23 Imperfect dielectrics, good conductors

$e^{j\tau}$

$\gamma = \alpha + j\beta.$

$\eta = |\eta| e^{j\tau}$

	Condition	β	α	$\lvert\eta\rvert$	τ	$\lambda = \frac{2\pi}{\beta}$	$\delta = \frac{1}{\alpha}$
Perfect dielectric	$\sigma = 0$	$\omega\sqrt{\epsilon\mu}$	0	$\sqrt{\frac{\mu}{\epsilon}}$	0	$\frac{2\pi}{\omega\sqrt{\epsilon\mu}}$	∞
Imperfect dielectric	$\frac{\sigma}{\omega\epsilon} \ll 1$	$\sim \omega\sqrt{\epsilon\mu}$	$\beta\frac{1}{2}\frac{\sigma}{\omega\epsilon} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$	$\sim \sqrt{\frac{\mu}{\epsilon}}$	$\sim \frac{\sigma}{2\omega\epsilon}$	$\sim \frac{2\pi}{\omega\sqrt{\epsilon\mu}}$	$\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$
Good conductor	$\frac{\sigma}{\omega\epsilon} \gg 1$	$\sim \sqrt{\pi f \mu\sigma}$	$\sim \sqrt{\pi f \mu\sigma}$	$\sqrt{\frac{\omega\mu}{\sigma}}$	$45°$	$\sim \frac{2\pi}{\sqrt{\pi f \mu\sigma}}$	$\sim \frac{1}{\sqrt{\pi f \mu\sigma}}$
Perfect conductor	$\sigma = \infty$	∞	∞	0	-	0	0

$\frac{2}{4}, \frac{\sqrt{4}}{120\pi}.$

$\sigma = 4\, S/m.$

$\sqrt{\frac{\mu}{\varepsilon}} \sim 120\pi.$

$\varepsilon = 4. \quad \sqrt{\frac{\varepsilon}{\mu}} = \frac{\sqrt{4}}{120\pi}.$

- The table above summarizes TEM wave parameters in homogeneous conducting media where the propagation velocity

$$v_p = \frac{\omega}{\beta}$$

(note that it can be frequency dependent) and field phasors can be expressed in formats similar to that shown in the margin, keeping in mind that propagation direction coincides with vector

$$\tilde{\mathbf{S}} \equiv \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*$$

such that

$$\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{S}}\}$$

is the average energy flux per unit area (time-average Poynting vector).

x-polarized phasor

$\tilde{\mathbf{E}} = \hat{x} E_o e^{\mp\alpha z} e^{\mp j\beta z}$

accompanied by

$\tilde{\mathbf{H}} = \pm\hat{y}\frac{E_o}{\eta} e^{\mp\alpha z} e^{\mp j\beta z}.$

Example 1: Consider the plane TEM wave

$$\tilde{\mathbf{E}} = \hat{y}2e^{-\alpha z}e^{-j\beta z}\,\frac{\mathrm{V}}{\mathrm{m}},$$

in an **imperfect dielectric**. Determine $\tilde{\mathbf{H}}$ and time-average Poynting vector $\langle\mathbf{S}\rangle$. Compute $\langle\mathbf{S}\rangle$ at $z=0$ and $z=10$ m, if $\epsilon = 4\epsilon_o$, $\mu=\mu_o$, $\sigma = 10^{-3}$ S/m, and $\omega = 2\pi\cdot 10^9$ rad/s

Solution: Using right hand rule, so that $\mathbf{E}\times\mathbf{H}$ points in propagation direction $\hat{z}$, we find that

$$\tilde{\mathbf{H}} = -\hat{x}\frac{2}{\eta}e^{-\alpha z}e^{-j\beta z} \approx -\hat{x}\frac{2}{\sqrt{\mu/\epsilon}}e^{-\alpha z}e^{-j\beta z}e^{-j\tau}\,\frac{\mathrm{A}}{\mathrm{m}}$$

using $|\eta| = \sqrt{\frac{\mu}{\epsilon}}$ from the table above for a perfect dielectric.

The avg. Poynting vector is

$$\begin{aligned}\langle\mathbf{S}\rangle &= \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^*\} = \frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z}e^{-j\beta z}\times(-\hat{x}\frac{2}{\sqrt{\mu/\epsilon}}e^{-\alpha z}e^{-j\beta z}e^{-j\tau})^*\}\\ &= -\frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z}\times\hat{x}\frac{2}{\sqrt{\mu/\epsilon}}e^{-\alpha z}e^{j\tau}\} = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}e^{-2\alpha z}\cos\tau.\end{aligned}$$

With the given parameters,

$$\begin{aligned}\frac{\sigma}{\omega\epsilon} &= \frac{10^{-3}\cdot 36\pi\times 10^9}{2\pi\cdot 10^9\cdot 4} = \frac{9}{2}10^{-3} \ll 1,\\ \tau &\approx \frac{\sigma}{2\omega\epsilon} \approx \frac{9}{4}10^{-3}\,\mathrm{rad}\\ |\eta| &\approx \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_o}{4\epsilon_o}} = \frac{\eta_o}{2} = 60\pi\,\Omega\\ \alpha &\approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = \frac{1}{2}10^{-3}60\pi = 30\pi\cdot 10^{-3}\,\frac{1}{\mathrm{m}}.\end{aligned}$$

Hence, at $z = 0$,

$$\langle \mathbf{S} \rangle = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}\cos\tau \approx \hat{z}\frac{2}{60\pi} = \hat{z}\frac{1}{30\pi}\,\frac{\mathrm{W}}{\mathrm{m}^2},$$

whereas, at $z = 10$ m,

$$\langle \mathbf{S} \rangle = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}e^{-2\cdot 30\pi\cdot 10^{-3}\cdot 10}\cos\tau \approx \hat{z}\frac{2}{60\pi}e^{-6\pi/10} \approx \hat{z}\frac{0.15}{30\pi}\,\frac{\mathrm{W}}{\mathrm{m}^2}.$$

- Note that in above example power transmitted per unit area has dropped to 15% of its value upon propapagating over a relatively short distance of 10 m.
  - In the physical terms, the lost power of the wave is gained by the propagation medium in the form of heat — average Joule heating $\langle \mathbf{J} \cdot \mathbf{E} \rangle$ in the medium will be positive and account for the loss of the wave power (as seen in a HW problem).

⇐This is what we want to happen in a microwave oven.

From a communications perspective, this rapid attenuation is problematic since it is evident that the signal energy is being wasted as heat in the medium rather than being transmitted efficiently to distant communication targets.

As the next example shows, we are better off using lower frequencies in under-water communcations.

Example 2: Repeat Example 1 for $\omega = 2\pi \cdot 10^3$ rad/s and $\sigma = 4$ S/m (sea water) in which case the propagation medium becomes a **good conductor**.

$\frac{\sigma}{\omega \varepsilon} \gg 1$.

Solution: Using right hand rule, so that $\mathbf{E} \times \mathbf{H}$ points in propagation direction $\hat{z}$, we have

$$\tilde{\mathbf{H}} = -\hat{x}\frac{2}{\eta}e^{-\alpha z}e^{-j\beta z} \approx -\hat{x}\frac{2}{|\eta|}e^{-\alpha z}e^{-j\beta z}e^{-j\tau}\,\frac{\text{A}}{\text{m}}$$

$\alpha \approx \beta \approx \sqrt{\pi f \mu \sigma}$. $\alpha \sqrt{f}$.

no ε

as well as

$$\begin{aligned}\langle \mathbf{S} \rangle &= \frac{1}{2}\text{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\} = \frac{1}{2}\text{Re}\{\hat{y}2e^{-\alpha z}e^{-j\beta z} \times (-\hat{x}\frac{2}{|\eta|}e^{-\alpha z}e^{-j\beta z}e^{-j\tau})^*\} \\ &= -\frac{1}{2}\text{Re}\{\hat{y}2e^{-\alpha z} \times \hat{x}\frac{2}{|\eta|}e^{-\alpha z}e^{j\tau}\} = \hat{z}\frac{2}{|\eta|}e^{-2\alpha z}\cos\tau.\end{aligned}$$

With the given parameters,

$$\frac{\sigma}{\omega\epsilon} = \frac{4 \cdot 36\pi \times 10^9}{2\pi \cdot 10^3 \cdot 4} = 18 \cdot 10^6 \gg 1,$$

which confirms that the medium behaves as a good conductor at this small ω, and using the appropriate formulae from the table,

$$\begin{aligned}\tau &\approx \frac{\pi}{4}\,\text{rad} \\ |\eta| &\approx \sqrt{\frac{\omega\mu}{\sigma}} = \sqrt{\frac{2\pi \cdot 10^3 \cdot 4\pi \cdot 10^{-7}}{4}} = \pi\sqrt{2 \times 10^{-4}} \approx \frac{\pi\sqrt{2}}{100}\,\Omega \\ \alpha &\approx \sqrt{\pi f \mu \sigma} = \sqrt{\pi \cdot 10^3 \cdot 4\pi \cdot 10^{-7} \cdot 4} = \sqrt{4^2\pi^2 10^{-4}} = \frac{\pi}{25}\,\frac{1}{\text{m}}.\end{aligned}$$

Hence, at $z = 0$,

$$\langle \mathbf{S} \rangle = \hat{z}\frac{2}{|\eta|}\cos\tau \approx \hat{z}\frac{200}{\pi\sqrt{2}}\cos\frac{\pi}{4} = \hat{z}\frac{100}{\pi}\,\frac{\text{W}}{\text{m}^2},$$

whereas, at $z = 10$ m,

$$\langle \mathbf{S} \rangle = \hat{z}\frac{100}{\pi} e^{-2\cdot\frac{\pi}{25}\cdot 10} \approx \hat{z}\frac{100}{\pi} 0.081 \, \frac{\mathrm{W}}{\mathrm{m}^2}.$$

- As Example 2 illustrates, at a frequency of $\omega = 2\pi \cdot 10^3$ rad/s or $f = 1$ kHz, wave power is reduced to about 8% over a 10 m distance in sea water. Less reduction in power is possible over the same distance if at a smaller frequency f since $\alpha \propto \sqrt{f}$.
  - The disadvantage of being forced to use smaler frequencies is of course having a smaller available bandwidth at small frequencies. Thus communication with submarines at great depths will only be possible at very slow rates.

The next example identifies the penetration depth in sea water at 1 kHz.

Example 3: What is the **penetration depth** $\delta = \alpha^{-1}$ in a medium with $\sigma = 4$ S/m, $\epsilon = 81\epsilon_o$, and $\mu = \mu_o$ for $\omega = 2\pi \cdot 10^3$ rad/s.

Solution: With the given parameters we have

$$\frac{\sigma}{\omega\epsilon} = \frac{4 \cdot 36\pi \times 10^9}{2\pi \cdot 10^3 \cdot 81} = \frac{72 \times 10^9}{81 \times 10^3} \approx 10^6 \gg 1,$$

i.e., good conductor situation. Hence the penetration depth is

$$\delta \approx \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi 10^3 \cdot 4\pi \cdot 10^{-7} 4}} = \frac{1}{\sqrt{4^2 \pi^2 \cdot 10^{-4}}} = \frac{100}{4\pi} = \frac{25}{\pi} \approx 7.95\,\text{m}.$$

$\frac{1}{\alpha}$.