1. (a) (3 points) The characteristic impedance of a co-axial cable is given by

$$Z_o = \frac{1}{GF}\sqrt{\frac{\mu}{\epsilon}}$$

where $GF = \frac{2\pi}{\ln(\frac{b}{a})}$. Given that $Z_o = 50\,\Omega$, $a = 1.2\,\text{mm}$, $\mu = \mu_o$, and $\epsilon = 2.25\epsilon_o$, we have

$$\begin{aligned} b &= a\exp(2\pi Z_o\sqrt{\frac{\epsilon}{\mu}}) \\ &= 1.2\times 10^{-3}\exp(2\pi\times 50\times\sqrt{\frac{2.25\times 8.854\times 10^{-12}}{1.257\times 10^{-6}}}) \\ &= 4.19\,[\text{mm}]. \end{aligned}$$

(b) (4 points) Considering $Z_o = 50\,\Omega$, $R_g = 50\,\Omega$, and $R_L = 25\,\Omega$, we can find the voltage reflection coefficients at the load and at the source as follows:

$$\Gamma_{LV} = \frac{R_L - Z_o}{R_L + Z_o} = -\frac{1}{3}$$

$$\Gamma_{gV} = \frac{R_g - Z_o}{R_g + Z_o} = 0$$

The corresponding current reflection coefficients are

$$\Gamma_{L_I} = -\Gamma_{L_V} = \frac{1}{3}, \text{ and}$$

$$\Gamma_{g_I} = -\Gamma_{g_V} = 0$$

Notice that there is no refected wave at the source due to the fact that $\Gamma_{gV} = \Gamma_{gI} = 0$. Therefore, for the entire transmission line to reach steady state, we need to wait until the reflected wave at the load to arrive at $z = 0$, which requires the two-way travel time. Thus, the steady state time is determined by

$$\begin{aligned} t_{ss} &= \frac{2l}{v} = 2l\sqrt{\epsilon\mu} \\ &= 2\times 0.15\times\frac{\sqrt{2.25}}{c} \\ &= 1.5\,[\text{ns}]. \end{aligned}$$

(c) (5 points) The initial voltage at the input of the transmission line is given by the voltage divider

$$\frac{Z_o}{R_g + Z_o}V_g = \frac{50}{50+50}\times 10u(t) = 5u(t)$$

Using the reflection coefficients found in (b), we can write down the expressions for voltage and current as

$$V(z,t) = 5u(t - \frac{z}{v}) - \frac{5}{3}u(t + \frac{z}{v} - 1.5ns)\,[\text{V}],$$

$$I(z,t) = \frac{1}{10}u(t-\frac{z}{v}) + \frac{1}{30}u(t+\frac{z}{v}-1.5ns)\,[\mathrm{A}].$$

At $z = \frac{l}{2}$,

$$V(\frac{l}{2},t) = 5u(t-0.375ns) - \frac{5}{3}u(t-1.125ns)\,[\mathrm{V}],$$
$$I(\frac{l}{2},t) = \frac{1}{10}u(t-0.375ns) + \frac{1}{30}u(t-1.125ns)\,[\mathrm{A}].$$

(d) (10 points) Applying the Gauss's law to the co-axial cable, we have

$$\epsilon(2\pi rl)E(r,t) = (2\pi al)\rho_s(t)$$
$$\mathbf{E}(r,t) = \frac{a}{\epsilon r}\rho_s(t)\hat{r}$$

where ρ_s is the surface charge density on the inner conductor, and $a < r < b$.
Now consider the voltage drop between the two cylinders as the line integral of the electric field. At $z = \frac{l}{2}$,

$$\begin{aligned} V(\frac{l}{2},t) &= -\int_a^b \mathbf{E}(r,t)\cdot d\mathbf{r} \\ &= -\int_a^b \frac{a}{\epsilon r}\rho_s(t)dr \\ &= -\frac{a}{\epsilon}\ln(\frac{b}{a})\rho_s(t) \end{aligned}$$

Therefore, the charge density as a function of time at $z = \frac{l}{2}$ is given by

$$\begin{aligned} \rho_s(\frac{l}{2},t) &= -\frac{\epsilon V(\frac{l}{2},t)}{a\ln(\frac{b}{a})} \\ &= -1499.2\epsilon_o V(\frac{l}{2},t) \\ &= -7496.0\epsilon_o u(t-0.375ns) + 2498.7\epsilon_o u(t-1.125ns)\,[\mathrm{C/m^2}]. \end{aligned}$$

Then the time dependent electric field is characterized by

$$\begin{aligned} \mathbf{E}(r,t) &= -\frac{1}{r\ln(\frac{b}{a})}V(\frac{l}{2},t)\hat{r} \\ &= \frac{1}{r}(-4.00u(t-0.375ns) + 1.33u(t-1.125ns))\hat{r}\,[\mathrm{V/m}]. \end{aligned}$$

By the Ampere's law, we have

$$(2\pi r)B(r,t) = \mu I(\frac{l}{2},t)$$

which gives

$$\begin{aligned} \mathbf{B}(r,t) &= \frac{\mu_o}{2\pi r}I(\frac{l}{2},t)(-\widehat{\phi}) \\ &= (-\widehat{\phi})\frac{\mu_o}{r}(1.59e^{-2}u(t-0.375ns) + 5.31e^{-3}u(t-1.125ns))\,[\mathrm{Wb/m^2}]. \end{aligned}$$

The direction of the magnetic field is $-\widehat{\phi}$ because the current in the inner conductor flows in $-\widehat{z}$ direction.

(e) (4 points) See (f) for the expression for $\rho_s(\frac{l}{2}, t)$. Notice that for the conducting cylinder, charges and current reside uniformly on the outer surface.
Therefore, the surface charge density J_s is given by

$$\begin{aligned}\mathbf{J}_s(\frac{l}{2}, t) &= \frac{I(\frac{l}{2}, t)}{2\pi a}(-\hat{z}) \\ &= -\hat{z}(13.26u(t - 0.375ns) + 4.42u(t - 1.125ns))\,[\text{A/m}].\end{aligned}$$

(f) (2 points) FALSE. By the boundary conditions at $r = a$ and $r = b$, since there is no electric field inside conductors, we have

$$\rho_s(r = a, \frac{l}{2}, t) = \epsilon E(r = a, \frac{l}{2}, t)$$

$$\rho_s(r = b, \frac{l}{2}, t) = -\epsilon E(r = b, \frac{l}{2}, t)$$

It is clear that the sign of the outer conductor surface charge density is reversed. However, the two charge densities have different magnitudes since $\mathbf{E}(r, t)$ is a function of r.
For the suface current density of the outer conductor, we can perform the same calculations as in (e). Then we obtain the following expressions:

$$\mathbf{J}_s(r = a, \frac{l}{2}, t) = \frac{I(\frac{l}{2}, t)}{2\pi a}(-\hat{z})$$

$$\mathbf{J}_s(r = b, \frac{l}{2}, t) = \frac{I(\frac{l}{2}, t)}{2\pi b}\hat{z}$$

Obviously, they have opposite signs and different magnitudes.

Intuitively, you might be thinking of "equal and opposite charge" on two pieces of parallel metal. This is true, however, for the total amount of charge, not for the charge density. Apparently the inner conductor surface and outer conductor surface have different areas; therefore due to "equal and opposite charge", their charge densities cannot be of the same magnitude.

(g) (2 points) Once the transmission line reaches steady state, it functions as an ordinary wire. By Ohm's law, the steady state current is given by

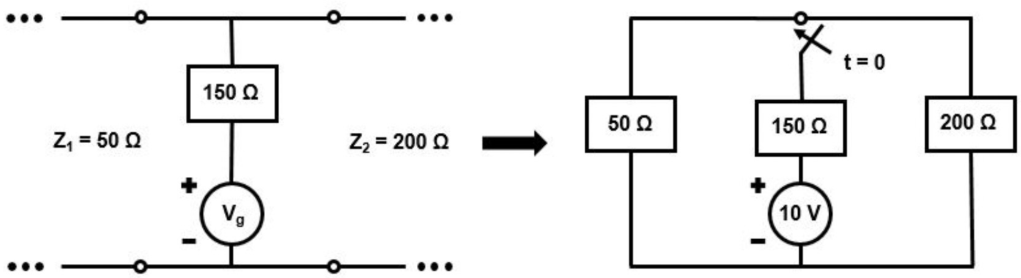
$$I_{ss} = \frac{V_g}{R_g + R_L} = \frac{10}{50 + 25} = \frac{2}{15}\,[\text{A}].$$

We can double check the answer from the expression of $I(z, t)$, which gives $I_{ss} = \frac{1}{10} + \frac{1}{30} = \frac{2}{15}$ A.
Thus, the power delivered to the load is calculated as

$$P_L = I_{ss}^2 R_L = (\frac{2}{15})^2 \times 25 = 0.44\,[\text{W}].$$

2. (6 points) We have examined the case of a terminated transmission line connected to a two terminal circuit modeled by a Thevenin equivalent voltage $V_g(t)$ and resistance R_g where the injection coefficient $\tau_g = \frac{Z_o}{R_g+Z_o}$ is found by applying voltage division. To derive this relation we had to combine the constraint imposed by the two terminal source $V = R_gIV_g$ (using KVL) and the transmission line $V = Z_oI$. Assuming that $V = V^+$ the two equations combine to give the result $V^+ = \tau_g V_g = \frac{Z_o}{R_g+Z_o}V_g$. At the moment the source turns on it is only influenced by the relation $V = Z_oI$ inside the transmission line at $z = 0$. This leads to the familiar voltage divider result.

For this problem we have two infinite lines connected in parallel with the same two terminal source with resistance $R_g = 150$ and $V_g(t) = 10u(t)$ V. To find the injection coefficient for



this configuration the voltage boundary conditions dictate that $V = V_g - R_gI$, $V = Z_1I$, and $V = Z_2I$. The currents must have the relation $I = I_1 + I_2$ if I is the current flowing from the source, I_1 and I_2 are the currents flowing into each transmission line. These constraints are exactly the same as a circuit with the same source configuration connected to two parallel resistors. This greatly simplifies finding the injection coefficient. Now

$$\tau_g = \frac{Z_{eq}}{R_g + Z_{eq}},$$

where

$$Z_{eq} = Z_1//Z_2 = (\frac{1}{50} + \frac{1}{200})^{-1} = 40\,[\Omega].$$

Then

$$\tau_g = \frac{Z_{eq}}{R_g + Z_{eq}} = \frac{40}{150 + 40} = \frac{4}{19}$$

is the total injection coefficient i.e. the ratio of the generator voltage injected into the two parallel lines. Since the lines are combined in parallel, the magnitude of the voltage waves along them are equal so

$$V(t = 0^+, z = 0^+) = V(t = 0^+, z = 0) = \tau_g V_g = \frac{4}{19} \times 10 = \frac{40}{19}\,[\text{V}],$$

and we find the current from each line's characteristic impedance

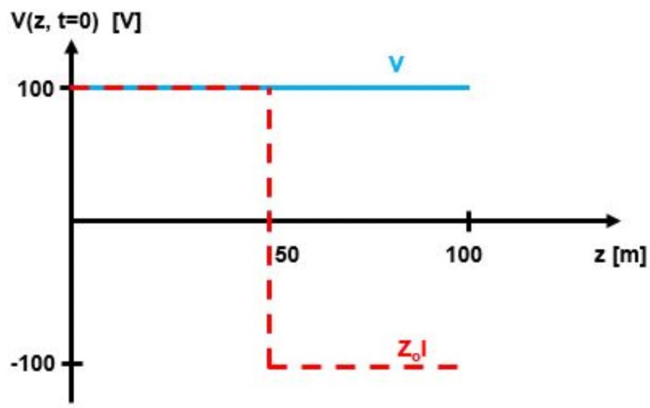
$$I(t = 0^+, z = 0^+) = \frac{\tau_g V_g}{Z_2} = \frac{40}{19 \times 200} = 10.5\,[\text{mA}],$$

$$I(t = 0^+, z = 0) = \frac{\tau_g V_g}{Z_1} = \frac{40}{19 \times 50} = 42.41\,[\text{mA}].$$

3. (14 points) We try to decompose the initial conditions into a forward wave and a backward wave as :

$$
\begin{aligned}
V(z,t) &= V^+ + V^- = f(t-\frac{z}{v}) + g(t+\frac{z}{v}) \\
I(z,t) = I^+ - I^- &= \frac{V^+ - V^-}{Z_o} = \frac{1}{Z_o}[f(t-\frac{z}{v}) - g(t+\frac{z}{v})]
\end{aligned}
\tag{1}
$$

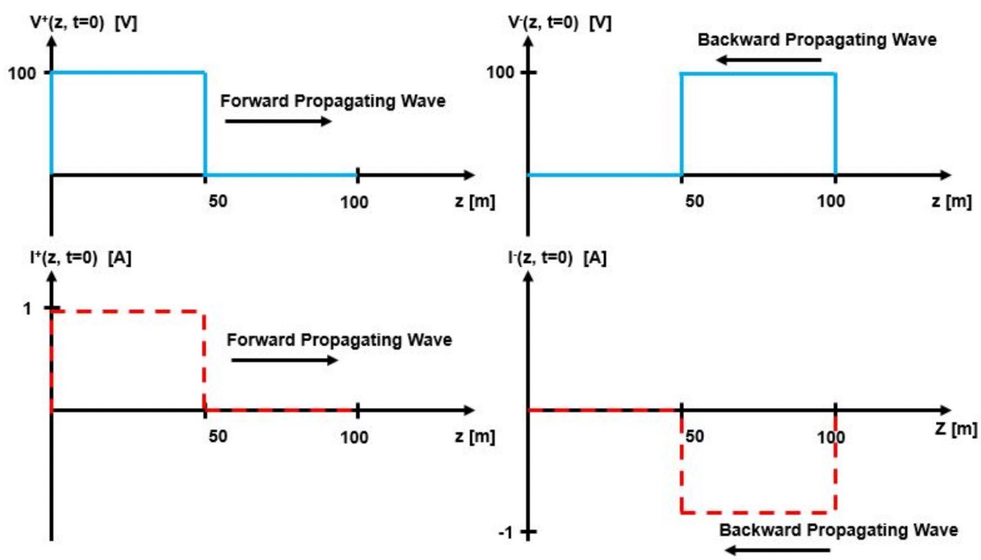
and seek a solution.



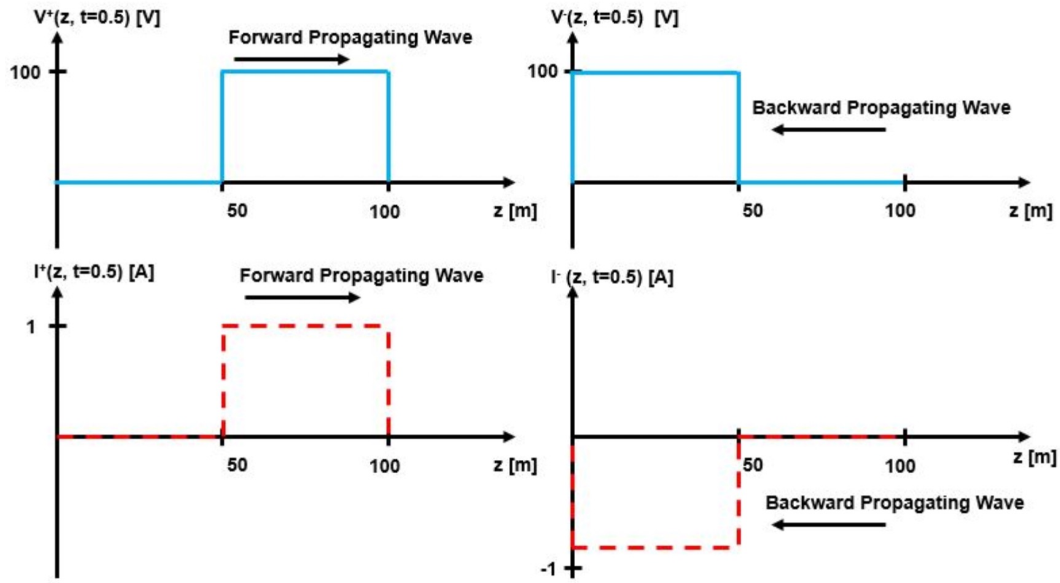
It is obvious that

$$
\begin{aligned}
V^+ &= \frac{1}{2}(V + IZ_o) \\
V^- &= \frac{1}{2}(V - IZ_o)
\end{aligned}
\tag{2}
$$

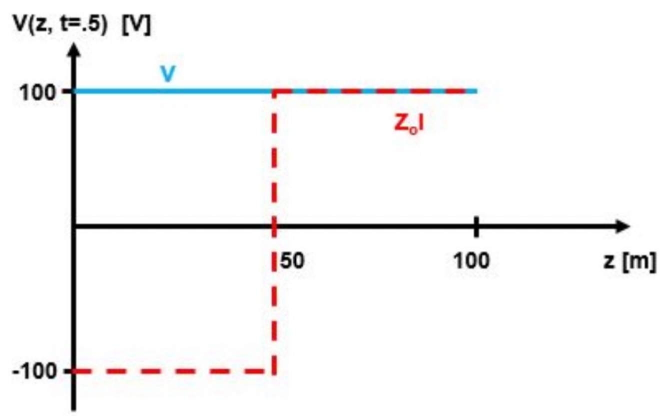
and we have the following decomposition.



Since the wave travels at $100\,\frac{\text{m}}{\mu\text{s}}$, after $0.5\,\mu$s the wave have traveled $50\,$m and have the following waveform.



Combining the waveforms we get the voltage and current as below.



4. Let us consider the following circuit diagram where Z_o, R_L, and L are unknowns.

   (a) (1 point) Looking at the voltage waveform plot given in the problem, it can be seen that the amplitude of the incident pulse is 60 V. Since the ratio between this amplitude and the source voltage is given by the voltage divider formula, we can write

$$\tau_g = \frac{60}{90} = \frac{2}{3} = \frac{Z_o}{R_g + Z_o} \Longrightarrow Z_o = 100\,[\Omega].$$

   (b) (2 points) Again looking at the plot in the problem, one can see that the amplitude of the first reflected pulse is -20 V. Therefore, the reflection coefficient is

$$\Gamma_L = -\frac{20}{60} = -\frac{1}{3} = \frac{R_L - Z_o}{R_L + Z_o}$$

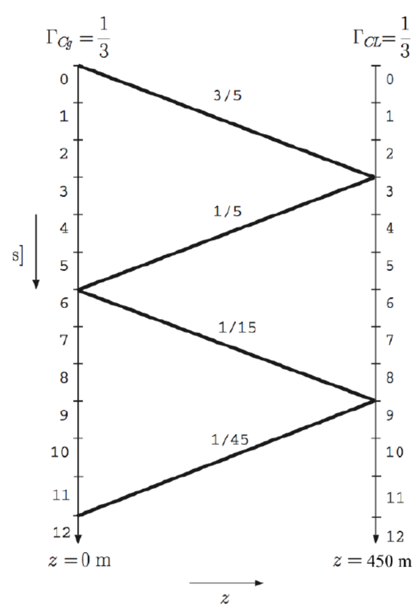
   from which we obtain the load resistance as $R_L = \frac{Z_o}{2} = 50\,[\Omega]$.

   (c) (1 point) The time interval between the incident pulse and the second reflected pulse is $6\,\mu$s, which is equal to the two-way travel time. Then, the time it takes the pulse to travel from one end of the line to the other is $T = 3\,\mu$s.

   (d) (2 points) Since at $z_0 = 300$ m, the incident pulse is delayed by $2\,\mu$s, the propagation speed $v_p = \frac{300}{2\times10^{-6}} = 1.5 \times 10^8$m/s. Thus, we can find that the length of the line is $L = v_p \times T = 150 \times 3 = 450$ m.

   (e) (3 points) The reflected coefficient at the source is $\Gamma_g = -\frac{1}{3}$. Therefore, the next two voltage impluses are $-\frac{20}{9}\delta(t-10)$ V and $\frac{20}{27}\delta(t-14)$ V.

   (f) (4 points) Bounce diagram for the *current* waveform $I(z,t)$ for $0 < t < 12\,\mu$s is given below:



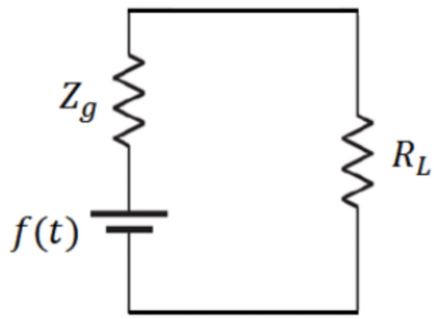
   (g) (2 points) The expression for the current waveform for $0 < z < L$ and $0 < t < 12\,\mu$s is

$$I(z,t) = \frac{3}{5}\delta(t - \frac{z}{v_p}) + \frac{1}{5}\delta(t + \frac{z}{v_p} - 6\,\mu\text{s}) + \frac{1}{15}\delta(t - \frac{z}{v_p} - 6\,\mu\text{s}) + \frac{1}{45}\delta(t + \frac{z}{v_p} - 12\,\mu\text{s})\,[\text{A}].$$

5. Using the figures, we can identify and compute the following parameters of the circuit.
   (a) (2 points) Transmission time $T_1 = 2\,[\mu s]$ and $T_2 = 3\,[\mu s]$.
   (b) (2 points) $\upsilon_{p1} = \frac{400}{2\times10^{-6}} = 2\times10^8\,[\text{m/s}]$, and $\upsilon_{p2} = \frac{300}{3\times10^{-6}} = 1\times10^8\,[\text{m/s}]$.
   (c) (1 point) Since there is no reflection at the loadR_L, the characteristic impedance of the line 2 is $Z_2 = R_L = 60\,[\Omega]$.
   (d) (2 points) The transmission coefficient from line 1 to line 2 is seen to be $\tau_{12} = \frac{2}{3}$. Hence, the reflection coefficient from line 1 to line 2 is $\Gamma_{12} = \tau_{12} - 1 = -\frac{1}{3}$.
   (e) (2 points) Impedance of line 2 has already been found in part (c) as $Z_2 = 60\,[\Omega]$. Thus, using $\Gamma_{12} = \frac{Z_2-Z_1}{Z_2+Z_1} = -\frac{1}{3}$, we find $Z_1 = 120\,[\Omega]$.
   (f) (2 points) The reflection coefficient at the source is given by $\Gamma_g = \frac{V^{-,+}}{V^-} = \frac{12}{-20} = -\frac{3}{5}$. Then, utilizing this result in $\Gamma_g = \frac{R_g-Z_1}{R_g+Z_1}$, we find $R_g = \frac{1}{4}Z_1 = 30\,[\Omega]$.
   (g) (1 point) Using the voltage divider rule, the incident voltage is expressed as $60\,\text{V} = V_o\frac{Z_1}{R_g+Z_1}$ where V_o is the source voltage. Therefore, $V_o = 60\frac{R_g+Z_1}{Z_1} = 75\,[\text{V}]$.
   (h) (1 point) Reflected voltage $V^{-,+,-} = -4\,[\text{V}]$.
   (i) (1 point) Since, as $t \to \infty$, transmission lines become ordinary wires, the steady-state voltage on line 1 is $V_1 = V_o\frac{R_L}{R_g+R_L} = 75 \times \frac{60}{30+60} = 50\,[\text{V}]$.
   (j) (1 point) Same as above, the steady-state voltage on line 2 is $V_2 = 50\,[\text{V}]$

6. Equivalent circuit of the problem can be seen from that of the previous problem.

   (a) (2 points) At steady-state, the presence of the TL doesn't matter, it acts like a short. Hence $V(z=0,t=\infty) = V_L$ and $I(z=0,t=\infty) = I_L$. The circuit simplifies to

   Hence,

   $$R_g = \frac{6\,\mathrm{V} - 3\,\mathrm{V}}{30\,\mathrm{mA}} = 0.1\,\mathrm{K\Omega} = 100\,\Omega.$$

   (b) (2 points) If the defect is a short, $R_L = 0$, hence $V(z=0,t=\infty) = 0\,[\mathrm{V}]$, $I(z=0,t=\infty) = \frac{6}{100} = 60\,[\mathrm{mA}]$

   (c) (2 points) If the defect is an open, $R_L \to \infty$, hence $V(z=0,t=\infty) = 6\,[\mathrm{V}]$, $I(z=0,t=\infty) = 0\,[\mathrm{mA}]$

   (d) (4 points) $2\,\mathrm{V}$ read from the input end of the line is due to the injection of incident wave ($V_s = 1\,\mathrm{V}$) from the source. Hence we get

   $$\begin{aligned} V(z=0,t=0) &= \tau_g V_s(t=0) \\ 2 &= \frac{Z_o}{Z_g + Z_o} \times 3. \end{aligned}$$

   Thus, the impedance of the tested line is

   $$Z_0 = 2Z_g = 200\,\Omega.$$

   The incident wave (2V) gets injected into the TL and starts to move down along the line for a length L then it hits the defect. The reflected wave comes back and reach the input end of the TL, superposes with the injected incident wave. That's why we see a change in response observed later. That means $t = 5\,\mu s$ is the "round-trip" time for the wave to travel from the input end of the TL to the defect and return back to the input end of the TL. L is then found as

   $$L = v \times \frac{t}{2} = 2 \times 10^8\,\mathrm{m/s} \times \frac{5}{2}\,\mu\mathrm{s} = 500\,[\mathrm{m}].$$

   Also, the total voltage observed at the input end of TL increases after the reflected wave reaches this end. It means the reflected wave is positive (or in phase) with the incident wave. Hence, it must be an open-circuit type of defect.

7. (a) (5 points) Suppose the incident wave is V^+ at the interface, the reflected and transmitted waves can be expressed as

$$\begin{aligned} V^- &= \Gamma V^+ = \frac{\alpha-1}{\alpha+1}V^+ \\ V^{++} &= \tau V^+ = (1+\Gamma)V^+ = \frac{2\alpha}{\alpha+1}V^+ \end{aligned} \tag{3}$$

and the corresponding current waves are

$$\begin{aligned} I^+ &= \frac{V^+}{Z_1} \\ I^- &= -\Gamma I^+ = -\frac{\alpha-1}{\alpha+1}\frac{V^+}{Z_1} \\ I^{++} &= \frac{V^{++}}{Z_2} = \frac{2}{\alpha+1}\frac{V^+}{Z_1} \end{aligned} \tag{4}$$

(b) (2 points) We need to show $P^- = P^+$.

$$\begin{aligned} P^- &= \frac{1}{2}\mathrm{Re}\{(V^+ + V^-)(I^+ + I^-)^*\} \\ &= \frac{1}{2}\mathrm{Re}\{(1+\Gamma)V^+(1-\Gamma)\frac{(V^+)^*}{Z_1}\} \\ &= \frac{1}{2}\mathrm{Re}\{(1-\Gamma^2)\frac{|V^+|^2}{Z_1}\} \\ &= \frac{1}{2}\mathrm{Re}\{\frac{4\alpha}{(1+\alpha)^2}\frac{|V^+|^2}{Z_1}\} \\ P^+ &= \frac{1}{2}\mathrm{Re}\{V^{++}I^{++*}\} \\ &= \frac{1}{2}\mathrm{Re}\{(1+\Gamma)V^+(1+\Gamma)\frac{V^+}{Z_2}\} \\ &= \frac{1}{2}\mathrm{Re}\{(1+\Gamma)^2\frac{|V^+|^2}{\alpha Z_1}\} \\ &= \frac{1}{2}\mathrm{Re}\{\frac{4\alpha}{(1+\alpha)^2}\frac{|V^+|^2}{Z_1}\} \end{aligned} \tag{5}$$

(c) (3 points) $\alpha = 0$: The T.L. is terminated with a short circuit, so $\Gamma = -1$ and $\tau = 0$. A standing wave is formed with V_{min} at the load.

$\alpha = 1$: $Z_1 = Z_2$, so $\Gamma = 0$ and $\tau = 1$. All the wave is transmitted into region 2.

$\alpha = \infty$: The T.L. is terminated with an open circuit, so $\Gamma = 1$ and $\tau = 2$. A standing wave is formed with V_{max} at the load.