28 Distributed circuits and bounce diagrams

$$\eta_o = \sqrt{\frac{u_o}{\varepsilon_o}} \approx 120\pi \approx 377\Omega.$$

Last lecture we learned that voltage and current variations on TL's are governed by telegrapher's equations and their d'Alembert solutions — the latter can be expressed as

$$V(z,t) = f(t - \frac{z}{v}) + g(t + \frac{z}{v})$$

$Z_o = 50\Omega.$ $75\Omega.$

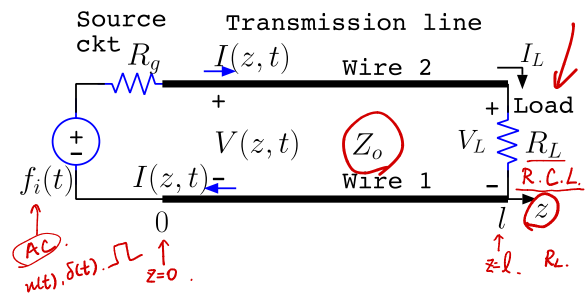
and

$$I(z,t) = \frac{f(t-\frac{z}{v})}{Z_o} - \frac{g(t+\frac{z}{v})}{Z_o}$$

$I \leftarrow H.$ $I = \frac{f}{50} - \frac{g}{50}.$

in terms of

$$v = \frac{1}{\sqrt{\mathcal{LC}}} \quad \left(\frac{1}{\sqrt{\mu\varepsilon}}\right) \quad \text{and} \quad Z_o = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \frac{1}{GF}\sqrt{\frac{u}{\varepsilon}}.$$



and functions $f(t)$ and $g(t)$ corresponding to signal waveforms propagated in $+z$ and $-z$ directions, respectively.

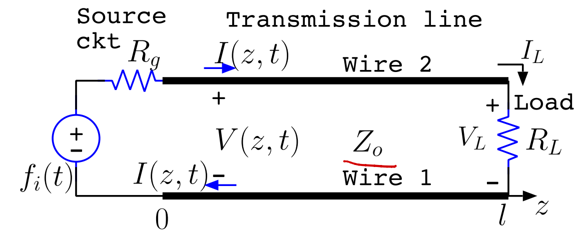
- In this lecture we will learn how to solve **distributed circuit problems** containing TL segments *and* two terminal elements such as resistors and voltage (or current) sources. In solving the problems, we will apply the usual rules of lumped circuit analysis at element terminals and treat the TL's in terms of d'Alembert solutions above.

$R_L = Z_o.$

$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}.$$

$= 0.$

- Consider a TL with a characteristic impedance Z_o extending from $z = 0$ to $z = l$, where a two-terminal *source* circuit (e.g., a receiving antenna) modeled by a Thevenin equivalent with voltage $f_i(t)$ and resistance R_g is connected between the TL terminals at $z = 0$ and a *load* (e.g., a receiver circuit) modeled by a resistance R_L terminates the line at $z = l$ (see margin).



$\frac{V_L}{I_L} = R_L = Z_o.$

  - We want to determine voltage and current signals $V(z,t)$ and $I(z,t)$ on the TL and the load in terms of source signal $f_i(t)$.

- Let us first consider the case when

$V(z,t)\big|_{z=\ell}$.

$$\frac{V(\ell,t)}{I(\ell,t)} = \frac{V_L}{I_L} = R_L = Z_o.$$

In that case

$$R_L = \frac{V(\ell,t)}{I(\ell,t)} = \frac{f(t-\frac{\ell}{v}) + g(t+\frac{\ell}{v})}{\frac{f(t-\frac{\ell}{v})}{Z_o} - \frac{g(t+\frac{\ell}{v})}{Z_o}} = Z_o \frac{f(t-\frac{\ell}{v}) + \cancel{g(t+\frac{\ell}{v})}}{f(t-\frac{\ell}{v}) - \cancel{g(t+\frac{\ell}{v})}} = Z_o = \frac{V_L}{I_L},$$

which is only possible if $g(t+\frac{\ell}{v}) = 0$ for all t. +1.

  - Hence

$$V(z,t) = f(t - \frac{z}{v}) \text{ and } I(z,t) = \frac{1}{Z_o} f(t - \frac{z}{v})$$

in this simplified circuit with $R_L = Z_o$.

$\checkmark$ $V(z,t) = f(t - \frac{z}{v})$. $\rightarrow$ $f_i(t)$.

$z = 0$.

– Now, for $z = 0$,

$$V(0,t) = f(t) \;\text{ and }\; I(0,t) = \frac{1}{Z_o} f(t)$$

and since

$$I(0,t) = \frac{f_i(t) - V(0,t)}{R_g},$$

$\checkmark$ $I(z,t) = \dfrac{f(t-\frac{z}{v})}{Z_o}$.

it follows that

$$f(t) = \boxed{\frac{Z_o}{R_g + Z_o}} f_i(t).$$

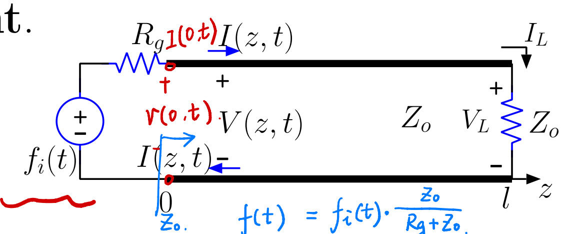
matched case

$R_L = Z_o$.

$g = 0$.

– We recognize this result as "voltage division" of the source voltage $f_i(t)$ across the transmission line terminals having an effective "input resistance" of Z_o. We will also write this result as

$$f(t) = \tau_g f_i(t) \;\text{ with }\; \tau_g = \frac{Z_o}{R_g + Z_o}$$ called an **injection coefficient**.



- In summary, in the circuit examined above having a "matched load" (see margin)

$$V(z,t) = \tau_g f_i(t - \frac{z}{v}) \;\text{ and }\; I(z,t) = \frac{\tau_g}{Z_o} f_i(t - \frac{z}{v}).$$

– These apply with any input function $f_i(t)$.

$Z = 0$.

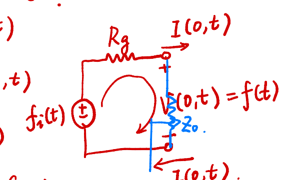
KVL

$$f_i(t) = R_g \underbrace{I(0,t)}_{\frac{f(t)}{Z_o}} + \underbrace{v(0,t)}_{f(t)}$$

$$= \frac{R_g}{Z_o} f(t) + f(t)$$

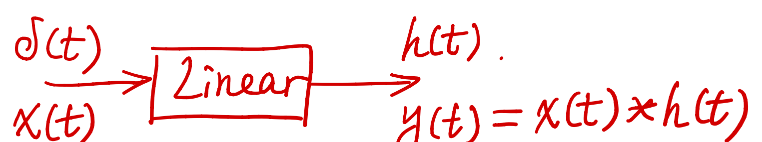
$$f(t) = \frac{Z_o}{R_g + Z_o} f_i(t).$$

voltage division.



- It is useful to consider the case $f_i(t) = \delta(t)$, and refer to the corresponding voltage solution

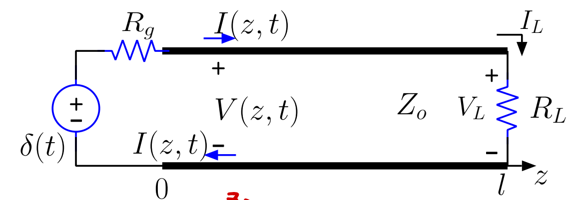
$$V(z,t) = \tau_g \delta(t - \frac{z}{v}) \equiv h_z(t)$$



as an **impulse response**, extending an important concept we learned about in ECE 210.

– Knowing an impulse response $h_z(t)$ is useful since convolving it with any $f_i(t)$ gives us the system response to input $f_i(t)$.

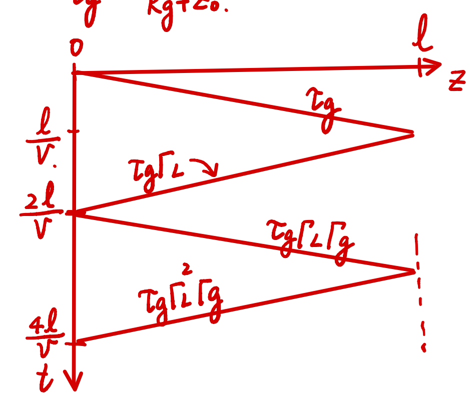
Clearly, we need the ckt impulse response for an *arbitrary* R_L.



$\tau_g = \frac{Z_o}{R_g + Z_o}.$

$R_L \neq Z_o.$ Γ_L

$R_g \neq Z_o$ Γ_g



- In our circuit with an arbitrary R_L and an impulse input (see margin), our earlier solutions

$$V(z,t) = \tau_g \delta(t - \frac{z}{v}) \text{ and } I(z,t) = \frac{\tau_g}{Z_o}\delta(t - \frac{z}{v})$$

will be inadequate to satisfy the constraint

$$\frac{V(\ell,t)}{I(\ell,t)} = R_L = \frac{V_L}{I_L}$$

at all times t. However, it can be easily verified that d'Alembert series solutions of the form

$$V(z,t) = \tau_g[\delta(t - \frac{z}{v}) + \Gamma_L \delta(t + \frac{z}{v} - \frac{2\ell}{v}) + \cdots]$$

and

$$I(z,t) = \frac{\tau_g}{Z_o}[\delta(t - \frac{z}{v}) - \Gamma_L \delta(t + \frac{z}{v} - \frac{2\ell}{v}) + \cdots],$$

can be fitted to the required boundary conditions at both ends if

$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}.$$

- In these series, the second terms represent the "reflected" counterparts of the first terms where Γ_L is a reflection coefficient, and
- We assume that other terms not explicitly shown (i.e., $+\cdots$'s in the series) vanish for $t < \frac{2\ell}{v}$ (and play a role at later times).

Verification: In the time interval $0 < t < \frac{2\ell}{v}$, the assumed series expressions evaluate, at $z = \ell$, to

$$V(\ell,t) = \tau_g[\delta(t - \frac{\ell}{v}) + \Gamma_L\delta(t + \frac{\ell}{v} - \frac{2\ell}{v})] = \tau_g\delta(t - \frac{\ell}{v})[1 + \Gamma_L]$$

and

$$I(\ell,t) = \frac{\tau_g}{Z_o}[\delta(t - \frac{\ell}{v}) - \Gamma_L\delta(t + \frac{\ell}{v} - \frac{2\ell}{v})] = \frac{\tau_g}{Z_o}\delta(t - \frac{\ell}{v})[1 - \Gamma_L],$$

respectively. Applying the boundary constraint for $z = \ell$ with these, we find

$$R_L = \frac{V_L}{I_L} = \frac{V(\ell,t)}{I(\ell,t)} = Z_o\frac{1+\Gamma_L}{1-\Gamma_L},$$

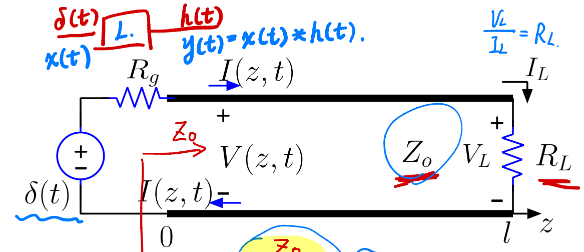
from which it follows that

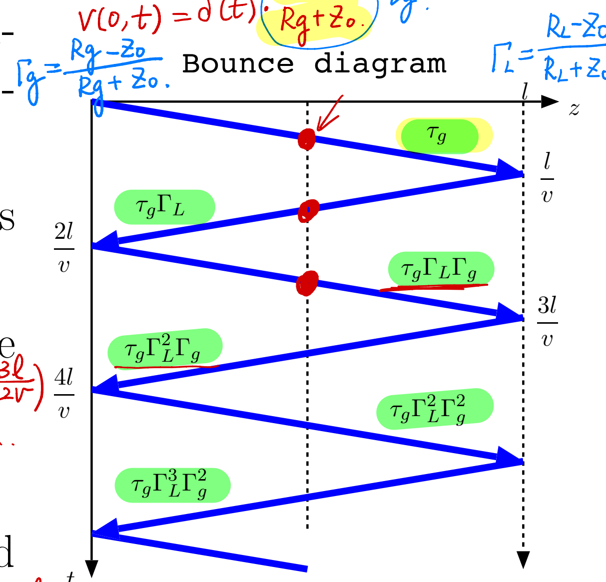
$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

as claimed.

This effectively verifies the assumed series solutions above for the time interval $0 < t < \frac{2\ell}{v}$ — *re*-confirming that the boundary condition at $z = 0$ is also met is left as an optional exercise.

- The solution obtained above (for $t < \frac{2\ell}{v}$) can be better appreciated and extended to *all* times with the help of so-called **bounce diagrams** — see margin.





  - A *bounce diagram* is a plot of the "trajectories" of traveling impulses found on transmission line segments excited by impulse inputs.
  - The horizontal axis represents position z of the traveling impulses while time t is represented by a downward pointing axis.
  - The first slanted line on the top of the diagram, representing the traveling impulse

$V(\frac{\ell}{2},t) = \tau_g\delta(t-\frac{\ell}{2v}) + \tau_g\Gamma_L\delta(t-\frac{3\ell}{2v}) + \tau_g\Gamma_L\Gamma_g\delta(t-\frac{5\ell}{2v}) + \cdots$

$$\tau_g\delta(t - \frac{z}{v}),$$

(first term of $h_z(t) = V(z,t)$) is "reflected" at time $t = \frac{\ell}{v}$ from load R_L to turn into a backward propagating impulse

$V(z,t) = \tau_g\delta(t-\frac{z}{v}) + \tau_g\Gamma_L\delta(t+\frac{z}{v}-\frac{2\ell}{v}) + \tau_g\Gamma_L\Gamma_g\delta(t-\frac{z}{v}-\frac{2\ell}{v}) + \tau_g\Gamma_L^2\Gamma_g\delta(t+\frac{z}{v}-\frac{4\ell}{v}) + \cdots$

$\tau_g = \frac{Z_o}{R_g+Z_o}$. $\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$.

$$\tau_g\Gamma_L\delta(t + \frac{z}{v} - \frac{2\ell}{v})$$

represented by the second line of the diagram.

$\frac{V}{Z_o}$ Forward. sign is the same
Backward change sign.

$V(z,t) = f + g$.
$\uparrow$ $t - \frac{z}{v}$ $\uparrow$ $t + \frac{z}{v}$.

  - The backward propagating impulse reaches $z = 0$ at $t = \frac{2\ell}{v}$ and is reflected once more with a reflection coefficient

$$\Gamma_g = \frac{R_g - Z_o}{R_g + Z_o}$$

$I(z,t) = \frac{f}{Z_o} - \frac{g}{Z_o}$.

$\frac{V}{I}\Big|_{z=\ell} = Z_o\frac{f+g}{f-g}\Big|_{z=\ell} = \frac{V_L}{I_L} = R_L$.

to become a forward propagating impulse

$Z_o f + Z_o g = R_L f - R_L g$.

$g = f\cdot\frac{R_L - Z_o}{R_L + Z_o}$.

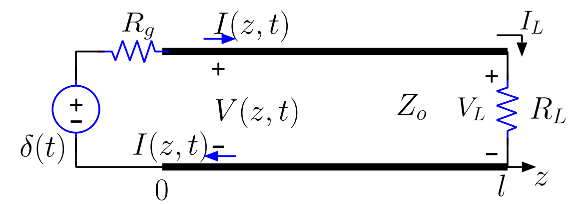
$$\tau_g\Gamma_L\Gamma_g\delta(t - \frac{z}{v} - \frac{2\ell}{v})$$

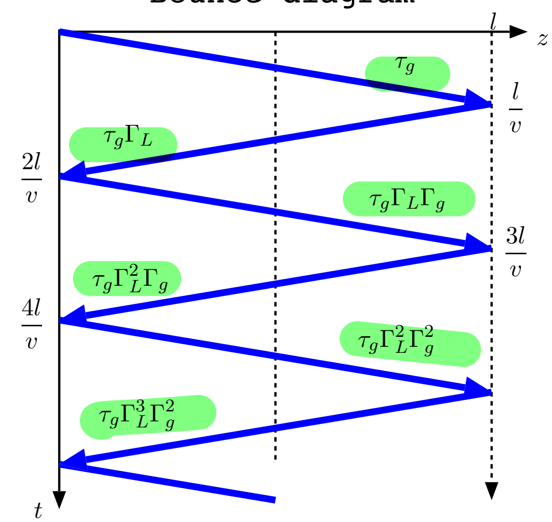
represented by the third line of the diagram.

  - Reflection at R_g is in effect the same physical process as reflection at R_L and therefore its coefficient Γ_g is identical with Γ_L except for the replacement of R_L by R_g.

- The bounce diagram is advanced in time with further reflections occurring at both ends.

- We show the calculated weights of traveling impulses directly on the diagram just above the slanted lines representing the trajectories of each traveling impulse (each having a lifetime of ℓ/v)





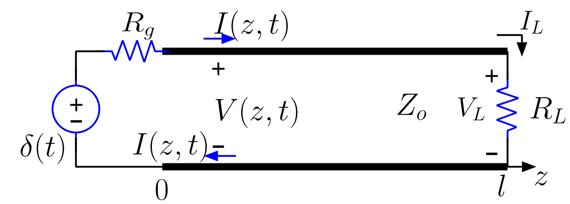
- Using the bounce diagram, the full expressions for the voltage and current impulse response functions of the circuit can be written as

$$\begin{aligned} V(z,t) &= \tau_g \sum_{n=0}^{\infty} (\Gamma_L\Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2\ell}{v}) \\ &\quad + \tau_g\Gamma_L \sum_{n=0}^{\infty} (\Gamma_L\Gamma_g)^n \delta(t + \frac{z}{v} - (n+1)\frac{2\ell}{v}) \end{aligned}$$

and

$$\begin{aligned} I(z,t) &= \frac{\tau_g}{Z_o} \sum_{n=0}^{\infty} (\Gamma_L\Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2\ell}{v}) \\ &\quad - \frac{\tau_g}{Z_o}\Gamma_L \sum_{n=0}^{\infty} (\Gamma_L\Gamma_g)^n \delta(t + \frac{z}{v} - (n+1)\frac{2\ell}{v}). \end{aligned}$$

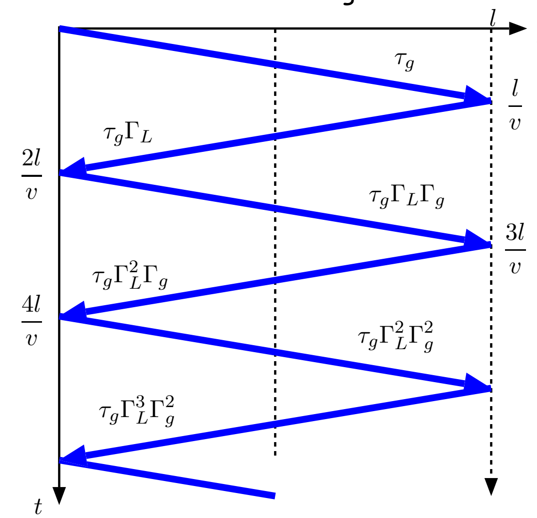
- Although these series formulae[1] look daunting, only the lower order terms usually matter — that is true because $|\Gamma_L| \leq 1$ and $|\Gamma_g| \leq 1$ and thus $(\Gamma_L\Gamma_g)^n$ is typically a rapidly diminishing function of n (unless the ckt is "dissipation free" and resonant, a concept explored in Lecture 31).



- We typically rely on the bounce diagram technique more so than the series expressions developed above. This will be illustrated by several examples in the next lecture.

  - The main idea is to combine *delayed* versions of the circuit input $f_i(t)$ with the impulse weights indicated on the bounce diagram, since, in general, the convolution $\delta(t - T_z) * f_i(t) = f_i(t - T_z)$ for any z-dependent delay such as $\frac{z}{v}$, $\frac{z}{v} - \frac{2\ell}{v}$, etc...

Bounce diagram



[1]The first term of $V(z,t)$ in the series formula can also be obtained from the formal solution of the equation

$$f(t) = \tau_g\delta(t) + \Gamma_L\Gamma_g f(t - \frac{2\ell}{v})$$

which, in turn, is obtained from

$$I(0,t) = \frac{\delta(t) - V(0,t)}{R_g} \quad \text{and} \quad I(\ell,t) = \frac{V(\ell,t)}{R_L}$$

enforced at $z = 0$ and $z = \ell$ at *all* times t. The second term in $V(z,t)$ comes from the requirement that

$$g(t) = \Gamma_L f(t + \frac{2\ell}{v}),$$

which is a consequence of the boundary condition at $z = \ell$. We have effectively by-passed such a formal derivation by using the bounce diagram technique.