1. (a) (3 points) For the wave described by $\mathbf{E}_1 = \cos(\omega t - \beta z + \frac{\pi}{4})\hat{x}\,\frac{\text{V}}{\text{m}}$,
   i. the corresponding phasors are given by

$$\tilde{\mathbf{E}}_1 = \hat{x}e^{-j\beta z}e^{j\frac{\pi}{4}}\,\frac{\text{V}}{\text{m}}$$

$$\tilde{\mathbf{H}}_1 = \hat{y}\frac{1}{\eta_o}e^{-j\beta z}e^{j\frac{\pi}{4}}\,\frac{\text{A}}{\text{m}}.$$

   ii. It is **linearly polarized** in $\hat{x}$-direction.

   (b) (3 points) For the wave described by $\mathbf{H}_2 = 2\cos(\omega t + \beta z + \frac{\pi}{3})\hat{x} + 2\sin(\omega t + \beta z + \frac{5\pi}{6})\hat{y}\,\frac{\text{A}}{\text{m}}$,
   i. the corresponding phasors are given by

$$\tilde{\mathbf{H}}_2 = \hat{x}2e^{j\beta z}e^{j\frac{\pi}{3}} + \hat{y}2e^{j\beta z}e^{j(-\frac{\pi}{2}+\frac{5\pi}{6})} = 2e^{j(\beta z+\frac{\pi}{3})}\left(\hat{x}+\hat{y}\right)\,\frac{\text{A}}{\text{m}}$$

$$\tilde{\mathbf{E}}_3 = 2\eta_o e^{j(\beta z+\frac{\pi}{3})}\left(-\hat{x}+\hat{y}\right)\,\frac{\text{V}}{\text{m}}.$$

   ii. Thus the wave is **linearly polarized** in $\frac{-\hat{x}+\hat{y}}{\sqrt{2}}$ direction.

   (c) (3 points) For the wave described by $\mathbf{E}_3 = 4\cos(\omega t - \beta x)\hat{z} - 3\sin(\omega t - \beta x)\hat{y}\,\frac{\text{V}}{\text{m}}$,
   i. the corresponding phasors are given by

$$\tilde{\mathbf{E}}_3 = \hat{z}4e^{-j\beta x} + \hat{y}3je^{-j\beta x} = e^{-j\beta x}\left(4\hat{z}+3j\hat{y}\right)\,\frac{\text{V}}{\text{m}}$$

$$\tilde{\mathbf{H}}_4 = \frac{1}{\eta_o}e^{-j\beta x}\left(-4\hat{y}+3j\hat{z}\right)\,\frac{\text{A}}{\text{m}}.$$

   ii. Since the two components have different magnitudes, the wave is **elliptical** polarized.

   (d) (3 points) For the wave described by $\mathbf{E}_4 = 5\cos(\omega t - \beta y)\hat{x} + 5\sin(\omega t - \beta y)\hat{z}\,\frac{\text{V}}{\text{m}}$,
   i. the corresponding phasors are given by

$$\tilde{\mathbf{E}}_4 = \hat{x}5e^{-j\beta y} - \hat{z}5je^{-j\beta y} = 5e^{-j\beta y}\left(\hat{x}-j\hat{z}\right)\,\frac{\text{V}}{\text{m}}$$

$$\tilde{\mathbf{H}}_4 = \frac{5}{\eta_o}e^{-j\beta y}\left(-\hat{z}-j\hat{x}\right)\,\frac{\text{A}}{\text{m}}.$$

   ii. Given that the wave propagates along $\hat{y}$ direction, it is seen that the wave is **left-hand-circularly** polarized.

   (e) (3 points) For the wave described by $\mathbf{H}_5 = 2\sin(\omega t + \beta y)\hat{x} - 2\sin(\omega t + \beta y - \frac{\pi}{4})\hat{z}\,\frac{\text{A}}{\text{m}}$,
   i. the corresponding phasors are given by

$$\tilde{\mathbf{H}}_5 = -\hat{x}2je^{j\beta y} + \hat{z}2je^{j\beta y - j\frac{\pi}{4}} = 2je^{j\beta y}\left(-\hat{x}+e^{-j\frac{\pi}{4}}\hat{z}\right)\,\frac{\text{A}}{\text{m}}$$

$$\tilde{\mathbf{E}}_5 = 2\eta_o je^{j\beta y}\left(\hat{z}+e^{-j\frac{\pi}{4}}\hat{x}\right)\,\frac{\text{V}}{\text{m}}$$

   ii. Since the phase angle between $\hat{x}$ and $\hat{z}$ components of $\tilde{\mathbf{E}}_5$ is $\frac{\pi}{4}$, not an integer multiple of $\frac{\pi}{2}$, it is seen that the wave is **elliptical** polarized.

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2. (5 points) From Faraday's Law,

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{H}$$

$$\nabla \times \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 e^{-\gamma z} & 0 & 0 \end{vmatrix} = \hat{y}\,\frac{\partial E_0 e^{-\gamma z}}{\partial z} = -\hat{y}\,\gamma E_0 e^{-\gamma z} = -j\omega\mu\tilde{H}$$

So we have

$$\tilde{H} = \hat{y}\frac{\gamma}{j\omega\mu}E_0 e^{-\gamma z}$$

If we still want $|\tilde{\mathbf{E}}| = |\eta||\tilde{H}|$,then

$$|\eta| = |\frac{j\omega\mu}{\gamma}|$$

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3. (a) (3 points) We have

$$\gamma^2 = j\omega\mu(j\omega\epsilon + \sigma) = (j\omega\mu \times j\omega\epsilon) \times (1 + \frac{\sigma}{j\omega\epsilon}) = -\omega^2\mu\epsilon(1 - j\frac{\sigma}{\omega\epsilon})$$

if $\frac{\sigma}{\omega\epsilon} \gg 1$, then $\gamma^2 \approx -\omega^2\mu\epsilon \times -j\frac{\sigma}{\omega\epsilon} = j\omega\mu\sigma$. On the other hand, we have

$$\gamma^2 = (\alpha + j\beta)^2 = \alpha^2 - \beta^2 + 2j\alpha\beta$$

Comparing the above two equations, we have

$$\alpha^2 = \beta^2,\ \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

(b) (3 points) For $\sigma = 0.1\,\mathrm{S},\ f = 20\,\mathrm{MHz},$ we have $\frac{\sigma}{\omega\epsilon} = 89.91 \gg 1$, so we have

$$\beta = \sqrt{\pi f\mu\sigma} = 2.81\,\mathrm{m}^{-1}$$

$$\lambda = \frac{2\pi}{\beta} = 2.23\,\mathrm{m}$$

$$v_p = \frac{\lambda}{T} = 4.46 \times 10^7\,\frac{\mathrm{m}}{\mathrm{s}}$$

(c) (1 point) η can be find by

$$|\eta| = \sqrt{\frac{\omega\mu}{\sigma}} = 39.79\,\Omega$$

$$\eta = 39.79e^{j\frac{\pi}{4}}\,\Omega$$

(d) (2 points) We know that $\alpha = \beta = 2.81\frac{1}{m}$, we have

$$\tilde{H} = \frac{1}{2}\hat{z} \times \tilde{J}e^{-\gamma z} = -\hat{y}\frac{1}{2}e^{-2.81z}e^{-j2.81z+j\pi}\,\frac{\mathrm{A}}{\mathrm{m}}$$

$$\tilde{E} = \eta\tilde{H} \times \hat{z} = -\hat{x}\frac{39.79}{2}e^{-2.81z}e^{-j2.81z+j\frac{5\pi}{4}}\,\frac{\mathrm{V}}{\mathrm{m}}$$

(e) (1 point) The location is given by

$$z = \delta = \frac{1}{\alpha} = 0.36\,\mathrm{m}$$

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4.

$$\tilde{E} = Ae^{-j\beta x}\hat{y} + Be^{-j\beta x}e^{-j\frac{\pi}{2}}\hat{z}$$

$$\tilde{H} = \frac{1}{\eta_o}(Ae^{-j\beta x}\hat{y} + Be^{-j\beta x}e^{-j\frac{\pi}{2}}\hat{z})$$

$$< \mathbf{S} >= \frac{1}{2}Re(\tilde{\mathbf{E}} \times \tilde{H}*)$$

(a) (3 points) The time-averaged Poynting vector:

$$< \mathbf{S} >= \frac{1}{2\eta_0}[A^2 + B^2]\hat{x}$$

If $A = B = 1/\sqrt{2}\, V/m$,

$$< \mathbf{S} >= \frac{1}{2\eta_0}\hat{x}\,[\mathrm{W/m^2}]$$

(b) (1 point) For $A = 1$ and $B = 0$,

$$< \mathbf{S} >= \frac{1}{2\eta_0}\hat{x}\,[\mathrm{W/m^2}]$$

(c) (1 point) The total time-averaged poynting vector is the sum of the two field components' time-averaged poynting vectors. So for linearly and circularly polarized TEM waves having equal instantaneous peak electric field magnitudes, the ratio of $< S >$ would be 1:2.

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5. (10 points) When a wave is incident on a boundary between two different media, a reflected wave is produced. In addition, if the second medium is not a perfect conductor, a transmitted wave is set up. Together, these waves satisfy the boundary conditions at the interface of the two media. We shall assume that a (+) wave is incident from medium 1 ($z < 0$) onto the interface, thereby setting up a reflected ($-$) wave in that medium, and a transmitted wave in medium 2 ($z > 0$). Then we can write the solution for the complex field components in medium 1 to be

$$\tilde{\mathbf{E}}_{1x} = E_1^+ e^{-j\beta_1 z}\hat{x} + E_1^- e^{j\beta_1 z}\hat{x}$$

$$\begin{aligned}\tilde{\mathbf{H}}_{1y} &= H_1^+ e^{-j\beta_1 z}\hat{y} + H_1^- e^{j\beta_1 z}\hat{y} \\ &= \frac{1}{\eta_1}(E_1^+ e^{-j\beta_1 z} - E_1^- e^{j\beta_1 z})\hat{y}\end{aligned}$$

where $\beta_1 = \frac{\omega}{v_1} = \omega\sqrt{\mu_1\epsilon_1}$ and $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$. The complex field components in medium 2 are given by

$$\tilde{\mathbf{E}}_{2x} = E_2^+ e^{-j\beta_2 z}\hat{x}$$

$$\begin{aligned}\tilde{\mathbf{H}}_{2y} &= H_2^+ e^{-j\beta_2 z}\hat{y} \\ &= \frac{E_2^+ e^{-j\beta_2 z}}{\eta_2}\hat{y}\end{aligned}$$

and $\beta_2 = \frac{\omega}{v_2} = \omega\sqrt{\mu_2\epsilon_2}$ and $\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$.

To satisfy the boundary conditions at $z = 0$, we note that both electric and magnetic fields are tangential to the surface and no current exists on the surface. Hence, we have

$$\tilde{\mathbf{E}}_{1x}(z = 0) = \tilde{\mathbf{E}}_{2x}(z = 0)$$

$$\tilde{\mathbf{H}}_{1y}(z = 0) = \tilde{\mathbf{H}}_{2y}(z = 0)$$

Applying these to the solution pairs,

$$E_1^+ + E_1^- = E_2^+$$

$$\frac{1}{\eta_1}(E_1^+ - E_1^-) = \frac{1}{\eta_2}E_2^+$$

The E component of the incident wave is given by $E(z,t) = A_1\cos(\omega t - \beta_1 z)\hat{x}$, therefore, $E_1^+ = A_1$. Solving for the equations above, the phasors of the reflected TEM wave at the interface are shown as

$$\tilde{\mathbf{E}}_1^- = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}A_1 e^{j\beta_1 z}\hat{x}$$

$$\tilde{\mathbf{H}}_1^- = -\frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}\frac{A_1}{\eta_1}e^{j\beta_1 z}\hat{y}$$

Retrieving the time-dependent forms of the reflected wave, we obtain

$$E_1^-(z,t) = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}A_1\cos(\omega t + \beta_1 z)\hat{x} = \frac{\sqrt{\frac{\mu_2}{\epsilon_2}} - \sqrt{\frac{\mu_1}{\epsilon_1}}}{\sqrt{\frac{\mu_1}{\epsilon_1}} + \sqrt{\frac{\mu_2}{\epsilon_2}}}A_1\cos(\omega t + \omega\sqrt{\mu_1\epsilon_1}z)\hat{x}$$

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$$H_1^-(z,t) = -\frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}\frac{A_1}{\eta_1}\cos(\omega t + \beta_1 z)\hat{y} = -\frac{\sqrt{\frac{\mu_2}{\epsilon_2}} - \sqrt{\frac{\mu_1}{\epsilon_1}}}{\sqrt{\frac{\mu_1}{\epsilon_1}} + \sqrt{\frac{\mu_2}{\epsilon_2}}} A_1 \sqrt{\frac{\epsilon_1}{\mu_1}}\cos(\omega t + \omega\sqrt{\mu_1\epsilon_1}z)\hat{y}.$$

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6. For a wave propagating in a vacuum in $x < 0$, we are given an incident electric field phasor

$$\tilde{\mathbf{E}}_i = (j\hat{z} - \hat{y})e^{-j2\pi x}\,\frac{\text{V}}{\text{m}}.$$

The wave encounters a boundary at $x = 0$ with $\mu = \mu_o$ and the reflected phasor is given by

$$\tilde{\mathbf{E}}_r = -\frac{1}{2}(j\hat{z} - \hat{y})e^{j2\pi x}\,\frac{\text{V}}{\text{m}}.$$

(a) (2 points) The incident wave is RHCP because $\hat{y}$ leads $\hat{z}$ and the wave is propagating in the $\hat{x}$ direction. The reflected wave is therefore LHCP.

(b) (2 points) The frequency can be calculated from $f = \frac{v_p\beta}{2\pi}$, where $\beta = 2\pi$ and $v_p = c$ in a vacuum. Thus, $f = 300\,\text{MHz}$.

(c) (2 points) The permittivity of the dielectric can be calculated using the reflection coefficient, Γ.

$$\begin{aligned}\Gamma &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -\frac{1}{2}\\ \eta &= \frac{1}{3}\eta_0\\ \epsilon_r &= 9\\ \epsilon &= 9\epsilon_o\end{aligned}$$

(d) (2 points) The transmitted electric phasor can be derived from the incident electric phasor with updated β and τ. So, $1 + \Gamma = \tau = \frac{1}{2}$ and $\beta = \frac{2\pi f}{v_p} = \frac{6\pi\times10^8}{3\times10^8\sqrt{9}} = 6\pi$ and therefore $\tilde{\mathbf{E}}_t = \frac{1}{2}(j\hat{z} - \hat{y})e^{-j6\pi x}\,\frac{\text{V}}{\text{m}}$.

(e) (2 points) The ratio of time-averaged incident power to time-averaged transmitted power can be found using the conservation of energy: $\frac{\eta_0}{\eta}\tau^2 = 1 - \Gamma^2 = 75\,\%$. The proof is presented below.

The magnetic fields in both the regions are given as,

$$\begin{aligned}\tilde{\mathbf{H}}_i &= \tfrac{1}{\eta_o}(-j\hat{y} - \hat{z})e^{-j2\pi x}\\ \tilde{\mathbf{H}}_r &= \tfrac{\Gamma}{\eta_o}(-j\hat{y} + \hat{z})e^{j2\pi x}\\ \tilde{\mathbf{H}}_t &= \tfrac{\tau}{\eta}(-j\hat{y} - \hat{z})e^{-j6\pi x}\end{aligned}$$

The corresponding time averaged poynting vectors ,

$$\begin{aligned}\langle\mathbf{S}_i\rangle &= \tfrac{1}{2}\text{Re}\left\{\tilde{\mathbf{E}}_i \times \tilde{\mathbf{H}}_i{}^*\right\} = \tfrac{1}{\eta_o}\hat{x}\\ \langle\mathbf{S}_r\rangle &= \tfrac{1}{2}\text{Re}\left\{\tilde{\mathbf{E}}_r \times \tilde{\mathbf{H}}_r{}^*\right\} = \tfrac{\Gamma^2}{\eta_o}\hat{x}\\ \langle\mathbf{S}_t\rangle &= \tfrac{1}{2}\text{Re}\left\{\tilde{\mathbf{E}}_t \times \tilde{\mathbf{H}}_t{}^*\right\} = \tfrac{\tau^2}{\eta}\hat{x}\end{aligned}$$

It can be shown that the power density calculated for the incident, reflected, and transported waves will satisfy

$$|\langle\mathbf{S}_i\rangle| = |\langle\mathbf{S}_r\rangle| + |\langle\mathbf{S}_t\rangle|\,,$$

which shows that the calculations are in compliance with energy conservation principle. Also, note that

$$\begin{aligned}\tfrac{1}{\eta_o} &= \tfrac{\Gamma^2}{\eta_o} + \tfrac{\tau^2}{\eta}\\ \tfrac{\eta_0}{\eta}\tau^2 &= 1 - \Gamma^2\end{aligned}$$

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7. (a) (5 points) Assuming the conductor lies along z axis, $E_z = B_z = 0$ for TEM waves. Since charge density for a perfect conductor is non-zero only at the surface, from Gauss's Law,

$$\nabla \cdot \mathbf{E} = 0$$

Expanding the divergence we obtain:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \tag{1}$$

Additionally, the curl of the electric field is given by,

$$\nabla \times \mathbf{E} = -\hat{x}\,\frac{\partial E_y}{\partial z} + \hat{y}\,\frac{\partial E_x}{\partial z} + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\mu\left(\hat{x}\,\frac{\partial H_x}{\partial t} + \hat{y}\,\frac{\partial H_y}{\partial t}\right)$$

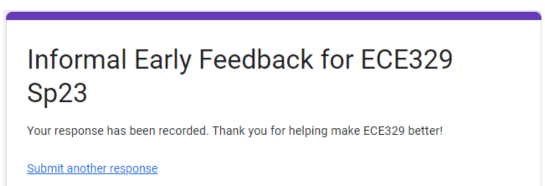
The z component gives:

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \tag{2}$$

Therefore, the solution for the electric field from (1) and (2) is zero after applying boundary conditions. Hence, a TEM wave cannot propagate through a hollow longitudinally symmetric metallic tube.

(b) (5 points) An ideal coax has no external field and the outer conductor acts as a shield against external electromagnetic waves, so it suffers less intereference problems than a pair of copper wires. Additionally, any change in shape or distance between the copper wires changes the impedance of the transmition line and cause losses due to reflection. A coax cable is specifically designed to keep spacing uniform to mitigate these issues.

(c) (5 points) Please refer to the following picture.



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