] \\ \mathbf{D}_2 &= -\hat{a}_n \frac{\rho_0}{2}\,[\frac{\mathrm{C}}{\mathrm{m}^2}] \end{aligned}$$

from which it is easy to see that $\hat{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_0$.

7. (a) (2 points) We use the boundary condition outlined in question 6(b),

$$\rho_0^{'} \;=\; \hat{y}\cdot(\mathbf{D}\,|_{y=2^+} - \mathbf{D}\,|_{y=2^-}\,) = -1\,[\frac{\mathrm{C}}{\mathrm{m}^2}].$$

(b) (2 points) Since, there is no static charge distribution dependence in z, the field component in $\hat{z}$ will be equal across the $y = 2$ static charge sheet. Therefore,

$$\mathbf{D}(0, y, 0) = \hat{y} - 2\hat{z}\,[\frac{\mathrm{C}}{\mathrm{m}^2}] \;\text{ for } y > 2$$

(c) (2 points) The y component of the field is given by

$$\rho\,|_{y=0} \;=\; \hat{y}\cdot(\mathbf{D}\,|_{y=0^+} - \mathbf{D}\,|_{y=0^-}\,),$$

which results in

$$\mathbf{D}\,|_{y=0^-} \;= -1\,[\frac{\mathrm{C}}{\mathrm{m}^2}].$$

And since the $\hat{z}$ component is unchanged, we have

$$\mathbf{D}(0, y, 0) = -\hat{y} - \hat{z}2\,[\frac{\mathrm{C}}{\mathrm{m}^2}] \;\text{ for } y < 0.$$

(d) (4 points) Possible volumetric charge densities for infinite charge sheets can be either $-4\delta(z + 30)\,[\frac{\mathrm{C}}{\mathrm{m}^3}]$ or $4\delta(z - 30)\,[\frac{\mathrm{C}}{\mathrm{m}^3}]$.

8. (10 points) The given scalar field is $f = x+yz$ and the given vector field $\mathbf{A} = (x+z)y\hat{x}+(x-z)\hat{z}$. The proof of the first identity $\nabla \times \nabla f = 0$ is given below. For

$$\nabla f = (1, z, y),$$

taking curl,

$$\begin{aligned} \nabla \times \nabla f &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & z & y \end{vmatrix} \\ &= 0 \end{aligned}$$

Hence, L.H.S = R.H.S.
The second identity $\nabla \cdot \nabla \times \mathbf{A} = 0$ is proved below:

$$\begin{aligned} \nabla \times \mathbf{A} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+z)y & 0 & x-z \end{vmatrix} \\ &= (0, y-1, -x-z) \end{aligned}$$

Taking divergence,

$$\nabla \cdot \nabla \times \mathbf{A} = 1 - 1 = 0$$

L.H.S=R.H.S.
The third identity is $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$,

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & y-1 & -x-z \end{vmatrix} \\ &= (0, 1, 0). \end{aligned}$$

Evaluating R.H.S,

$$\nabla(\nabla \cdot \mathbf{A}) = \nabla(y-1) = \hat{y}$$

and

$$\nabla^2 \mathbf{A} = 0.$$

We see that L.H.S = R.H.S = $\hat{y}$.