1. Consider the 3D vectors

$$\begin{aligned}\mathbf{A} &= 3\hat{x}+\hat{y}-2\hat{z},\\ \mathbf{B} &= \hat{x}+\hat{y}-\hat{z},\\ \mathbf{C} &= \hat{x}-2\hat{y}+3\hat{z},\end{aligned}$$

(a) (1 point) The vector

$$\mathbf{D}=\mathbf{A}+\mathbf{B}=4\hat{x}+2\hat{y}-3\hat{z}.$$

(b) (1 point) The vector

$$\mathbf{A}+\mathbf{B}-4\mathbf{C}=4\hat{x}+2\hat{y}-3\hat{z}-4(\hat{x}-2\hat{y}+3\hat{z})=1\hat{0}y-15\hat{z}.$$

(c) (2 points) The vector *magnitude*

$$|\mathbf{A}+\mathbf{B}-4\mathbf{C}|=\sqrt{10^2+15^2}=18.03.$$

(d) (2 points) The unit vector $\hat{u}$ along vector

$$\mathbf{A}+2\mathbf{B}-\mathbf{C}=(3\hat{x}+\hat{y}-2\hat{z})+2(\hat{x}+\hat{y}-\hat{z})-(\hat{x}-2\hat{y}+3\hat{z})=4\hat{x}+5\hat{y}-7\hat{z}$$

$$|\mathbf{A}+2\mathbf{B}-\mathbf{C}|=\sqrt{4^2+5^2+7^2}=9.49$$

$$\hat{u}=\frac{\mathbf{A}+2\mathbf{B}-\mathbf{C}}{|\mathbf{A}+2\mathbf{B}-\mathbf{C}|}=\frac{4\hat{x}+5\hat{y}-7\hat{z}}{9.49}=0.42\hat{x}+0.53\hat{y}-0.74\hat{z}.$$

(e) (2 points) The *dot product*

$$\mathbf{A}\cdot\mathbf{B}=(3\hat{x}+\hat{y}-2\hat{z})\cdot(\hat{x}+\hat{y}-\hat{z})=3\times 1+1\times 1+(-2)\times(-1)=6.$$

(f) (2 points) The *cross product*

$$\begin{aligned}\mathbf{B}\times\mathbf{C} &= (\hat{x}+\hat{y}-\hat{z})\times(\hat{x}-2\hat{y}+3\hat{z})\\ &= \hat{x}\times(\hat{x}-2\hat{y}+3\hat{z})+\hat{y\times}(\hat{x}-2\hat{y}+3\hat{z})-\hat{z}\times(\hat{x}-2\hat{y}+3\hat{z})\\ &= (-2\hat{z}-3\hat{y})+(-\hat{z}+3\hat{x})-(\hat{y}+2\hat{x})\\ &= \hat{x}-4\hat{y}-3\hat{z}.\end{aligned}$$

2. (7 points) In the three cases, we have the same $\mathbf{E}$ and $\mathbf{B}$ at the origin, but different $\mathbf{v}$. Thus, we have different $\mathbf{F}$. In each case, $\mathbf{v}$ and $\mathbf{F}$ must satisfy $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Therefore, we have the following three equations

$$\begin{cases} 3\hat{z} & = \mathbf{E} \\ \hat{z} & = \mathbf{E} + \hat{y} \times \mathbf{B} \\ 3\hat{z} + 4\hat{y} & = \mathbf{E} + 2\hat{z} \times \mathbf{B}, \end{cases}$$

from which we obtain

$$\begin{cases} \mathbf{E} & = 3\hat{z} \\ \hat{y} \times \mathbf{B} & = -2\hat{z} \\ \hat{z} \times \mathbf{B} & = 2\hat{y}. \end{cases}$$

If we assume $\mathbf{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$, then

$$\begin{cases} \hat{y} \times (\hat{x}B_x + \hat{y}B_y + \hat{z}B_z) & = -2\hat{z} \\ \hat{z} \times (\hat{x}B_x + \hat{y}B_y + \hat{z}B_z) & = 2\hat{y}. \end{cases}$$

Further simplify into

$$\begin{cases} -\hat{z}B_x + \hat{x}B_z & = -2\hat{z} \\ \hat{y}B_x - \hat{x}B_y & = 2\hat{y}. \end{cases}$$

Therefore, we have

$$\begin{cases} B_x & = 2 \\ B_y & = B_z = 0. \end{cases}$$

In summary, $\mathbf{E} = 3\hat{z}$ [V/m] and $\mathbf{B} = 2\hat{x}$ [Wb/m^2].

3. (13 points) Let us use S_1, S_2, S_3, S_4, S_5 and S_6 to denote the surfaces $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$, respectively. We consider S_1 and S_2 first. The unit vector on S_1 pointing away from the volume is along $-\hat{x}$ direction, so we have

$$\int_{S_1} J \cdot dS = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]\,|_{x=0} \cdot (-\hat{x}) dydz = -\frac{1}{3}\,[\text{A}].$$

For S_2, the unit vector pointing away from the volume is along $+\hat{x}$ direction, therefore

$$\int_{S_2} J \cdot dS = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]\,|_{x=1} \cdot (+\hat{x}) dydz = \frac{1}{3}\,[\text{A}].$$

Similarly, for S_3, S_4, S_5 and S_6, we can obtain

$$\int_{S_3} J \cdot dS = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]\,|_{y=0} \cdot (-\hat{y}) dxdz = -\frac{1}{3}\,[\text{A}],$$

$$\int_{S_4} J \cdot dS = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]\,|_{y=1} \cdot (+\hat{y}) dxdz = \frac{1}{3}\,[\text{A}],$$

$$\int_{S_5} J \cdot dS = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]\,|_{z=0} \cdot (-\hat{z}) dxdy = 0\,[\text{A}],$$

$$\int_{S_6} J \cdot dS = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]\,|_{z=1} \cdot (+\hat{z}) dxdy = 1\,[\text{A}].$$

Therefore,

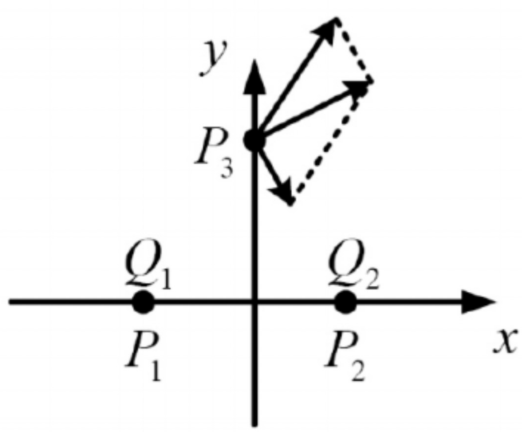
$$\oint_S J \cdot dS = \sum_{i=1}^{6} \int_{s_i} J \cdot dS = 1\,[\text{A}].$$

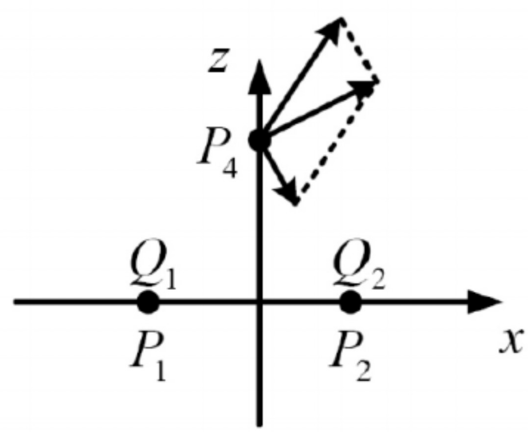
4. (10 points) We know $Q_1 = 8\pi\varepsilon_0$ C, so $Q_2 = -Q_1/2 = 4\pi\varepsilon_0$ C. The electric field at the point P_3 is the superposition of those induced by Q_1 and Q_2, namely

$$\begin{aligned}
\mathbf{E}_3 &= \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0 |r_3 - r_i|^2} \cdot \frac{r_3 - r_i}{|r_3 - r_i|} \\
&= \frac{2}{|\hat{x}+\hat{y}|^3}(\hat{x}+\hat{y}) - \frac{1}{|-\hat{x}+\hat{y}|^3}(-\hat{x}+\hat{y}) \\
&= \frac{3\hat{x}+\hat{y}}{2\sqrt{2}} \text{ [V/m].}
\end{aligned}$$

The electric field at the point P_4 can be obtained in a similar way

$$\begin{aligned}
\mathbf{E}_4 &= \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0 |r_4 - r_i|^2} \cdot \frac{r_4 - r_i}{|r_4 - r_i|} \\
&= \frac{2}{|\hat{x}+\hat{z}|^3}(\hat{x}+\hat{z}) - \frac{1}{|-\hat{x}+\hat{z}|^3}(-\hat{x}+\hat{z}) \\
&= \frac{3\hat{x}+\hat{z}}{2\sqrt{2}} \text{ [V/m].}
\end{aligned}$$





5. (10 points) Firstly, recall that divergence theorem states:

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS$$

We let $\mathbf{F} = \psi \nabla \phi$ and apply divergence theorem:

$$\int_V \nabla \cdot (\psi \nabla \phi)\, dV = \int_{\partial V} \psi \nabla \phi \cdot \mathbf{n}\, dS$$

The expression can be simplified using a vector identity: (This is identity (7) in the "List of Vector Identities" sheet found on ECE 329 web site)

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

by letting $f = \psi$ and $\mathbf{A} = \nabla \phi$:

$$\int_V (\psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi)\, dV = \int_{\partial V} \psi(\nabla \phi \cdot \mathbf{n})\, dS.$$