1. (5 points) For twin-lead transmission lines, the geometrical factor is given by

$$\mathrm{GF} = \frac{\pi}{\cosh^{-1}\left(\frac{D}{2a}\right)}.$$

Since

$$Z_o = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \frac{1}{\mathrm{GF}}\sqrt{\frac{\mu}{\epsilon}} = \frac{\cosh^{-1}\left(\frac{D}{2a}\right)}{\pi}\sqrt{\frac{\mu}{\epsilon}},$$

we can can find that

$$D = 2a\,\cosh\left(Z_o\pi\sqrt{\frac{\epsilon}{\mu}}\right).$$

Assuming $\epsilon = \epsilon_o$, $\mu = \mu_o$, and $2a = 1\,\mathrm{mm}$, let us calculate D as follows:

(a) For $Z_o = 50\,\Omega$,

$$D = 1\times 10^{-3}\,\cosh\left(\frac{50\pi}{120\pi}\right) \approx 1.088\times 10^{-3}\,\mathrm{m} = 1.088\,[\mathrm{mm}].$$

(b) For $Z_o = 300\,\Omega$,

$$D = 1\times 10^{-3}\,\cosh\left(\frac{300\pi}{120\pi}\right) \approx 6.132\times 10^{-3}\,\mathrm{m} = 6.132\,[\mathrm{mm}].$$

2. There is a resistor in-between two TLs, so:

   (a) (2 points) The voltage in the first line will be a sum of incident $V^+$and reflected voltages V^-, i.e. $V_1 = V^+ + V^-$ whereas after the first line it will be $V_2 = V_R + V^{++}$ where V_R is the voltage across the resistor. Therefore, the KVL equation at the junction is

$$V_1 = V_2 \implies V^+ + V^- = V_R + V^{++}$$

   The current flowing in the first line is the sum of the incident and reflected currents, i.e. $I_1 = \frac{V^+}{Z_1} - \frac{V^-}{Z_1}$. However, this current must also be equal to the current flowing through the resistor and also to the current transmitted to the second line. Thus, the KCL equation at the junction is given by

$$\frac{V^+ - V^-}{Z_1} = \frac{V_R}{R} = \frac{V^{++}}{Z_2}$$

   (b) (4 points) Combining the previous equations by eliminating V_R we have

$$V^+ - V^- = \left(\frac{Z_1}{Z_2}V^{++}\right)$$

$$V^+ + V^- = \left(\frac{R + Z_2}{Z_2}V^{++}\right)$$

   Thus, the reflection coefficient is

$$\Gamma_{12} = \frac{V^-}{V^+} == \frac{R + Z_2 - Z_1}{R + Z_2 + Z_1}$$

   Similarly, transmission coefficient is simply found by referring to $\tau_{12} = \dfrac{V^{++}}{V^+}$. Hence, we have

$$\begin{aligned} \tau_{12} &= (1 + \Gamma_{12})\frac{Z_2}{R + Z_2} \\ &= \frac{2Z_2}{R + Z_2 + Z_1} \end{aligned}$$

   where $\dfrac{Z_2}{R + Z_2}$ is a voltage division factor. Note in this case $\tau_{12} - \Gamma_{12} \neq 1$(because we have an extra resistor.)

   (c) (3 points) D'Alembert traveling waves for incident voltage waves V^+:

$$V^+ = f\left(t - \frac{z}{v_1}\right)$$

   Then for V^-:

$$V^- = \Gamma_{12} f\left(t + \frac{z}{v_1}\right)$$

   and for V^{++}:

$$V^{++} = \tau_{12} f\left(t - \frac{z}{v_2}\right)$$

The total wave at $z < 0$:

$$V|_{z<0} = V^+ + V^- = f\left(t - \frac{z}{v_1}\right) + \Gamma_{12} f\left(t + \frac{z}{v_1}\right)$$

and at $z > 0$:

$$V|_{z>0} = V^{++} = \tau_{12} f\left(t - \frac{z}{v_2}\right)$$

And for current waves:

$$I^+ = \frac{1}{Z_1} f\left(t - \frac{z}{v_1}\right)$$

Then for I^-:

$$I^- = \frac{\Gamma_{12}}{Z_1} f\left(t + \frac{z}{v_1}\right)$$

and for I^{++}:

$$I^{++} = \frac{\tau_{12}}{Z_2} f\left(t - \frac{z}{v_2}\right)$$

Thus the total current at $z < 0$:

$$I|_{z<0} = I^+ - I^- = \frac{1}{Z_1} f\left(t - \frac{z}{v_1}\right) - \frac{\Gamma_{12}}{Z_1} f\left(t + \frac{z}{v_1}\right)$$

and at $z > 0$:

$$I_{z>0} = I^{++} = \frac{\tau_{12}}{Z_2} f\left(t - \frac{z}{v_2}\right)$$

(d) (2 points) Considering $Z_1 = 50\,\Omega$, $Z_2 = 100\,\Omega$, and $R = 200\,\Omega$, we can find that $Z_{eq} = R + Z_2 = 300\,\Omega$. Then, we calculate

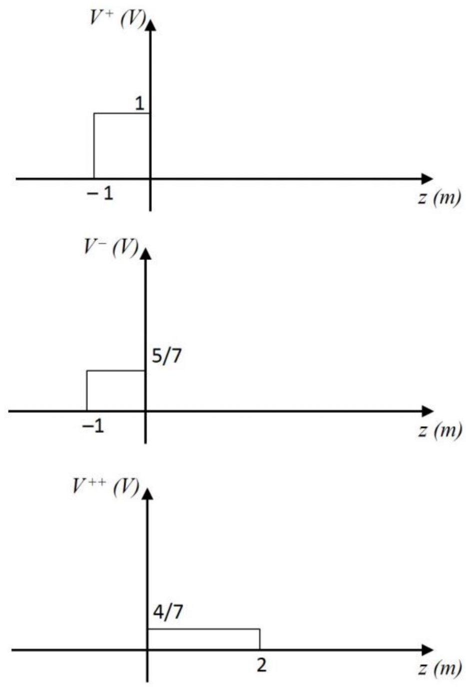
$$\Gamma_{12} = \frac{Z_{eq} - Z_1}{Z_{eq} + Z_1} = \frac{300 - 50}{300 + 50} = \frac{5}{7}$$

and

$$\tau_{12} = \frac{2Z_2}{Z_{eq} + Z_1} = \frac{200}{300 + 50} = \frac{4}{7}$$

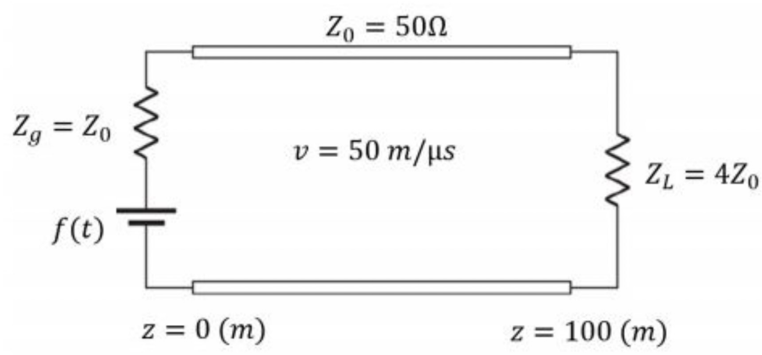
Note that $1 + \Gamma_{12} \neq \tau_{12}$.

(e) (4 points) At $t = 0$, the square pulse starts to hit the resistor. As time advances, the pulse moves toward the $+\hat{z}$ direction, a part of it will be reflected back toward $-\hat{z}$ direction, that is V^-, another part gets transmitted toward $+\hat{z}$ direction, which results in V^{++}.
At $t = 10\,\text{ns}$, the incident and reflected wave have been moving a distance of $10\,\text{ns} \times 1 \times 10^8 = 1\,\text{m}$ (of course, in opposite directions) whereas the transmitted wave has been moving a distance of $10\,\text{ns} \times 2 \times 10^8 = 2\,\text{m}$ (into $+\hat{z}$ direction). Hence, for each of them, their distributions over the location z are as follow



$V(t = 10ns, z)$ can be found by summation of plots of V^+ and V^- above.

3. The associated circuit is shown in the following figure:



(a) (11 points) In this circuit, the **injection coefficient** is found as

$$\tau_g = \frac{Z_o}{Z_g + Z_o} = \frac{1}{2}$$

The load voltage reflection coefficient is given by

$$\Gamma_{L_V} = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{3}{5}$$

whereas the generator voltage reflection coefficient is

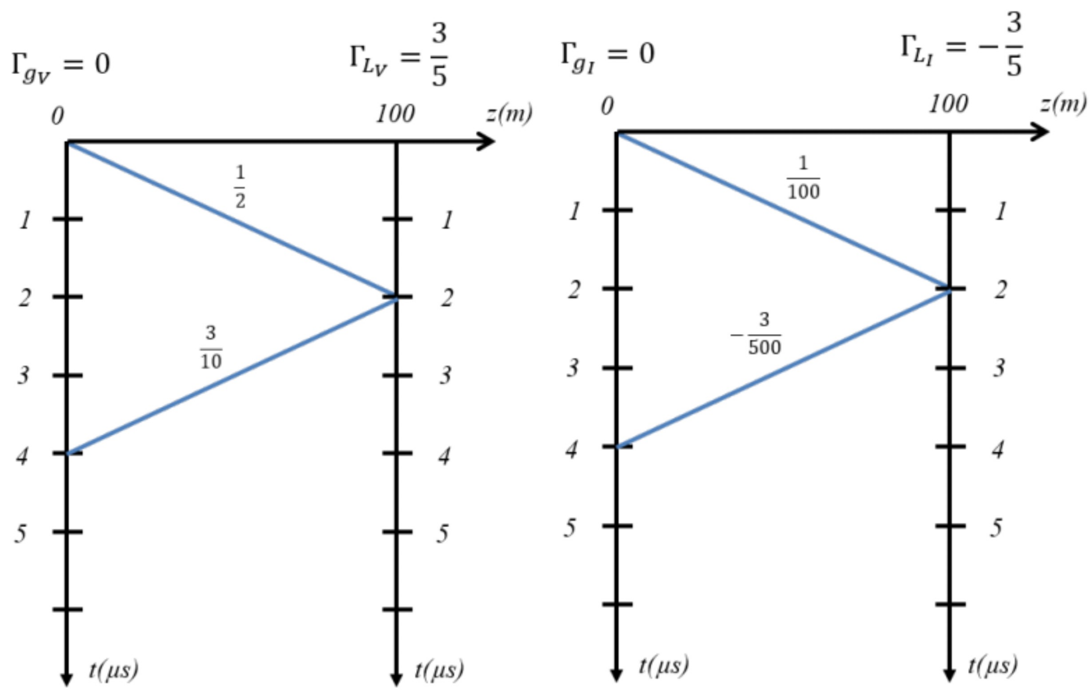
$$\Gamma_{g_V} = \frac{R_g - Z_o}{R_g + Z_o} = 0$$

The corresponding current reflection coefficients are

$$\Gamma_{L_I} = -\Gamma_{L_V} = -\frac{3}{5}, \text{ and}$$

$$\Gamma_{g_I} = -\Gamma_{g_V} = 0$$

Using these information, we can build the following bounce diagrams for the voltage $V(z,t)$ and current $I(z,t)$ on the transmission line.



Voltage (left) and current (right) bounce diagrams for voltage source $f(t) = \delta(t)$. The expressions of $V(z,t)$ and $I(z,t)$ are given by

$$V(z,t) = \frac{1}{2}\delta(t - \frac{6z}{c}) + \frac{3}{10}\delta(t + \frac{6z}{c} - 4\,\mu\text{s})\,[\text{V}]$$
$$I(z,t) = \frac{1}{100}\delta(t - \frac{6z}{c}) - \frac{3}{500}\delta(t + \frac{6z}{c} - 4\,\mu\text{s})\,[\text{A}].$$

(b) (2 points) Evaluating these expressions at $z = \frac{l}{2} = 50\,\text{m}$, we have

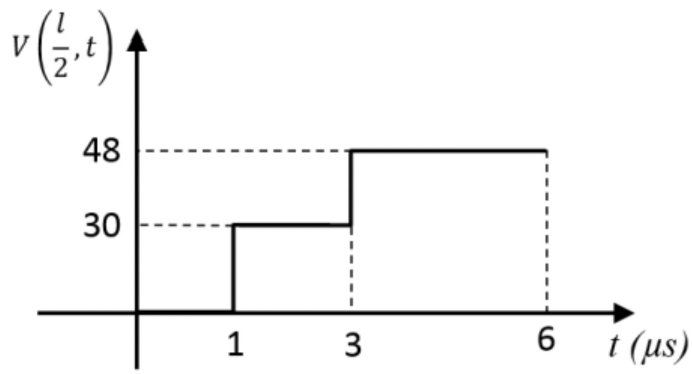
$$V(\frac{l}{2},t) = \frac{1}{2}\delta(t - 1\,\mu\text{s}) + \frac{3}{10}\delta(t - 3\,\mu\text{s})\,[\text{V}]$$
$$I(\frac{l}{2},t) = \frac{1}{100}\delta(t - 1\,\mu\text{s}) - \frac{3}{500}\delta(t - 3\,\mu\text{s})\,[\text{A}].$$

(c) (2 points)

$$V(z, 5\mu s) = 0\,[\text{V}]$$
$$I(z, 5\mu s) = 0\,[\text{A}].$$

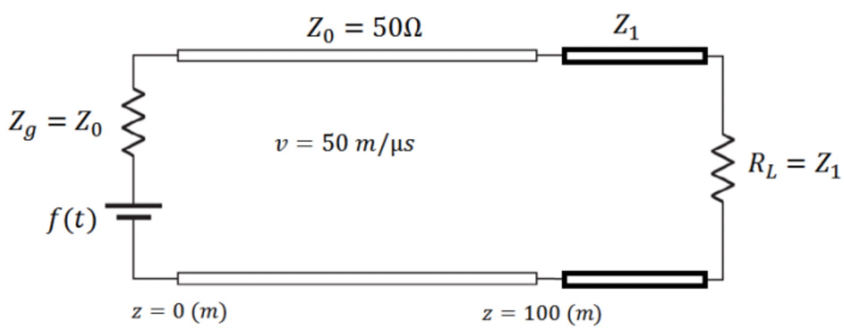
(d) (2 points) Given that the voltage source is $f(t) = 60u(t)$, then

$$\begin{aligned} V(\tfrac{l}{2},t) &= 60u(t) * \left(\tfrac{1}{2}\delta(t - 1\,\mu\text{s}) + \tfrac{3}{10}\delta(t - 3\,\mu\text{s})\right) \\ &= 30u(t - 1\,\mu\text{s}) + 18u(t - 3\,\mu\text{s})\,[\text{V}]. \end{aligned}$$



(e) (2 points) $V(\frac{l}{2}, t) = 48\,[\text{V}]$.

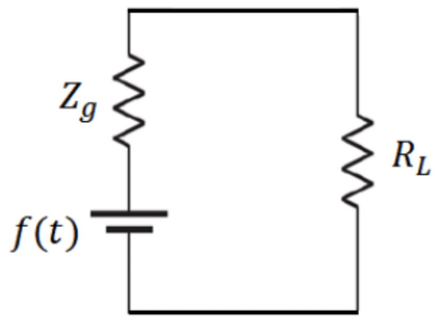
(f) (1 point) The circuit looks like one depicted in the following figure



The equivalent impedance seen at $z = 100\,\text{m}$ looking from the source is Z_1. If no reflection at $z = 100\,\text{m}$ is desired, the equivalent impedance must be as same as that of the transmission line, hence $Z_1 = Z_0 = 50\,\Omega$.

4. Equivalent circuit of the problem can be seen from that of the previous problem.

   (a) (2 points) At steady-state, the presence of the TL doesn't matter, it acts like a short. Hence $V(z=0,t=\infty) = V_L$ and $I(z=0,t=\infty) = I_L$. The circuit simplifies to



   Hence,

$$R_g = \frac{6\,\mathrm{V} - 3\,\mathrm{V}}{30\,\mathrm{mA}} = 0.1\,\mathrm{K}\Omega = 100\,\Omega.$$

   (b) (2 points) If the defect is a short, $R_L = 0$, hence $V(z=0,t=\infty) = 0\,[\mathrm{V}]$, $I(z=0,t=\infty) = \frac{6}{100} = 60\,[\mathrm{mA}]$

   (c) (2 points) If the defect is an open, $R_L \to \infty$, hence $V(z=0,t=\infty) = 6\,[\mathrm{V}]$, $I(z=0,t=\infty) = 0\,[\mathrm{mA}]$

   (d) (4 points) 2 V read from the input end of the line is due to the injection of incident wave ($V_s = 1\,\mathrm{V}$) from the source. Hence we get

$$\begin{aligned} V(z=0,t=0) &= \tau_g V_s(t=0) \\ 2 &= \frac{Z_o}{Z_g + Z_o} \times 3. \end{aligned}$$

   Thus, the impedance of the tested line is

$$Z_0 = 2Z_g = 200\,\Omega.$$

   The incident wave (2V) gets injected into the TL and starts to move down along the line for a length L then it hits the defect. The reflected wave comes back and reach the input end of the TL, superposes with the injected incident wave. That's why we see a change in response observed later. That means $t = 5\,\mu s$ is the "round-trip" time for the wave to travel from the input end of the TL to the defect and return back to the input end of the TL. L is then found as

$$L = v \times \frac{t}{2} = 2 \times 10^8\,\mathrm{m/s} \times \frac{5}{2}\,\mu\mathrm{s} = 500\,[\mathrm{m}].$$

   Also, the total voltage observed at the input end of TL increases after the reflected wave reaches this end. It means the reflected wave is positive (or in phase) with the incident wave. Hence, it must be an open-circuit type of defect.

5. (10 points) **Bonus problem:** The voltage and current waves $V(z,t)$ and $I(z,t)$ that propagate on a transmission line satisfy the following set of partial differential equations (PDE's)

$$-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t}, \quad (1)$$

$$-\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t}. \quad (2)$$

Given that $V(z,t) = 4\cos(\omega t + \beta z)$, we get

$$\frac{\partial I}{\partial t} = -\frac{1}{\mathcal{L}}\frac{\partial V}{\partial z} = 4\frac{\beta}{\mathcal{L}}\sin(\omega t + \beta z),$$

by utilizing (1). Then, after integrating it over time, we get

$$I = -4\frac{\beta}{\omega\mathcal{L}}\cos(\omega t + \beta z).$$

Inserting this result into (2), we get

$$\frac{\partial V}{\partial t} = -\frac{1}{\mathcal{C}}\frac{\partial I}{\partial z} = -4\frac{\beta^2}{\omega\mathcal{L}\mathcal{C}}\sin(\omega t + \beta z),$$

and integrating over time, we finally obtain

$$V = 4\frac{\beta^2}{\omega^2\mathcal{L}\mathcal{C}}\cos(\omega t + \beta z).$$

This result implies that

$$\frac{\beta^2}{\omega^2\mathcal{L}\mathcal{C}} = 1, \qquad \text{then} \qquad \beta = \omega\sqrt{\mathcal{L}\mathcal{C}}.$$

In addition, the expression for the current becomes

$$I = -4\sqrt{\frac{\mathcal{C}}{\mathcal{L}}}\cos(\omega t + \beta z).$$