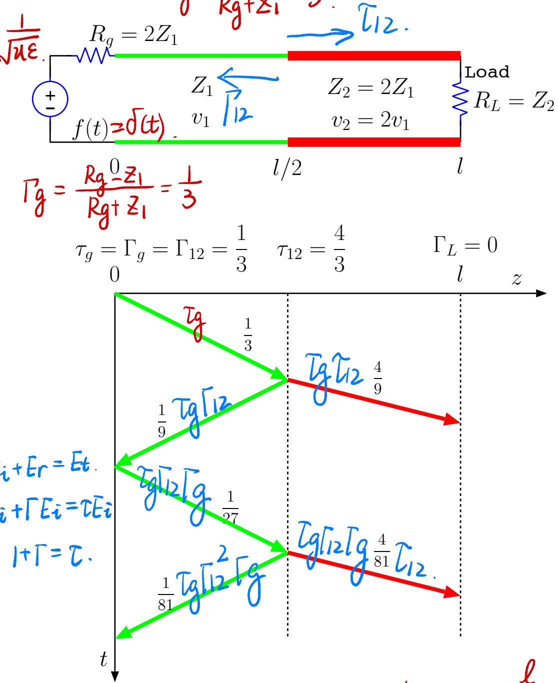
30 Multi-line circuits



$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{GF}\sqrt{\frac{\mu}{\epsilon}}$

$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}}$

$\tau_g = \frac{Z_1}{R_g + Z_1} = \frac{1}{3}$

$\Gamma_g = \frac{R_g - Z_1}{R_g + Z_1} = \frac{1}{3}$

- In this lecture we will extend the bounce diagram technique to solve distributed circuit problems involving multiple transmission lines.

- One example of such a circuit is shown in the margin where two distinct TL's of equal lengths have been joined directly at a distance $\frac{l}{2}$ away from the generator.

  1 | 2

  - The impulse response of the system can be found by first constructing the bounce diagram for the TL system as shown in the margin.
  - In this bounce diagram, $z = \frac{l}{2}$ happens to be the location of additional reflections as well as transmissions because of the sudden change of Z_o from Z_1 to $Z_2 = 2Z_2$.

These reflections and transmissions between line j and k — transmission from j to k, and reflection from k back to j — can be computed with **reflection coefficient**

$$\Gamma_{jk} = \frac{Z_k - Z_j}{Z_k + Z_j}$$

and **transmission coefficient**

$$\tau_{jk} = 1 + \Gamma_{jk}$$

that ensure the voltage and current continuity at the junction

$V_1(z,t) = V^+ + V^-$ on line 1 $\quad t = \frac{l}{2v}$

$V_2(z,t) = V^{++}$ on line 2

$I_1(z,t) = \frac{V^+}{Z_1} - \frac{V^-}{Z_1}$ on line 1

$I_2(z,t) = \frac{V^{++}}{Z_2}$

$\Gamma_{12}V^+$ $\quad \tau_{12}V^+$

$$\begin{cases} V^+ + V^- = V^{++} \\ \frac{V^+}{Z_1} - \frac{V^-}{Z_1} = \frac{V^{++}}{Z_2} \end{cases}$$

$$\begin{cases} V^+(1 + \Gamma_{12}) = \tau_{12}V^+ \\ \frac{V^+}{Z_1}(1 - \Gamma_{12}) = \frac{V^+}{Z_2}\tau_{12} \end{cases}$$

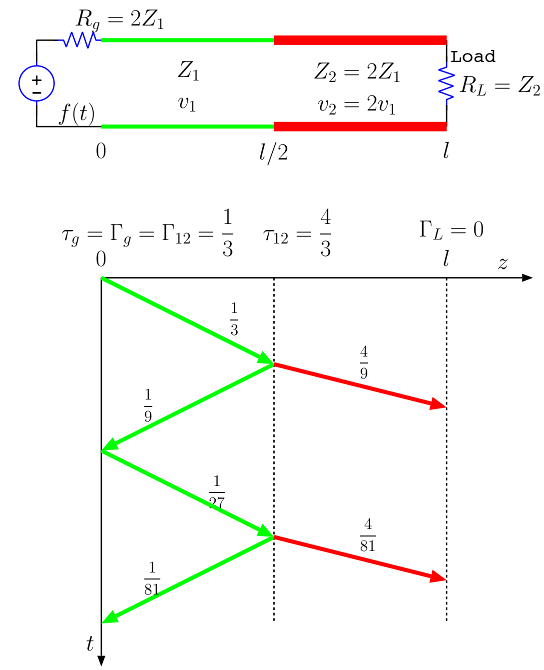
$\frac{2Z_1 - Z_1}{2Z_1 + Z_1} = \frac{1}{3}$

$1 + \Gamma_{12} = \tau_{12}$

$1 - \Gamma_{12} = \frac{Z_1}{Z_2}\tau_{12}$

$$\begin{cases} \Gamma_{12} = \frac{Z_2 - Z_1}{Z_2 + Z_2} \\ \tau_{12} = 1 + \Gamma_{12} \end{cases}$$

- Z_j is the characteristic impedance of the line of the incident pulse, while
- Z_k is the impedance of the cascaded line into which the transmitted pulse is injected.



Verification:

- Let
$$V_j^+(1+\Gamma_{jk}) \quad \text{and} \quad V_j^+(1-\Gamma_{jk})/Z_j$$
denote the total voltage and current on line Z_j expressed in terms of incident voltage wave V^+ , and
- let
$$V_j^+\tau_{jk} \quad \text{and} \quad V_j^+\tau_{jk}/Z_k$$
the voltage and current on line Z_k.

This notation identifies Γ_{jk} and τ_{jk} as reflection and transmission coefficients at the junction.

- Taking
$$V_j^+(1+\Gamma_{jk}) = V_j^+\tau_{jk}$$
and
$$V_j^+(1-\Gamma_{jk})/Z_j = V_j^+\tau_{jk}/Z_k$$
in order to enforce voltage and current continuity, we can solve for Γ_{jk} and τ_{jk} given above.

Example 1: In the circuit shown in the margin with two TL segments, line 2 has twice the characteristic impedance and propagation velocity of line 1, i.e.,

$$Z_2 = 2Z_1 \quad \text{and} \quad v_2 = 2v_1.$$

Determine $\mathcal{L}_2$ and $\mathcal{C}_2$ in terms of $\mathcal{L}_1$ and $\mathcal{C}_1$.

Solution: We have

$Z_0 \equiv \sqrt{\frac{\mathcal{L}}{C}}$.

$$Z_2 = 2Z_1 \quad \Rightarrow \quad \frac{\mathcal{L}_2}{\mathcal{C}_2} = 4\frac{\mathcal{L}_1}{\mathcal{C}_1}$$

and

$v = \frac{1}{\sqrt{\mathcal{L}C}}$

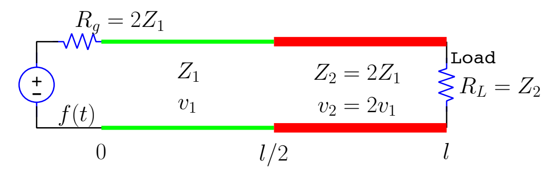
$$v_2 = 2v_1 \quad \Rightarrow \quad \frac{1}{\mathcal{L}_2\mathcal{C}_2} = 4\frac{1}{\mathcal{L}_1\mathcal{C}_1}.$$

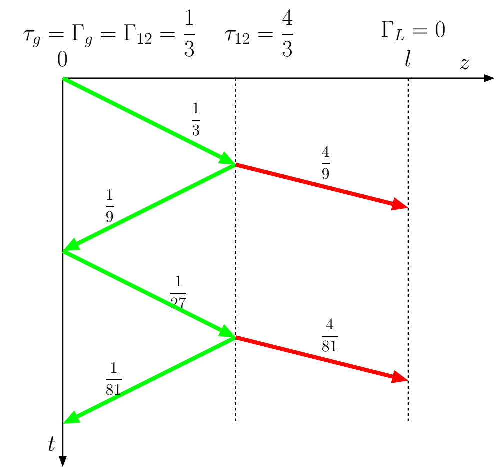
The product of the two equations gives

$$\frac{1}{\mathcal{C}_2^2} = 16\frac{1}{\mathcal{C}_1^2} \quad \Rightarrow \quad \mathcal{C}_2 = \frac{1}{4}\mathcal{C}_1,$$

while their ratio leads to

$$\mathcal{L}_2 = \mathcal{L}_1.$$





Example 2: In the circuit of Example 1, determine $V(z,t)$ and $I(z,t)$ if

$$f(t) = \sin(2\pi t)u(t), \ \ t \text{ in } \mu s,$$

and $l = 2400$ m, $v_1 = 150\,\text{m}/\mu\text{s}$, and $Z_1 = 25\,\Omega$.

Solution: From the bounce diagram we infer the following impulse-response for the voltage variable:

$$V(z,t) = \frac{1}{3}\sum_{n=0}^{\infty}(\frac{1}{3})^{2n}[\delta(t - \frac{z}{v_1} - n\frac{l}{v_1}) + \frac{1}{3}\delta(t + \frac{z}{v_1} - (n+1)\frac{l}{v_1})]$$

$\Gamma_g\Gamma_{12}$ Γ_{12}

for $z < \frac{l}{2}$, and

$$V(z,t) = \frac{1}{3}\sum_{n=0}^{\infty}(\underline{\frac{1}{3}})^{2n}\frac{4}{3}\delta(t - \frac{z}{v_2} - (4n+\cancel{1}2)\frac{l/2}{v_2})$$

τ_g $\Gamma_g\Gamma_{12}$ τ_{12}

for $\frac{l}{2} < z < l$. The impulse response for the current is

$$I(z,t) = \frac{1}{3Z_1}\sum_{n=0}^{\infty}(\frac{1}{3})^{2n}[\delta(t - \frac{z}{v_1} - n\frac{l}{v_1}) - \frac{1}{3}\delta(t + \frac{z}{v_1} - (n+1)\frac{l}{v_1})]$$

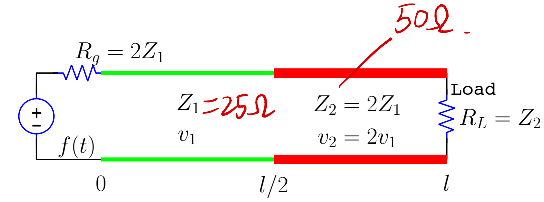
for $z < \frac{l}{2}$, and

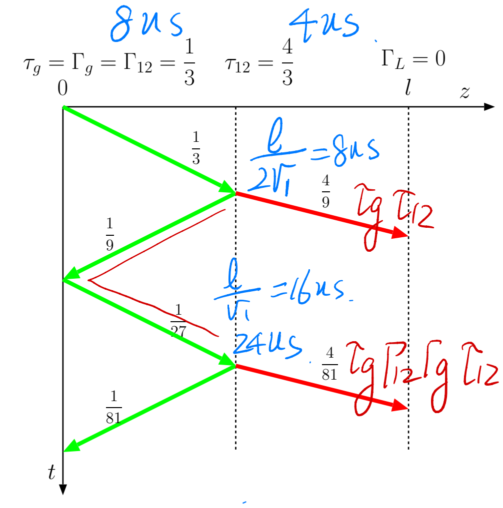
$$I(z,t) = \frac{1}{3Z_2}\sum_{n=0}^{\infty}(\frac{1}{3})^{2n}\frac{4}{3}\delta(t - \frac{z}{v_2} - (4n+1)\frac{l/2}{v_2})$$

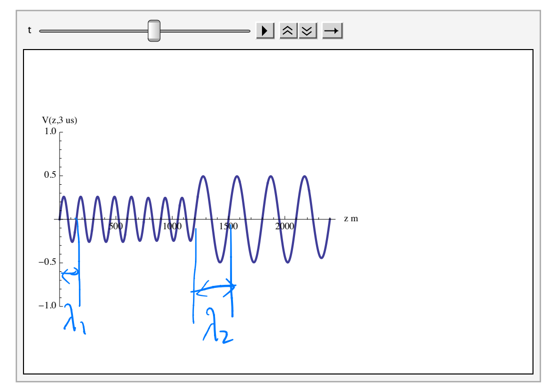
for $\frac{l}{2} < z < l$. Using

$$\frac{l}{v_1} = \frac{2400}{150} = 16\,\mu\text{s}$$

and replacing $\delta(t)$ with $f(t) = \sin(2\pi t)u(t)$ the plot depicted in the margin was obtained.







$\lambda = \frac{2\pi}{\beta}$ $\beta = \frac{\omega}{v}$

$= \frac{2\pi}{\omega}v$

Example 3: Two TL's with characteristic impedances Z_1 and Z_2 are joined at a junction that also includes a "shunt" resistance R as shown in the diagram in the margin. Determine the reflection coefficient Γ_{12} and transmission coefficient τ_{12} at the junction.

Solution: Consider a voltage wave

$$V^+(t - \frac{z}{v_1})$$

coming from the left producing reflected and transmitted waves

$$V^-(t + \frac{z}{v_1}) \text{ and } V^{++}(t - \frac{z}{v_2})$$

on lines 1 and 2 traveling to the left and right, respectively, on two sides of the junction. Using an abbreviated notation, KVL and KCL applied at the junction can be expressed as

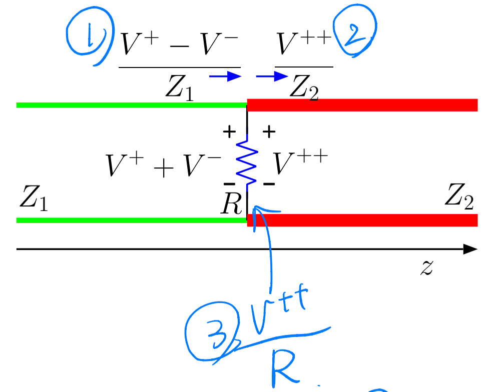
$$V^+ + V^- = V^{++} \text{ and } \frac{V^+}{Z_1} - \frac{V^-}{Z_1} = \frac{V^{++}}{R} + \frac{V^{++}}{Z_2},$$

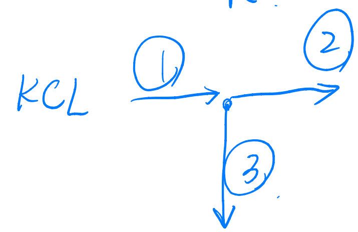
where in the KCL equation the first term is the coming down the resistor R, and the second term is the TL current on line 2 (as marked in the circuit diagrams in the margin). The equations can be rearranged as

$$\begin{aligned} V^+ + V^- &= V^{++} \\ V^+ - V^- &= \frac{Z_1}{Z_{eq}} V^{++}, \end{aligned}$$

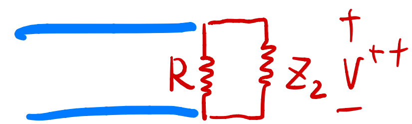
where

$$Z_{eq} \equiv \frac{RZ_2}{R + Z_2}$$

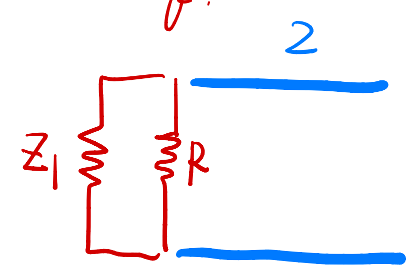




① = ② + ③



$$\frac{V^{++}}{\underbrace{R // Z_2}_{Z_{eq}}}$$



$Z_{eq2} = R // Z_1$

is the parallel combination of R and Z_2. Solving these equations, we find that

$$\Gamma_{12} \equiv \frac{V^-}{V^+} = \frac{Z_{eq} - Z_1}{Z_{eq} + Z_1}$$

and

$$\tau_{12} = \frac{V^{++}}{V^+} = \frac{2Z_{eq}}{Z_{eq} + Z_1}. = 1 + \Gamma_{12}$$

By, symmetry, the coefficients

$$\Gamma_{21} = \frac{Z_{eq} - Z_2}{Z_{eq} + Z_2}$$

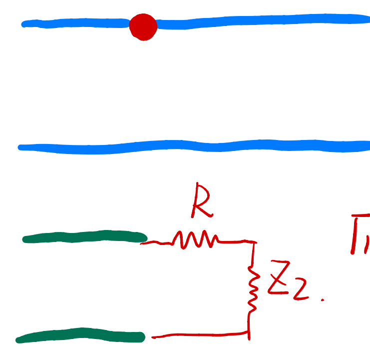
and

$$\tau_{21} = \frac{2Z_{eq}}{Z_{eq} + Z_2} = 1 + \Gamma_{21}.$$

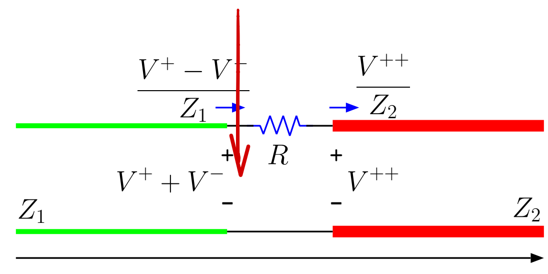
would describe reflection and transmission when a wave is incident from right provided that

$$Z_{eq} \equiv \frac{RZ_1}{R + Z_1}$$

is used.



$$\Gamma_{12} = \frac{Z_{eq} - Z_1}{Z_{eq} + Z_1}$$



$$\begin{cases} V^+ + V^- = \dfrac{V^{++}}{Z_2} \cdot R + V^{++} \\ \dfrac{V^+ - V^-}{Z_1} = \dfrac{V^{++}}{Z_2}. \end{cases}$$

Exercise: Two TL's with characteristic impedances Z_1 and Z_2 are joined at a junction that also includes a series resistance R as shown in the margin. Determine the reflection coefficient Γ_{12} and transmission coefficient τ_{12} at the junction.

Hint: in this ckt Γ_{12} has the usual form in terms of $Z_{eq} \equiv R + Z_2$. For τ_{12} we need $1 + \Gamma_{12}$ multiplied by a voltage division factor $Z_2/(R + Z_2)$.

$$\tau_{12} = (1 + \Gamma_{12}) \frac{Z_2}{R + Z_2}.$$