# 22 Phasor form of Maxwell's equations and damped waves in conducting media

$e^{st}$

- When the fields and the sources in Maxwell's equations are all monochromatic functions of time expressed in terms of their phasors, Maxwell's equations can be transformed into the phasor domain.

$$\begin{aligned}\nabla\cdot\mathbf{D} &= \rho\\ \nabla\cdot\mathbf{B} &= 0\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\times\mathbf{H} &= \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}.\end{aligned}$$

  – In the phasor domain all

$$\frac{\partial}{\partial t}\rightarrow j\omega$$

  and all variables $\mathbf{D}$, $\rho$, etc. are replaced by their phasors $\tilde{\mathbf{D}}$, $\tilde{\rho}$, and so on. With those changes Maxwell's equations take the form shown in the margin.

$\nabla\cdot\epsilon\tilde{E}=0.$

$$\begin{aligned}\nabla\cdot\tilde{\mathbf{D}} &= \tilde{\rho} = 0\\ \nabla\cdot\tilde{\mathbf{B}} &= 0\\ \nabla\times\nabla\times\tilde{\mathbf{E}} &= -j\omega\tilde{\mathbf{B}}\\ \nabla\times\tilde{\mathbf{H}} &= \tilde{\mathbf{J}}+j\omega\tilde{\mathbf{D}}\end{aligned}$$

$0$

$\underbrace{j\omega\epsilon\tilde{E}}$

  – Also in these equations it is implied that

$$\begin{aligned}\tilde{\mathbf{D}} &= \epsilon\tilde{\mathbf{E}}\\ \tilde{\mathbf{B}} &= \mu\tilde{\mathbf{H}}\\ \tilde{\mathbf{J}} &= \sigma\tilde{\mathbf{E}}\end{aligned}$$

  where $\epsilon$, $\mu$, and $\sigma$ could be a function of frequency $\omega$ (as, strictly speaking, they all are as seen in Lecture 11).

  – We can derive from the phasor form Maxwell's equations shown in the margin the TEM wave properties obtained earlier on using the time-domain equations by assuming $\tilde{\rho}=\tilde{\mathbf{J}}=0$.

We will do that, and and after that relax the requirement $\tilde{\mathbf{J}} = 0$ with $\tilde{\mathbf{J}} = \sigma \tilde{\mathbf{E}}$ to examine how TEM waves propagate in conducting media.

- With $\tilde{\rho} = \tilde{\mathbf{J}} = 0$ the phasor form Maxwell's equation take their simplified forms shown in the margin.

$$-j\omega\mu \cdot j\omega\varepsilon \tilde{\vec{E}} = \omega^2 \mu\varepsilon \tilde{\vec{E}}$$

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{E}} &= 0 \\ \nabla \cdot \tilde{\mathbf{H}} &= 0 \\ \nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}} \\ \nabla \times \tilde{\mathbf{H}} &= j\omega\epsilon\tilde{\mathbf{E}} \end{aligned}$$

$\tilde{\vec{J}} = \sigma \tilde{\vec{E}}$

  - Using

$$\nabla \times [\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}] \Rightarrow -\nabla^2\tilde{\mathbf{E}} = -j\omega\mu\nabla \times \tilde{\mathbf{H}}$$

$\nabla \times \nabla \times \tilde{\vec{E}}$ $\quad -j\omega\mu\nabla\times\tilde{\vec{H}}$

which combines with the Ampere's law to produce

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}.$$

$$\nabla^2\tilde{\mathbf{E}} + \omega^2\mu\epsilon\tilde{\mathbf{E}} = 0.$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\cancel{\nabla \cdot \vec{A}}) - \nabla^2 \vec{A}.$$

$0$

$\frac{\partial}{\partial t} \to j\omega.$

  - For $x$-polarized waves with phasors

$$\tilde{\mathbf{E}} = \hat{x}\tilde{E}_x(z),$$

the phasor wave equation above simplifies as

$$\frac{\partial^2}{\partial z^2}\tilde{E}_x + \omega^2\mu\epsilon\tilde{E}_x = 0.$$

$e^{\mp st}$

$\nabla^2 \tilde{E}_x$

  - Try solutions of the form

$$\tilde{E}_x(z) = e^{-\gamma z} \text{ or } e^{\gamma z}$$

$e^{\mp j\beta z}.$

where $\gamma$ is to be determined.

– Upon substitution into wave equation both of these lead to

$$(\gamma^2 + \omega^2\mu\epsilon)\tilde{E}_x = 0,$$

which yields

$$\gamma^2 + \omega^2\mu\epsilon = 0 \quad\Rightarrow\quad \gamma^2 = -\omega^2\mu\epsilon$$

from which one possibility is

$\gamma = -j\beta$.

$$\gamma = j\beta, \quad \text{with} \quad \beta \equiv \omega\sqrt{\mu\epsilon}.$$

Thus viable phasor solutions are

$$\tilde{E}_x(z) = e^{\mp j\beta z}$$

$\cos(\omega t \mp \beta z) \leftrightarrow e^{\mp j\beta z}$

as we already knew.

$\nabla \times \tilde{E} = -\frac{\partial \tilde{B}}{\partial t}$. $\quad \tilde{B} = \mu \tilde{H}$

$\hat{z}$

– Furthermore, using the phasor form Faraday's law it is easy to show that

$$\tilde{H}_y = \pm\frac{e^{\mp j\beta z}}{\eta} \quad \text{with} \quad \eta = \sqrt{\frac{\mu}{\epsilon}}.$$

$\sqrt{\frac{\mu_0}{\epsilon_0}} \approx 120\pi\ \Omega$.

$\tilde{H}_y = \frac{e^{-j\beta z}}{\eta}$. $\quad \hat{y}$

$-\hat{z}$.

$\tilde{H}_y = -\frac{e^{+j\beta z}}{\eta} \quad -\hat{y}$

**Note that we have recovered above the familiar properties of plane TEM waves using phasor methods.**

Next, the phasor method carries us to a new domain that cannot be easily examined using time-domain methods.

- With $\tilde{\rho} = 0$ but $\tilde{\mathbf{J}} = \sigma\tilde{\mathbf{E}} \neq 0$, implying non-zero conductivity $\sigma$, the pertinent phasor form equations are as shown in the margin.

lossy.

$$\begin{aligned}\nabla\cdot\tilde{\mathbf{E}} &= 0\\ \nabla\cdot\tilde{\mathbf{H}} &= 0\\ \nabla\times\tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}}\\ \nabla\times\tilde{\mathbf{H}} &= \sigma\tilde{\mathbf{E}} + j\omega\epsilon\tilde{\mathbf{E}}\\ &= (\sigma + j\omega\epsilon)\tilde{\mathbf{E}}\end{aligned}$$

  - This is the same set as before, except that

$$j\omega\epsilon \text{ has been replaced by } \sigma + j\omega\epsilon.$$

Thus, we can make use of phasor wave solutions above after applying the following modifications to $\gamma$ and $\eta$:

$\sigma + j\omega\varepsilon$. $\tilde{E} = E_0 e^{\mp\gamma z}$.

$v = \frac{1}{\sqrt{\mu\varepsilon}}$.

$\eta = \sqrt{\frac{\mu}{\varepsilon}}$.

1.
$$\gamma^2 = -\omega^2\mu\epsilon = (j\omega\mu)(j\omega\epsilon) \;\boxed{\begin{matrix}\Rightarrow\Rightarrow\\ \sigma\neq 0\end{matrix}}\; \gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)}$$

2.
$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\epsilon}} \;\boxed{\begin{matrix}\Rightarrow\Rightarrow\\ \sigma\neq 0\end{matrix}}\; \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}.$$

$\mu . \varepsilon .$

Note that the modified $\gamma$ and $\eta$ satisfy

$$\gamma\eta = j\omega\mu \text{ and } \frac{\gamma}{\eta} = \sigma + j\omega\epsilon$$

leading to useful relations shown in the margin (assuming real valued $\sigma$ and $\epsilon$).

$$\begin{aligned}\mu &= \frac{\gamma\eta}{j\omega}\\ \sigma &= \text{Re}\{\frac{\gamma}{\eta}\}\\ \epsilon &= \frac{1}{\omega}\text{Im}\{\frac{\gamma}{\eta}\}\end{aligned}$$

- In terms of $\gamma$ and $\eta$ above, we can express an $x$-polarized plane wave propagating in $z$ direction in terms of phasors

$e^{-\gamma z}$ $\quad \mathrm{Re}\{e^{-(\alpha+j\beta)z}E_0\hat{x}\}e^{j\omega t}$

$$\tilde{\mathbf{E}} = \hat{x}E_o e^{\mp\gamma z} \text{ and } \tilde{\mathbf{H}} = \pm\hat{y}\frac{E_o}{\eta}e^{\mp\gamma z}$$

$|\eta|e^{j\tau}$ $\quad e^{-\alpha z}\cos(\omega t-\beta z)$

where $E_o$ is an arbitrary complex constant (complex wave amplitude).

- In expanded forms $\gamma$ and $\eta$ appear as:

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)} \equiv \alpha + j\beta, \text{ so that } \alpha = \mathrm{Re}\{\gamma\} \text{ and } \beta = \mathrm{Im}\{\gamma\},$$

and $f=1\text{MHz}.\ \mu=\mu_0.\ \sigma=2\ \text{S/m}.\ \epsilon=4\epsilon_0.$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \equiv |\eta|e^{j\tau} \text{ so that } |\eta| = |\sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}| \text{ and } \tau = \angle\sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}.$$

1. In the special case of a **perfect dielectric** with $\sigma = 0$, we find

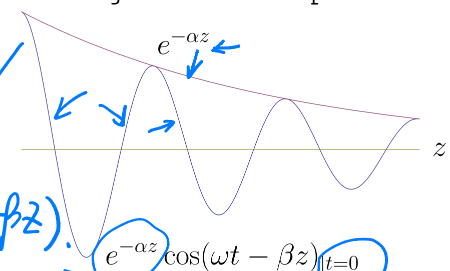
$$\gamma = j\omega\sqrt{\mu\epsilon} \equiv j\beta \text{ and } \eta = \sqrt{\frac{\mu}{\epsilon}},$$

and, therefore,

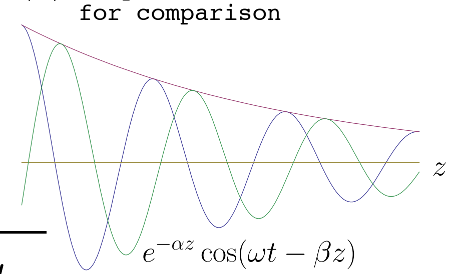
$$\tilde{\mathbf{E}} = \hat{x}E_o e^{\mp j\beta z} \text{ and } \tilde{\mathbf{H}} = \pm\frac{\hat{y}E_o e^{\mp j\beta z}}{\eta}$$

as before. In this case $\alpha = \tau = 0$.

(a) Damped wave snapshot at t=0 together with exponential envelope




(b) Snaphot at t>0, with t=0 waveform for comparison




– $\beta$ appears within cosine argument and determines the wavelength

$$\lambda = \frac{2\pi}{\beta}$$

and propagation speed

$$v_p = \frac{\omega}{\beta}.$$

– $\alpha$ controls wave attenuation by

$$e^{\mp\alpha z}$$

factor in propagation direction.

2. Another case of **imperfect dielectric** (or "lousy" conductor) occurs when $\sigma$ is not zero, but it is so small that are justified in using

$$(1 \pm a)^p \approx 1 \pm pa, \quad \text{if } |a| \ll 1,$$

with $p = \frac{1}{2}$ as follows: For $\frac{\sigma}{\omega\epsilon} \ll 1$,

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)} = j\omega\sqrt{\mu\epsilon}(1 - j\frac{\sigma}{\omega\epsilon})^{1/2} \approx j\omega\sqrt{\mu\epsilon}(1 - j\frac{\sigma}{2\omega\epsilon}) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} + j\omega\sqrt{\mu\epsilon};$$

hence

$$\tilde{\mathbf{E}} \approx \hat{x}E_o e^{\mp(\alpha + j\beta)z} \text{ with } \alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \text{ and } \beta = \omega\sqrt{\mu\epsilon};$$

also in the same case

$$\tilde{\mathbf{H}} \approx \pm\frac{\hat{y}E_o e^{\mp(\alpha + j\beta)z}}{\eta} \text{ with } \eta = \sqrt{\frac{\mu}{\epsilon(1 - j\frac{\sigma}{\omega\epsilon})}} \approx \sqrt{\frac{\mu}{\epsilon}}(1 + j\frac{\sigma}{2\omega\epsilon}) \approx \sqrt{\frac{\mu}{\epsilon}}e^{j\tan^{-1}\frac{\sigma}{2\omega\epsilon}},$$

such that

$$|\eta| \approx \sqrt{\frac{\mu}{\epsilon}} \text{ and } \tau = \angle\eta \approx \frac{\sigma}{2\omega\epsilon}.$$

**Note:** $\gamma$ and $\eta$ both are *complex* valued, the consequences of which will be examined later on.

3. A third case of **good conductor** corresponds to $\frac{\sigma}{\omega\epsilon} \gg 1$. In that case,

$$\gamma = j\omega\sqrt{\mu\epsilon(1 - j\frac{\sigma}{\omega\epsilon})} \approx \omega\sqrt{j\mu\frac{\sigma}{\omega}} = (1 + j)\sqrt{\frac{\omega\mu\sigma}{2}} \text{ and } \eta \approx \sqrt{\frac{\mu}{-j\frac{\sigma}{\omega}}} = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}e^{j\pi/4}.$$

Hence,

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \text{ while } |\eta| = \sqrt{\frac{\omega\mu}{\sigma}} \text{ and } \tau = \angle\eta = 45^o.$$

4. Finally, **perfect conductor** case corresponds to $\sigma \to \infty$, in which case $\tilde{E}_x \to 0$ as we will show later on. Wave fields cannot exist in perfect conductors.

- Summarizing, in a **homogeneous medium** with arbitrary but constant $\mu$, $\epsilon$, and $\sigma$, time-harmonic plane TEM waves are in terms of

$$\mathbf{E} = \hat{x}\text{Re}\{E_o e^{\mp(\alpha+j\beta)z} e^{j\omega t}\} = \hat{x}|E_o|e^{\mp\alpha z}\cos(\omega t \mp \beta z + \angle E_o)$$

and accompanying magnetic fields

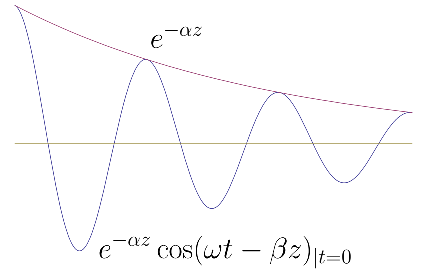
$$\mathbf{H} = \pm\hat{y}\text{Re}\{\frac{E_o}{\eta} e^{\mp(\alpha+j\beta)z} e^{j\omega t}\} = \pm\hat{y}\frac{|E_o|}{|\eta|}e^{\mp\alpha z}\cos(\omega t \mp \beta z + \angle E_o - \angle\eta).$$

- **Propagation velocity**

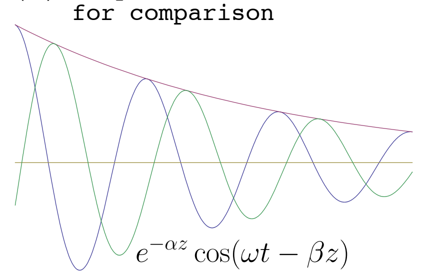
$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\text{Im}\{\sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)}\}},$$

now depends on frequency $\omega$ and it describes the speed of the **nodes** (zero-crossings, not modified by the attenuation factor) of the field waveform. Subscript $p$ is introduced to distinguish $v_p$ — also called *phase velocity* — from *group velocity* $v_g$ discussed in ECE 450 (velocity of narrowband wave packets).

(a) Damped wave snapshot at t=0 together with exponential envelope




(b) Snaphot at t>0, with t=0 waveform for comparison




- **$\beta$ appears within cosine argument and determines the wavelength**

$$\lambda = \frac{2\pi}{\beta}$$

**and propagation speed**

$$v_p = \frac{\omega}{\beta}.$$

- **$\alpha$ controls wave attenuation by**

$$e^{\mp\alpha z}$$

**factor in propagation direction.**

- **Wavelength**

$$\lambda = \frac{2\pi}{\beta} = \frac{v_p}{f}$$

now depends on frequency $f$ via both the numerator and the denominator, and measures twice the distance between successive nodes of the waveform.

- **Penetration depth** (also called *skin depth* if very small)

$$\delta \equiv \frac{1}{\alpha} = \frac{1}{\mathrm{Re}\{\sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)}\}}$$

is the distance for the field strength to be reduced by $e^{-1}$ factor in its direction of propagation.

  – For a fixed $\sigma$, and a sufficiently large $\omega$, the penetration depth

$$\delta \approx \frac{2}{\sigma\sqrt{\frac{\mu}{\epsilon}}} \text{ Imperfect dielectric formula}$$

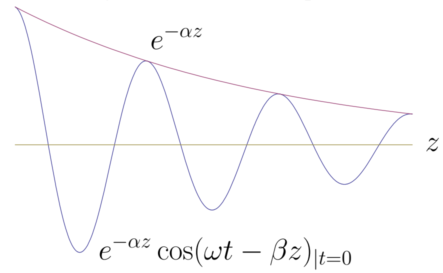
  which can be very small if $\sigma$ is large — with small $\delta$ the wave is severely attenuated as it propagates.

  – For a fixed $\sigma$, and a sufficiently small $\omega$,

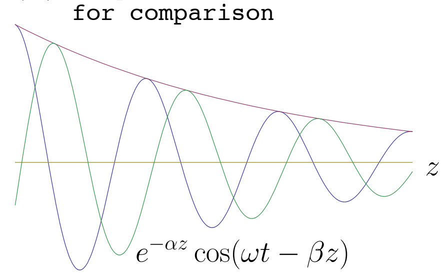
$$\delta \approx \sqrt{\frac{2}{\mu\omega\sigma}} = \frac{1}{\sqrt{\pi f \mu\sigma}} \text{ Good conductor "skin depth" formula}$$

  which, although small with large $\sigma$, increases as $\omega$ decreases, making *low frequencies to be preferable* in applications requiring propagating through lossy media with large $\sigma$, such as in sea-water.

(a) Damped wave snapshot at t=0 together with exponential envelope




(b) Snaphot at t>0, with t=0 waveform for comparison




– **$\beta$ appears within cosine argument and determines the wavelength**

$$\lambda = \frac{2\pi}{\beta}$$

**and propagation speed**

$$v_p = \frac{\omega}{\beta}.$$

– **$\alpha$ controls wave attenuation by**

$$e^{\mp\alpha z}$$

**factor in propagation direction.**