1. (a) (2 points) For the particle to levitate motionless at the position where it was placed at $\mathbf{t} = \mathbf{0}$, the total force exerted on the particle must be zero. There are two types of forces exerted on the particle : gravity $\mathbf{F}_g = m\mathbf{g}$ and electric force $\mathbf{F}_E = Q\mathbf{E}$. Since the particle remains motionless, the sheet is parallel to the earth and is laid over $xy-$plane, the total force will be

$$mg(-\hat{z}) + QE\hat{z} = 0$$

where g is the gravity constant and $E = \frac{\rho_s}{2\epsilon_o}$. Rearranging terms, the charge of the particle is calculated as

$$Q = mg\frac{2\epsilon_o}{\rho_s} = 18\pi \times 10^{-3} \times 9.8 \times 2 \times \frac{1}{36\pi \times 10^9} \times \frac{1}{9.8 \times 10^{-6}} = 1\,[\mu\text{C}].$$

(b) (3 points) In this case, the electric force is increased while the gravitation force remains the same. Therefore, the acceleration vector is in the $\hat{z}$ direction and its magnitude can be defined as

$$a\,(t) = \frac{QE}{m} - g \approx 19.6 \times 10^6\,[\frac{\text{m}}{\text{s}^2}],$$

which leads to

$$v\,(t) = v_0 + \int_0^t a\,(t)\,dt = 19.6 \times 10^6 t\,[\frac{\text{m}}{\text{s}}],$$

$$d\,(t) = d_0 + \int_0^t v\,(t)\,dt = d_0 + 9.8 \times 10^6 t^2\,[\text{m}],$$

where $v_0 = 0$ and $d_0 = 19.6\,\text{m}$.

2. (a) (1 point) According to Gauss's law, the electric flux Φ will be

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V \rho dV = \frac{1}{\epsilon_0} \rho \times V = \frac{1}{\epsilon_0} \rho \times L^3 = \frac{4}{\epsilon_0} \times 10^{-9}\,\text{Vm},$$

(b) (2 points) If $\rho(x, y, z) = x^2 + y^2 + z^2\,\frac{\text{C}}{\text{m}^3}$ (within the cube), the total electric flux can be computed as follows:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V x^2 + y^2 + z^2\, dV$$

$$= \frac{1}{\epsilon_0} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} x^2 + y^2 + z^2\, dx\, dy\, dz = \frac{2.5}{\epsilon_0} \times 10^{-16}\,\text{Vm}.$$

(c) (1 point) If the medium is homogeneous, then the displacement flux is

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \epsilon_0 \oint_S \mathbf{E} \cdot d\mathbf{S}$$
$$= 2.5 \times 10^{-16}\,\text{C}.$$

(d) (1 point) Since the charge distribution is symmetrical around the center, the flux for any of the square surfaces is one-sixth of the total value, i.e., 0.416×10^{-16} C.

3. (5 points) Two unknown charges, Q_1 and Q_2 are located at $(1,0,0)$ and $(-1,0,0)$, respectively. The displacement flux $\oint_S \mathbf{D}\cdot d\mathbf{S}$ through the $yz-$plane in the $\hat{x}$ direction is $-2\,\mathrm{C}$, thus,

$$-\frac{Q_1}{2}+\frac{Q_2}{2}=-2\,[\mathrm{C}]$$

Also, the displacement flux through the plane $y=1$ in the $-\hat{y}$ direction is $1\,\mathrm{C}$, which implies that

$$-\frac{Q_1}{2}-\frac{Q_2}{2}=3\,[\mathrm{C}]$$

Using these two equations, we get

$$Q_1=-1\,[\mathrm{C}] \qquad \text{and} \qquad Q_2=-5\,[\mathrm{C}].$$

4. (a) (3 points) According to Gauss's law and Example 2 in Lecture 2,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

Since the caps of the cylinder will have no electric field lines going through them, the flux of the caps will be 0. We only need to calculate the side of the cylinder

$$\epsilon_0 E_r 2\pi r L = \rho_0 L \pi R^2$$

$$\rho_0 = \frac{2\epsilon_0 E_r r}{R^2} = -200 \left[\frac{\mathrm{C}}{\mathrm{m}^3}\right]$$

(b) (3 points) Using Gauss's law, and consider a cylinder with length L

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

$$\epsilon_0 E_{in} 2\pi r L = \pi r^2 L \rho_0$$

$$E_{in} = \frac{r\rho_0}{2\epsilon_0} = -\frac{1}{3\epsilon_0} \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

(c) (4 points) From (a), for $r > R$

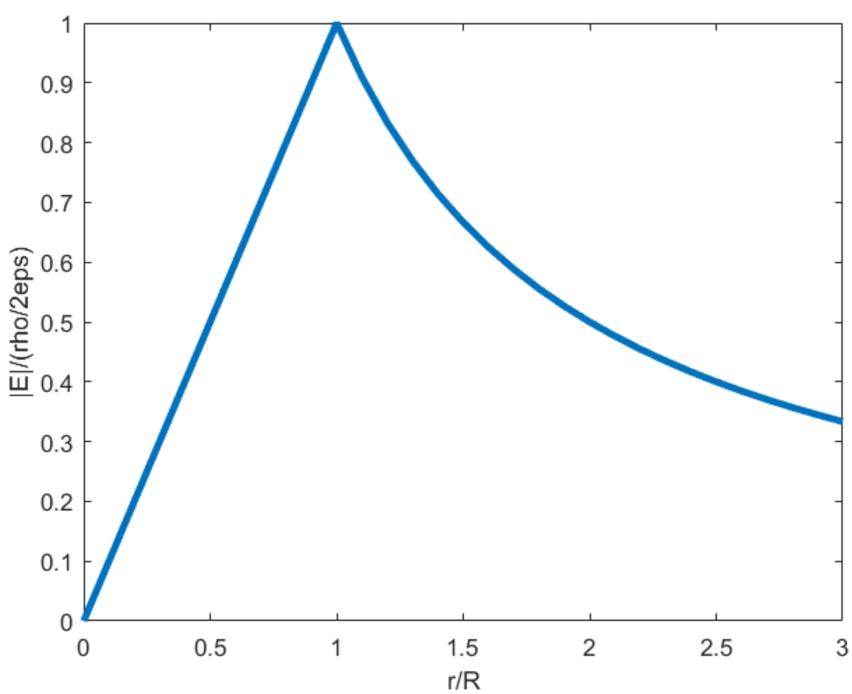
$$E_r(r) = \frac{\rho_0 R^2}{2\epsilon_0 r}$$

from (b), for $r < R$

$$E_{in}(r) = \frac{\rho_0 r}{2\epsilon_0}$$

at $r = R$

$$E_{in}(R) = \frac{\rho_0 R}{2\epsilon_0} = \frac{\rho_0 R^2}{2\epsilon_0 R} = E_r(R)$$



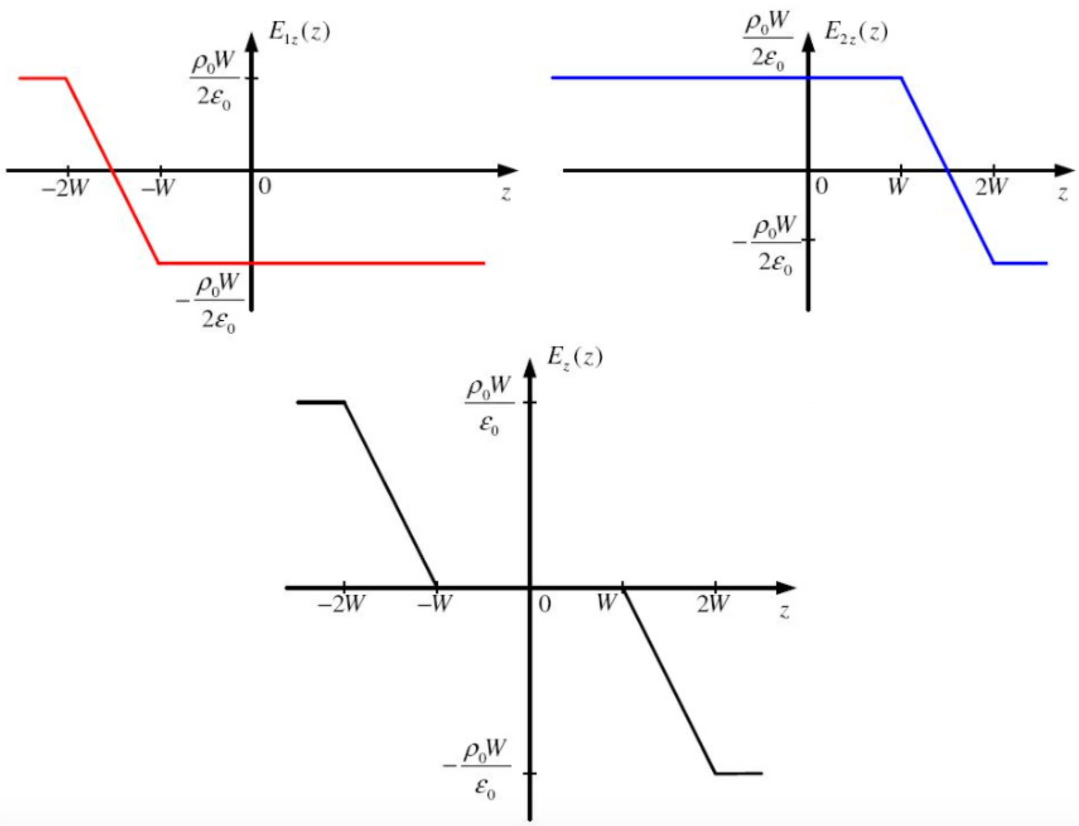
5. (a) (7 points) A two-slab geometry of two identical slabs of equal widths W in $z-$direction and expanding infinitely in x and y directions is considered here. The electric field generated by a single charged slab has been derived in the class notes. Making use of that result, we find that the electric fields generated by each of the two slabs are:

$$\mathbf{E}_1 = \begin{cases} \frac{\rho_0 W}{2\epsilon_0}\,\hat{z}, & z < -2W, \\ -\frac{\rho_0}{\epsilon_0}(z + \frac{3W}{2})\,\hat{z}, & -2W < z < -W, \\ -\frac{\rho_0 W}{2\epsilon_0}\,\hat{z}, & z > -W \end{cases}$$

$$\mathbf{E}_2 = \begin{cases} -\frac{\rho_0 W}{2\epsilon_0}\,\hat{z}, & z > 2W, \\ -\frac{\rho_0}{\epsilon_0}(z - \frac{3W}{2})\,\hat{z}, & W < z < 2W, \\ \frac{\rho_0 W}{2\epsilon_0}\,\hat{z}, & z < W \end{cases}$$

respectively. Adding these fields, we find that the total field in space

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} \frac{\rho_0 W}{\epsilon_0}\,\hat{z}, & z < -2W \\ -\frac{\rho_0}{\epsilon_0}(z + W)\,\hat{z}, & -2W < z < -W \\ 0, & -W < z < W \\ -\frac{\rho_0}{\epsilon_0}(z - W)\,\hat{z}, & W < z < 2W \\ -\frac{\rho_0 W}{\epsilon_0}\,\hat{z}, & z > 2W \end{cases}$$



(b) (3 points) From $\mathbf{D} = \epsilon_0 \mathbf{E}$,

$$\nabla \cdot \mathbf{D} = \begin{cases} 0, & z < -2W \\ -\rho_{0,} & -2W < z < -W \\ 0, & -W < z < W \\ -\rho_0 & W < z < 2W \\ 0 & z > 2W \end{cases}$$

6. (5 points) (a) For $\rho_{s_1} = 8\,\frac{\text{C}}{\text{m}^2}$, the displacement vectors at the origin due to charge sheets 1 and 2 can be written as

$$\mathbf{D}_1 = \frac{1}{2}\rho_{s_1}\hat{x} = 4\hat{x}\,\frac{\text{C}}{\text{m}^2},$$
$$\mathbf{D}_2 = \frac{1}{2}\rho_{s_2}(-\hat{x}) = -\frac{1}{2}\rho_{s_2}\hat{x},$$

respectively. The resultant displacement vector is given by $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 = (4 - \frac{1}{2}\rho_{s_2})\hat{x} = 6\,\frac{\text{C}}{\text{m}^2}$. Therefore, we can determine that the charge density ρ_{s_2} is $-4\,[\frac{\text{C}}{\text{m}^2}]$.

(b) Following the same procedure, the displacement vectors at the origin due to charge sheets 1 and 2 can be written as

$$\mathbf{D}_1 = \frac{1}{2}\rho_{s_1}\hat{x},$$
$$\mathbf{D}_2 = \frac{1}{2}\rho_{s_2}(-\hat{x}) = -\frac{1}{2}\rho_{s_2}\hat{x},$$

respectively. Given that $\rho_{s_1} = -\rho_{s_2}$, the resultant displacement vector is given by

$$\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 = -\rho_{s_2}\hat{x} = 6\hat{x}\,\frac{\text{C}}{\text{m}^2}.$$

As a result, the charge density ρ_{s_2} is determined to be $-6\,[\frac{\text{C}}{\text{m}^2}]$.

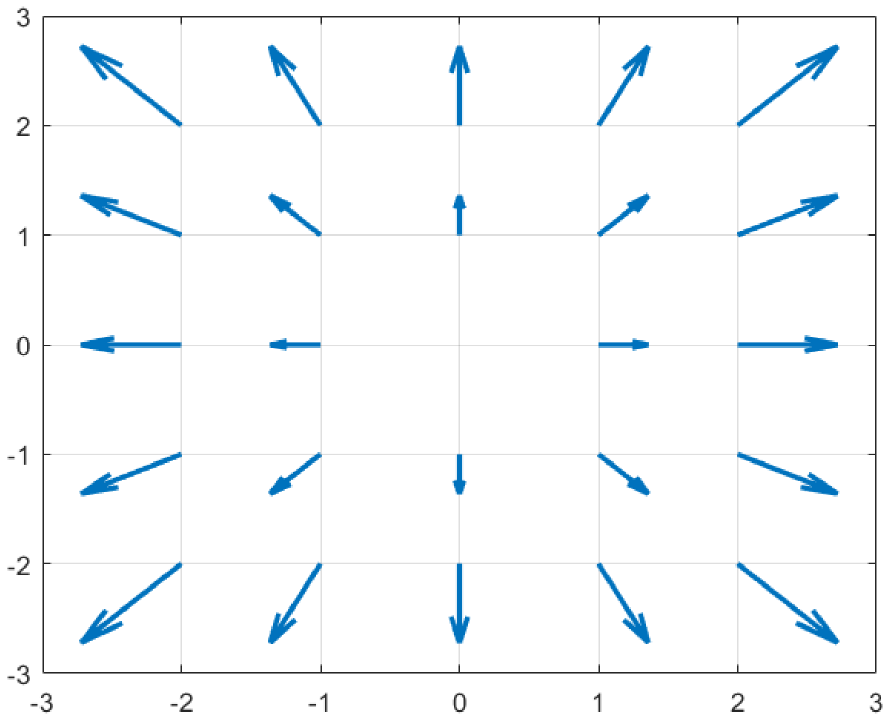
7. (a) (3 points) Curl and divergence of the vector field $\mathbf{F} = x\hat{x} + y\hat{y}$ are

$$\nabla \times \mathbf{F} = \nabla \times (x\hat{x} + y\hat{y}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0,$$

and

$$\nabla \cdot \mathbf{F} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (x\hat{x} + y\hat{y}) = 2,$$

respectively. When we sketch the vector field $\mathbf{F}$, we obtain the following figure.



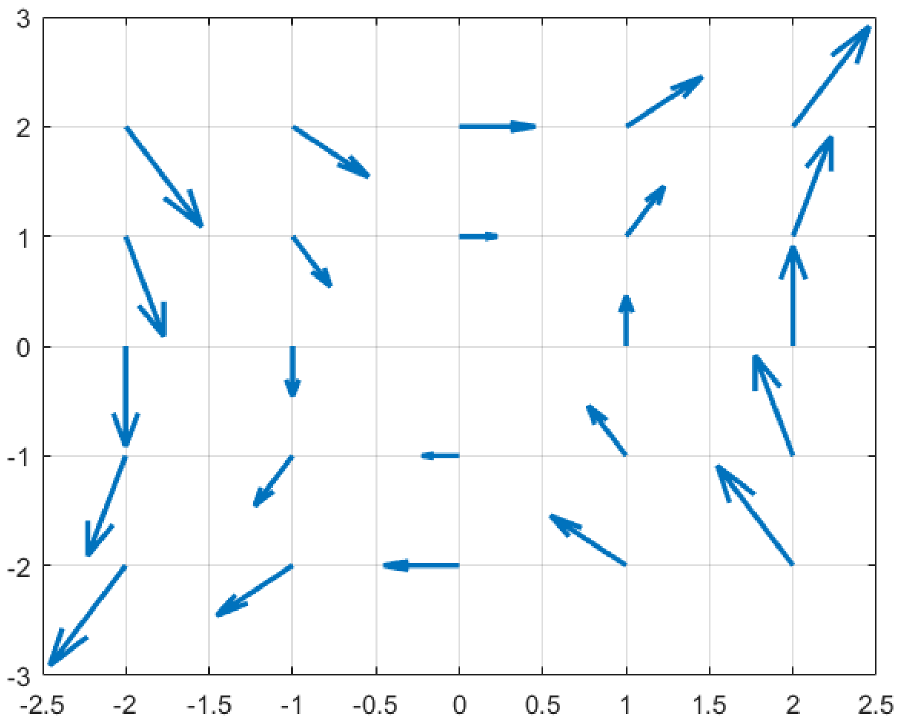
(b) (3 points) Curl and divergence of the vector field $\mathbf{F} = y\hat{x} + 2x\hat{y}$ are

$$\nabla \times \mathbf{F} = \nabla \times (y\hat{x} + 2x\hat{y}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2x & 0 \end{vmatrix} = \hat{z},$$

and

$$\nabla \cdot \mathbf{F} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (y\hat{x} + 2x\hat{y}) = 0,$$

respectively. When we sketch the vector field $\mathbf{F}$, we obtain the following figure.



(c) (2 points) **Curl** is a vector operator that describes the rotation of a 3-D vector field. In short, it is the vector representing circulation per unit area. Since the vector field $\mathbf{F}$ in part (b) is defined only by the curl operator (i.e. divergence-free), $\nabla \times \mathbf{F} \neq 0$ implies the field strength varies **across** the direction of the field.

(d) (2 points) **Divergence** is an operator picking up the variation of the field strength along the direction of the vector field. As seen in part (a), the divergence operator picks up the field strength of the vector $\mathbf{F} = x\hat{x} + y\hat{y}$ which is curl-free. Therefore, $\nabla \cdot \mathbf{F} \neq 0$ implies the field strength varies **along** the direction of the field.

8. Given that

$$\mathbf{E} = \hat{x}sin(y) + \hat{y}2cos(x)$$

Let us calculate the following.

(a) (3 points) Curl of $\mathbf{E}$,

$$\nabla\times\mathbf{E} = \left\{ \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ sin(y) & 2cos(x) & 0 \end{array} \right\} = (\frac{\partial}{\partial x}(2cos(x)) - \frac{\partial}{\partial y}(sin(y)))\hat{z} = (-2sin(x) - cos(y))\hat{z}.$$

Curl of curl of $\mathbf{E}$,

$$\nabla\times(\nabla\times\mathbf{E}) = \left\{ \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -2sin(x) - cos(y) \end{array} \right\} = sin(y)\hat{x} + 2cos(x)\hat{y}.$$

(b) (2 points) Applying Gauss law(in differential form)

$$\nabla\cdot\mathbf{D} = \epsilon_0\nabla\cdot\mathbf{E} = \rho,$$

we find that

$$\rho = \epsilon_0\nabla\cdot\mathbf{E} = 0.$$

9. Given an electrostatic potential V(x,y,z) $= 2x^2 - 3\,\text{V}$ in certain region of space, let us calculate the following.

   (a) (2 points) Electrostatic field **E**

$$\mathbf{E} = -\nabla V = -\frac{\partial}{\partial x}(2x^2-3)\hat{x} - \frac{\partial}{\partial y}(2x^2-3)\hat{y} - \frac{\partial}{\partial z}(2x^2-3)\hat{z} = -4x\hat{x}\,[\frac{\text{V}}{\text{m}}].$$

   (b) (1 point) Curl of **E**

$$\nabla \times \mathbf{E} = \left\{ \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -4x & 0 & 0 \end{array} \right\} = 0.$$

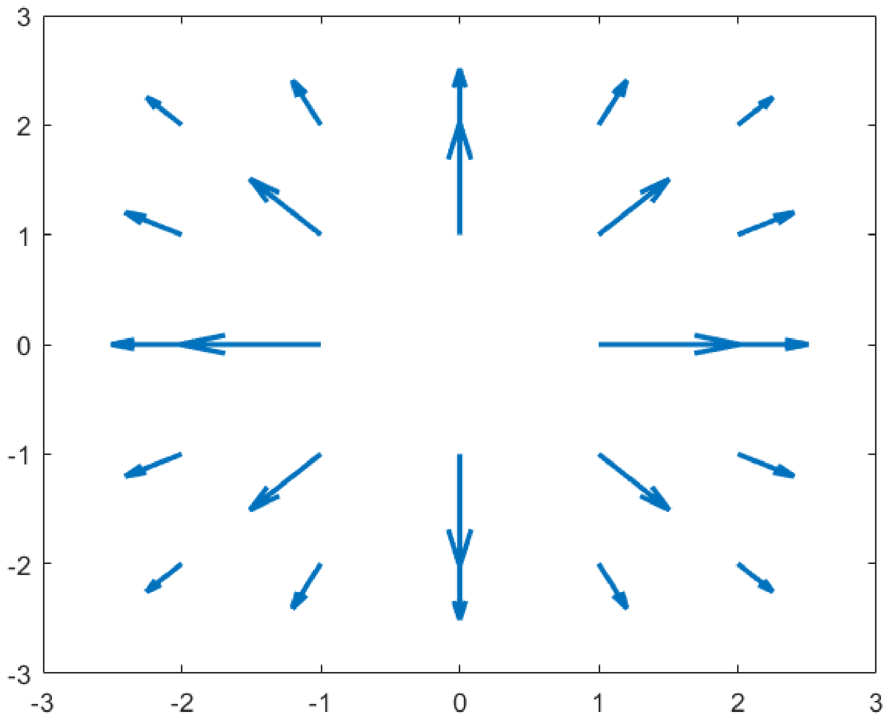
   (c) (1 point) Divergence of **E**

$$\nabla \cdot \mathbf{E} = \frac{\partial}{\partial x}(-4x) = -4.$$

   (d) (1 point) Charge density ρ

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = -4\epsilon_0\,[\frac{\text{C}}{\text{m}^3}].$$

10. (a) (4 points) By calculating the field value at each point, we obtain the following plot of the vector field



(b) (2 points) Since the field has a variation along the direction of the field, we would expect that $\nabla \cdot \mathbf{D} \neq 0$

(c) (4 points) $\nabla \cdot \mathbf{D}$ in the spherical coordinates can be calculated as

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{1}{r^2}) = 0$$

which contradicts our prediction from part (b). This is because the field has a singularity at $r = 0$, so that the differential expression cannot be directly applied to calculating the divergence.

Or, this is because the number of field lines for any concentric sphere surfaces are the same.