1. (a) (4 points) Since both ends of the transmission line (TL) are open, each resonant mode will have resonance frequency

$$f = \frac{v}{2l}n, \quad n \leq 1.$$

Plugging in numbers, we get

$$f = \frac{1 \times 10^8}{2 \times 5}n = 10n \,[\text{MHz}]$$

(b) (3 points) The general solution to TL problems are

$$\tilde{V}(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$
$$\tilde{I}(z) = \frac{V^+}{Z_o} e^{-j\beta z} - \frac{V^-}{Z_o} e^{j\beta z}$$

Since both ends are open, the current at the TL ends must be zero. At $z = 0$ end, plug in the current expression, we get:

$$V^+ = V^-.$$

Therefore, the voltage and the current expressions are simplified to

$$\tilde{V}(z) = 2V^+ \cos(\beta z)$$
$$\tilde{I}(z) = -\frac{V^+}{Z_o} 2j \sin(\beta z).$$

Using the relations $\beta = \frac{2\pi}{\lambda}$ and $\lambda = \frac{v}{f} = \frac{2l}{n}$, we can get the expression for current phasor:

$$\tilde{I}(z) = -\frac{V^+}{Z_o} 2j \sin(\frac{n\pi}{l} z).$$

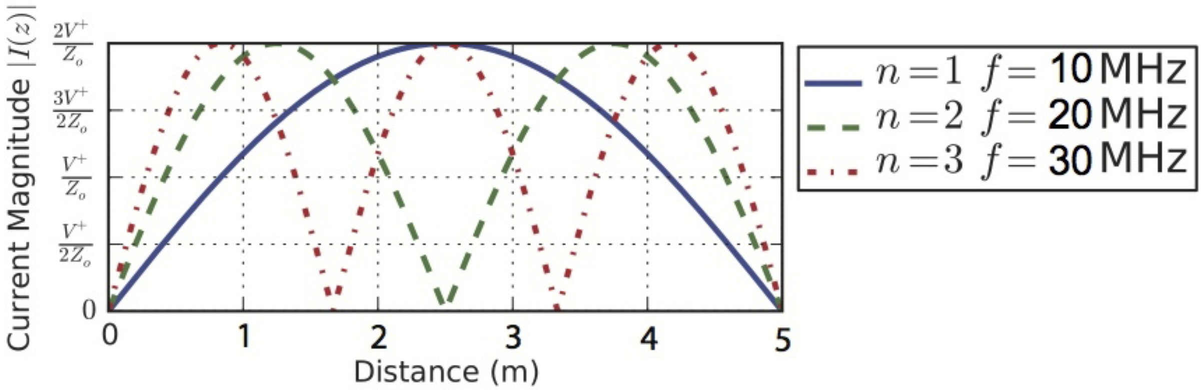
The current plot is shown in the following figure. Notice that each mode n has n half wavelengths fitted into the length of TL, and higher modes oscillates with higher frequencies.

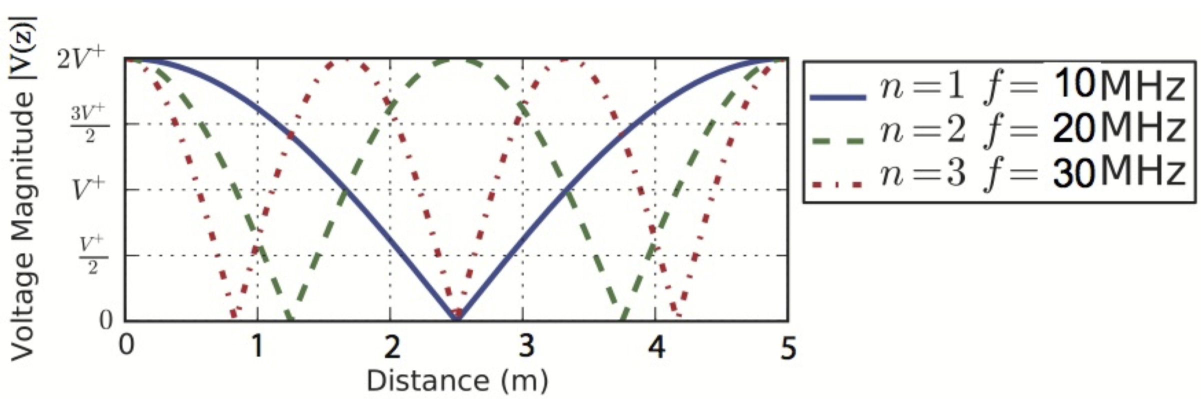
(c) (3 points) We repeat the same step from part (b) to the voltage expression. We get

$$\tilde{V}(z) = 2V^+ \cos(\frac{n\pi}{l} z),$$

and the voltage is printed in the following figure. We can see that there is also n half wavelengths for mode n. Additionally, we observe that the null of voltage and current on the TL are separately by a quarter wavelength.

Note: Keep in mind the convention used here is to place the load at $z = 0$ and source at $z = -L$. Alternatively, we can say the load is placed at $d = 0$ and source is $d = +L$. Remember to stay consistent!


Current Magnitude |I(z)|
$\frac{2V^+}{Z_o}$
$\frac{3V^+}{2Z_o}$
$\frac{V^+}{Z_o}$
$\frac{V^+}{2Z_o}$
0
0 1 2 3 4 5
Distance (m)
$n=1$ $f=$ 10 MHz
$n=2$ $f=$ 20 MHz
$n=3$ $f=$ 30 MHz


Voltage Magnitude |V(z)|
$2V^+$
$\frac{3V^+}{2}$
V^+
$\frac{V^+}{2}$
0
0 1 2 3 4 5
Distance (m)
$n=1$ $f=$ 10 MHz
$n=2$ $f=$ 20 MHz
$n=3$ $f=$ 30 MHz

2. (a) (2 points) From previous question, we know the voltage and current expressions on TL with load end open are:

$$\tilde{V}(z) = 2V^+ \cos(\beta z)$$
$$\tilde{I}(z) = -\frac{V^+}{Z_o} 2j \sin(\beta z).$$

Therefore we have the input impedance of the open TL as

$$Z_{in} = Z(-l) = \frac{\tilde{V}(-l)}{\tilde{I}(-L)} = -jZ_o \cot(\beta l)$$

(b) (3 points) By the steady state analysis, we can regard all circuit elements as general impedance. Thus, the impedance of the capacitor will be

$$Z_C = \frac{1}{j\omega C}.$$

Using the voltage division rule, we write

$$V_L = V_g \frac{R_L}{R_L + R_g + Z_{\text{in}} + Z_C}.$$

Given that $R_g = R_L$ and $V_L = \frac{1}{2} V_g$, we find $Z_{\text{in}} + Z_C = 0$. Therefore, we can write

$$-jZ_o \cot(\beta L) = \frac{1}{j\omega C}$$

from which we obtain

$$C = -\frac{1}{\omega Z_o \cot(\beta L)} = -\frac{1}{\omega Z_o \cot(\frac{\omega}{v} L)} = -63.66\,\text{pF}.$$

Note: Capacitance should be positive. The length in the problem is not practical.

3. (a) (1 point) The signal wavelength λ is $\lambda = \frac{v}{f} = \frac{1\times10^8}{\pi\times10^8/2\pi} = 2\,\text{m}$.

   (b) (1 point) Since the end of the line (i.e. at $d = 0$) is open circuited, $I(0) = 0$.

   (c) (2 points) Yes, because the magnitude of the reflection coefficient at the open termination is one (i.e., $|\Gamma_L| = 1$), then, the reflected wave has the same amplitude as the incident wave, which is enough to set up a standing wave.

   (d) (2 points) If $I_R = 2\,\text{A}$ then the current in the transmission line at $d = l$ is zero (i.e., $I(l) = 0$). Since successive current nulls are $\lambda/2$ apart, the smallest non-zero transmission line length is $l = \lambda/2 = 1\,\text{m}$.

   (e) (2 points) If $l = \frac{\lambda}{2}$ then the voltage at $d = l/2$ is zero (i.e, $V(\frac{l}{2}) = 0$), because half-way between successive current nulls there is a voltage null.

   (f) (2 points) If $l = \frac{\lambda}{2}$ then the current at $d = l/2$ can not be zero (i.e., $I(\frac{l}{2}) \neq 0$), because at this point the current is a maximum.

   (g) (1 point) If $I_R = 0$ then the voltage at $d = l$ is also zero (i.e., $V(l) = 0$). Since voltage and current nulls are $\lambda/4$ apart, the smallest non-zero transmission line length is $l = \frac{\lambda}{4} = 0.5\,\text{m}$.

   (h) (2 points) If $I(\frac{l}{2}) = 0$ and $I(0) = 0$ then $\frac{l}{2} = \frac{\lambda}{2}$ because the separation between two successive current nulls is $\lambda/2$ (non-trivial case). Therefore, $l = \lambda = 2\,\text{m}$.

   (i) (2 points) In part (h), $I(\frac{l}{2}) = 0$ and $l = \lambda$ which implies that $I(l) = 0$. Since $I_R = 2\angle 0°\,\text{A}$ then $V(l) = I_R \times 50\,\Omega = 100\angle 0°\,\text{V}$.

4. (a) (2 points) Referring to page 3 of Lecture 33, the input current is given by

$$I_{in} \equiv I(\frac{\lambda}{4}) = \frac{jV^{+} + jV^{-}}{Z_o} = j\frac{V(0)}{Z_o} = j\frac{V_L}{Z_o}$$

Therefore

$$V_L = -jI_{in}Z_o = -j50\,\mathrm{V}.$$

(b) (2 points) If the load end is shorted then $V_L = 0$. This implies that the current input must also be zero ($I_{in} = j\frac{V_L}{Z_o} = 0$), and therefore, $I_{in} = 0.5\angle 0^\circ$ A is not a possibility in this situation.

(c) (2 points) Since the voltage measured at any distance along the TL is given by

$$V(d) = V^{+}(e^{j\beta d} + \Gamma_L e^{-j\beta d}),$$

where $\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$, the voltages at one end is

$$V_L = V(0) = V^{+}(1 + \Gamma_L)$$

whereas the input voltage of the quarter-wave section at the other end is

$$V_{in} = V(\frac{\lambda}{4}) = jV^{+}(1 - \Gamma_L).$$

Therefore, we can write

$$\frac{V_L}{V_{in}} = -j\frac{1 + \Gamma_L}{1 - \Gamma_L} = -j\frac{Z_L}{Z_o},$$

from which we can obtain

$$V_{in} = j\frac{V_L Z_o}{Z_L} = 100\,\mathrm{V}.$$

5. (a) (1 point) Voltage input at the load end of the half-wave transformer

$$V_{L_1} = -V_{in} = -j10\,\text{V}.$$

(b) (1 point) Voltage input at the load end of the quarter-wave transformer

$$V_{L_2} = -j\frac{V_{in}Z_{L_2}}{Z_o} = 5\,\text{V}.$$

(c) (1 point) Current input at the load end of the half-wave transformer

$$I_{L_1} = \frac{V_{L_1}}{Z_{L_1}} = \frac{-j}{20}\,\text{A}.$$

(d) (1 point) Current input at the load end of the quarter-wave transformer

$$I_{L_2} = -j\frac{V_{in}}{Z_0} = \frac{V_{L_2}}{Z_{L_2}} = \frac{1}{10}\,\text{A}.$$

(e) (2 points) Input impedance of the combined network is given as

$$Z_{in} = Z_{in_1}||Z_{in_2} = Z_{L_1}||\frac{Z_o^2}{Z_{L_2}} = 100\,\Omega.$$

(f) (2 points) Total time-averaged power absorbed is

$$\begin{aligned} P_{avg} &= \tfrac{1}{2}Re\{V_{L_1}I_{L_1}^*\} + \tfrac{1}{2}Re\{V_{L_2}I_{L_2}^*\} \\ &= \tfrac{1}{2}Re\{-j10 \times \tfrac{j}{20}\} + \tfrac{1}{2}Re\{(5) \times (\tfrac{1}{10})\} = 0.5\,\text{W}. \end{aligned}$$

6. (a) (4 points) Since the $\frac{\lambda}{2}$-transformer inverts the sign of its voltage and current inputs at the load end, and two $\frac{\lambda}{2}$ transformers will be a double negative. We can write

$$V_{L1} = -V_g = -100 \text{ V}$$

which implies that

$$I_{L1} = \frac{V_{L1}}{R_{L1}} = -2 \text{ A}.$$

We know that a $\frac{\lambda}{4}$-transformer has a load current $I_L = -j\frac{V_{in}}{Z_o}$. It follows that

$$\begin{aligned} V_{L2} &= I_{L2}R_{L2} \\ &= -j\frac{V_{in}}{Z_{o2}}R_{L2} = -j50 \text{ V}, \end{aligned}$$

which yields

$$I_{L2} = \frac{V_{L2}}{R_{L2}} = -j\frac{V_{in}}{Z_{o2}} = -j \text{ A}.$$

(b) (2 points) The power dissipated in resistor R_{L1} is

$$P_{L1} = \frac{1}{2}\text{Re}\left\{V_{L1}I_{L1}^*\right\} = \frac{1}{2}Re\left\{\frac{|V_{L1}|^2}{R_{L1}^*}\right\} = 100 \text{ W}.$$

Then for the second line which is a $\frac{\lambda}{4}$-transformer, the time average dissipated power in resistor R_{L2} is given by

$$P_{L2} = \frac{1}{2}\text{Re}\left\{V_{L2}I_{L2}^*\right\} = 25 \text{ W}.$$

7. **Bonus Problem :** An air filled (i.e. $\mu = \mu_o$ and $\epsilon = \epsilon_o$) parallel-plate transmission line with length $l = 1\,\text{m}$, width $w = 5\,\text{cm}$, and plate separation $d = 0.5\,\text{cm}$ is shorted at both ends.

   (a) (1 point) As both ends of the TL has been "short circuited", and thus TL voltage $V(z,t)$ needs to vanish at $z = 0$ and $z = 1$. Since

$$V(z,t) = V^+(t - \frac{z}{v}) + V^-(t + \frac{z}{v})$$

from d'Alembert solutions, these boundary conditions

$$V(0,t) = V^+(t) + V^-(t) = 0$$

and

$$V(l,t) = V^+(t - \frac{l}{v}) + V^-(t + \frac{l}{v}) = 0$$

require that

$$V^+(t) = -V^-(t),$$

and

$$V^+(t - \frac{l}{v}) = -V^-(t + \frac{l}{v}) \;\Rightarrow\; V^+(t') = V^+(t' + \frac{2l}{v}),$$

where $t' = t - \frac{l}{v}$. Finally, we can conclude that period $T = \frac{2l}{v}$, and fundamental frequency $\omega_o = \frac{2\pi}{T} = \frac{\pi v}{l}$. At the lowest resonant frequency l should be half a wavelength long, therefore

$$f_o = \frac{\omega_o}{2\pi} = \frac{v}{2l} = 1.5 \times 10^8\,\text{Hz} = 150\,\text{MHz}.$$

   (b) (2 points) The phasor counterpart of a co-sinusoidal wave $V(z,t)$ is

$$\tilde{V}(z) = V^+e^{-j\beta z} + V^-e^{j\beta z}.$$

Using the result from part (a) $V^+(t) = -V^-(t)$, we get

$$\tilde{V}(z) = V^+e^{-j\beta z} - V^+e^{j\beta z} = -j2V^+\sin(\beta z),$$

which leads

$$V(z,t) = \text{Re}\left\{\tilde{V}(z)e^{j\omega t}\right\} = 2V^+\sin(\beta z)\sin(\omega t),$$

where $2V^+ = V_o$ (maximum voltage of the resonant mode), and therefore

$$V(z,t) = V_o\sin(\beta z)\sin(\omega t)\,\text{V}.$$

   (c) (2 points) Utilizing the Telegrapher's equation $-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t}$ together with the expressions for $V(z,t)$ obtained in part (b), we can find

$$-\frac{\partial V}{\partial z} = -\beta V_o\cos(\beta z)\sin(\omega t),$$

which yields

$$\frac{\partial I}{\partial t} = -\frac{\beta V_o}{\mathcal{L}}\cos(\beta z)\sin(\omega t).$$

Next, finding the anti-derivative of the above, we get

$$I(z,t) = \frac{\beta V_o}{\omega\mathcal{L}}\cos(\beta z)\cos(\omega t)\,\text{A}.$$

(d) (3 points) The stored energy is given by

$$W = \int_{x=0}^{d}\int_{y=0}^{w}\int_{z=0}^{\ell}(\frac{1}{2}\epsilon_o E_x^2 + \frac{1}{2}\mu_o H_y^2)\,dxdydz.$$

Given that $V = E_x d$ and $I = H_y w$ and substituting V and I with those expressions derived in parts (b) and (c), respectively, we obtain a more complicated expression for the stored energy

$$W = \int_{x=0}^{d}\int_{y=0}^{w}\int_{z=0}^{\ell}\left[\frac{1}{2}\epsilon_o\frac{V_o^2}{d^2}\sin^2(\beta z)\sin^2(\omega t) + \frac{1}{2}\mu_o\frac{\beta^2 V_o^2 w^2}{w^2\omega^2\mu_o^2 d^2}\cos^2(\beta z)\cos^2(\omega t)\right]\,dxdydz,$$

where $\beta = \omega\sqrt{\epsilon_o\mu_o} \implies \epsilon_o = \frac{\beta^2}{\omega^2\mu_o}$ which lets us work through the previous equation

$$W = \frac{1}{2}\epsilon_o\frac{V_o^2}{d^2}\left[\int_{x=0}^{d}\int_{y=0}^{w}\int_{z=0}^{\ell}\left\{\sin^2(\beta z)\sin^2(\omega t) + \cos^2(\beta z)\cos^2(\omega t)\right\}\,dxdydz\right],$$

$$W = \frac{1}{2}\epsilon_o\frac{wV_o^2}{d}\left[\int_{z=0}^{\ell}\left\{\sin^2(\beta z)\sin^2(\omega t) + \cos^2(\beta z)\cos^2(\omega t)\right\}\,dz\right],$$

$$W = \frac{1}{2}CV_o^2\left[\int_{z=0}^{\ell}\left\{\sin^2(\beta z)\sin^2(\omega t) + \cos^2(\beta z)\cos^2(\omega t)\right\}\,dz\right],$$

$$W = \frac{1}{2}CV_o^2\left[\sin^2(\omega t)\left(\frac{\ell}{2} - \frac{\sin(2\beta\ell)}{4\beta}\right) + \cos^2(\omega t)\left(\frac{\ell}{2} + \frac{\sin(2\beta\ell)}{4\beta}\right)\right].$$

Now, for the lowest resonant frequency $f_o = \frac{v}{2\ell}$ (from part (a)), we get $\lambda = 2\ell = 2$ (as $\ell = 1\,\text{m}$). Also, $\sin(2\beta\ell) = \sin(2\times 2\pi \times 2) = 0$. Therefore, the above equation will reduce to

$$W = \frac{1}{4}C\ell V_o^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right],$$

$$W = \frac{1}{4}C\ell V_o^2 \ \text{ J}.$$

The numerical value of proportionality constant is $\frac{1}{4}$.

(e) (2 points) From part (d), we know that $W = \frac{1}{4}C\ell V_o^2$. The expected value of V_o^2 will be

$$E\left[V_o^2\right] = \frac{E\left[W\right]}{\frac{1}{4}C\ell} = \frac{k_B T}{\frac{1}{4}C\ell} = \frac{4\times 1.38\times 10^{-23}\times 300}{10\times 8.85\times 10^{-12}\times 1},$$

which gives us

$$E\left[V_o^2\right] = 1.87\times 10^{-10}.$$

Therefore, the square root of the expected value of V_o^2 will be $\sqrt{E\left[V_o^2\right]} = 13.68\times 10^{-6}\,\text{V}$.