21 Monochromatic waves and phasor notation

- Recall that we reached the traveling-wave d'Alembert solutions

$$\mathbf{E}, \mathbf{H} \propto f(t \mp \frac{z}{v})$$

via the superposition of time-shifted and amplitude-scaled versions of

$$f(t) = \cos(\omega t),$$

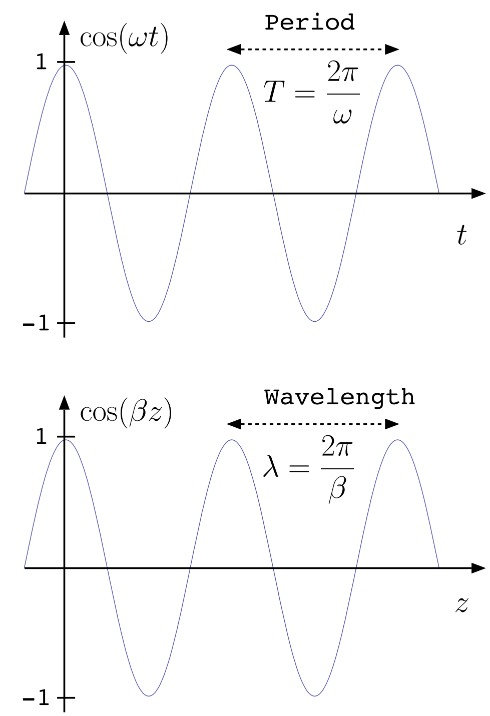
namely the **monochromatic waves**

$$A\cos[\omega(t \mp \frac{z}{v})] = A\cos(\omega t \mp \beta z),$$

with amplitudes A where

$$\beta \equiv \frac{\omega}{v} = \omega\sqrt{\mu\epsilon}$$

can be called **wave-number** in analogy with **wave-frequency** ω.



  - As depicted in the margin, monochromatic solutions $A\cos(\omega t \mp \beta z)$ are periodic in position and time, with the *wave-number* β being essentially a *spatial-frequency*, the spatial counterpart of ω.

This is an important point that you should try to understand well — it has implications for signal processing courses related to images and vision.

– *In general*, monochromatic solutions of *1D wave-equations* obtained in various branches of science and engineering can all be represented in the same format as above in terms of wave-frequency / wave-wavenumber pairs ω and β having a ratio

$$v \equiv \frac{\omega}{\beta}$$

recognized as the **wave-speed** and specific **dispersion relations** such as:

1. **TEM waves** in perfect dielectrics:

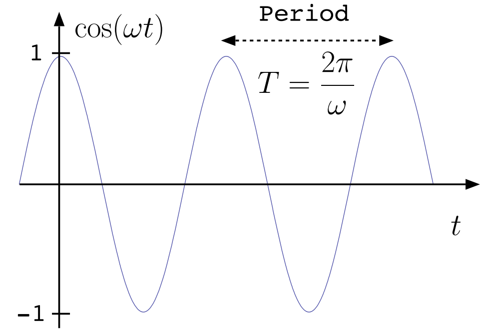
$$\beta = \omega\sqrt{\mu\epsilon},$$

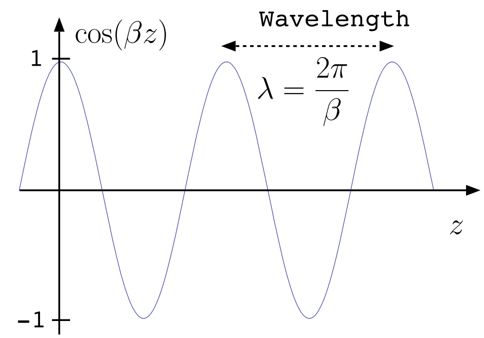
2. **Acoustic waves** in monoatomic gases with temperature T (K) and atomic mass m (kg):

$$\beta = \omega\sqrt{\frac{m}{\frac{5}{3}KT}},$$

3. TEM waves in collisionless **plasmas** (ionized gases) with plasma frequency $\omega_p = \sqrt{\frac{Ne^2}{m\epsilon_o}}$:

$$\beta = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}.$$





Dispersion relations between *wavefrequency* ω and *wavenumber* β determine the *propagation velocity*

$$v = \frac{\omega}{\beta} = \lambda f$$

for all types of wave motions.

– For any type of wave solution — TEM, acoustic, plasma wave — once the dispersion relation is available (meaning that it has been derived from fundamental physical laws governing the specific wave type), wave propagation velocity is always obtained as

$$v = \frac{\omega}{\beta}$$

or, equivalently, as

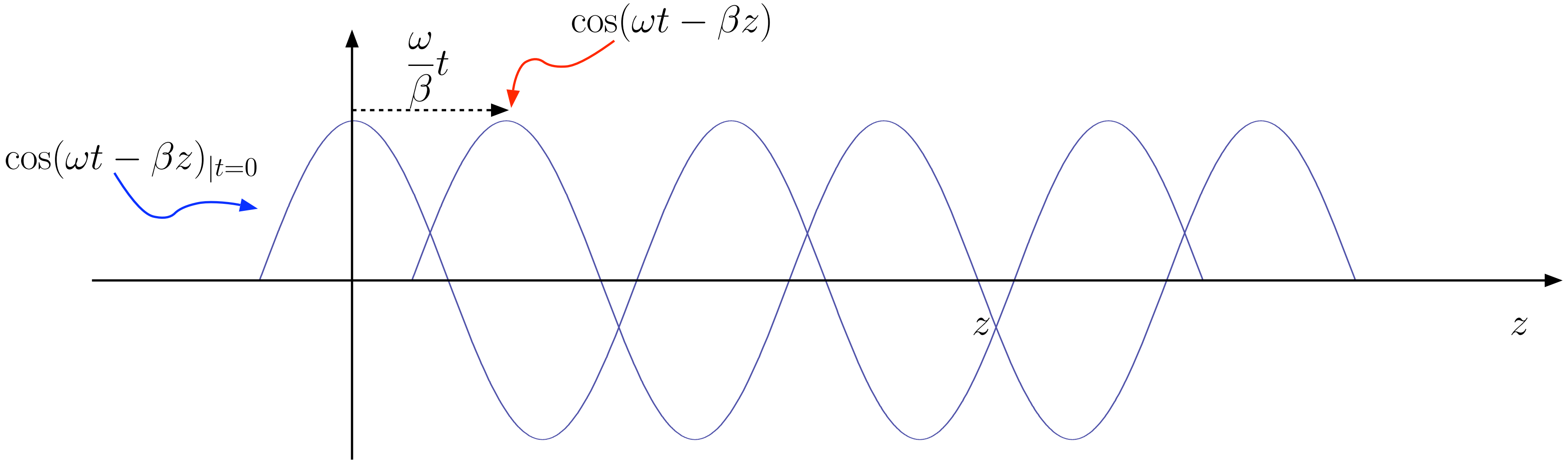
$$v = \frac{\lambda}{T} = \lambda f$$

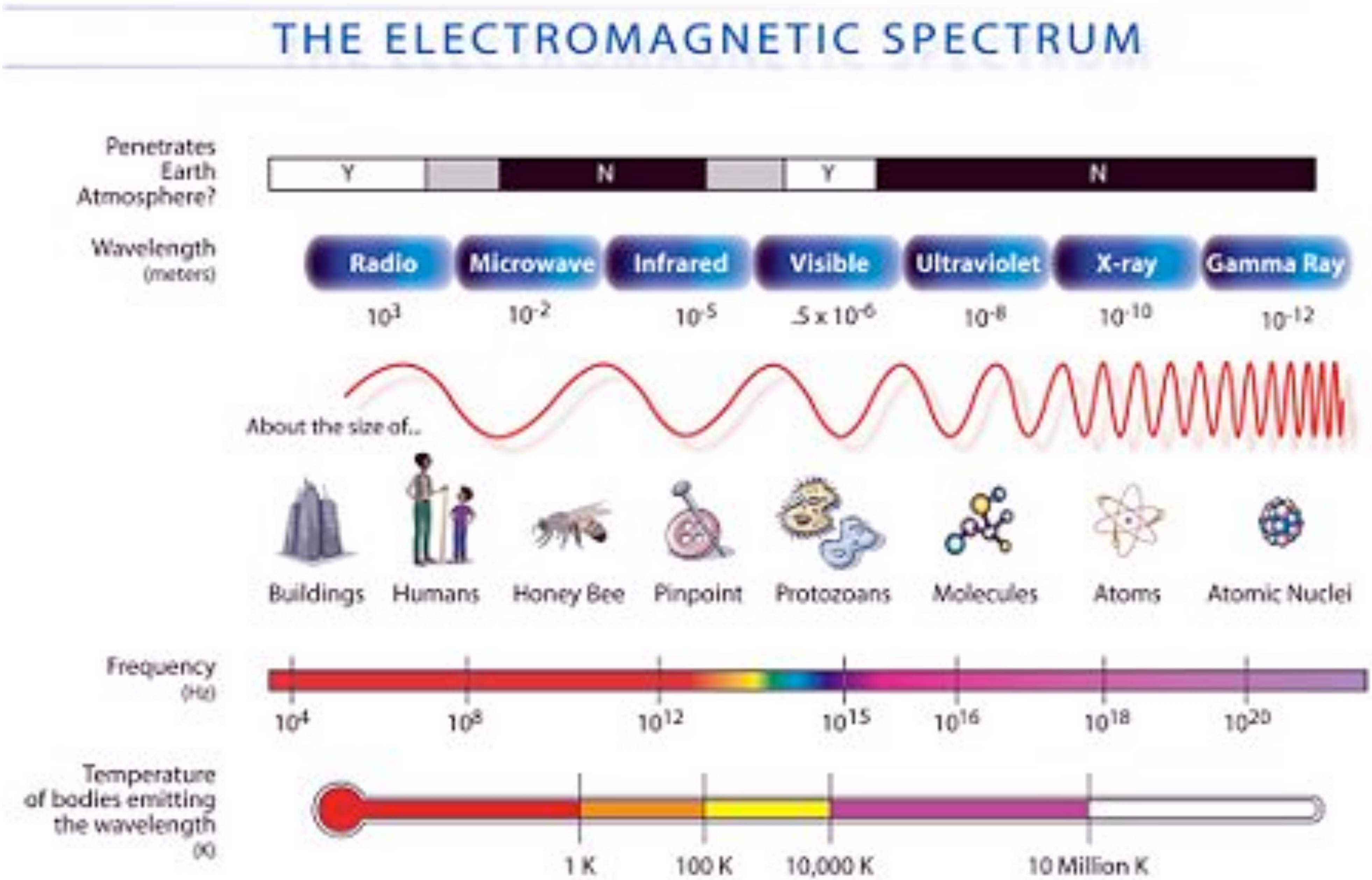
where

$$\lambda \equiv \frac{2\pi}{\beta} \quad \text{Wavelength}$$

and

$$T = \frac{2\pi}{\omega} \equiv \frac{1}{f} \quad \text{Waveperiod.}$$





THE ELECTROMAGNETIC SPECTRUM
Penetrates Earth Atmosphere?
Y
N
Y
N
Wavelength (meters)
Radio
Microwave
Infrared
Visible
Ultraviolet
X-ray
Gamma Ray
10^3
10^{-2}
10^{-5}
$.5 \times 10^{-6}$
10^{-8}
10^{-10}
10^{-12}
About the size of...
Buildings
Humans
Honey Bee
Pinpoint
Protozoans
Molecules
Atoms
Atomic Nuclei
Frequency (Hz)
10^4
10^8
10^{12}
10^{15}
10^{16}
10^{18}
10^{20}
Temperature of bodies emitting the wavelength (K)
1 K
100 K
10,000 K
10 Million K

- Monochromatic x-polarized waves

$$\mathbf{E} = E_o \cos(\omega t \mp \beta z)\, \hat{x}\, \frac{\mathrm{V}}{\mathrm{m}}$$

can also be expressed in phasor form as

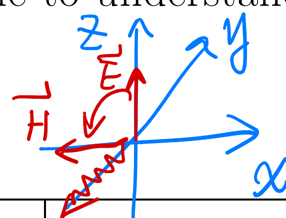
$$\tilde{\mathbf{E}} = E_o e^{\mp j\beta z} \hat{x}\, \frac{\mathrm{V}}{\mathrm{m}}$$

such that

$$\mathrm{Re}\{\tilde{\mathbf{E}} e^{j\omega t}\} = E_o \cos(\omega t \mp \beta z)\, \hat{x} = \mathbf{E}$$

in view of Euler's identity.

Example 1: Study the following table to understand monochromatic wave fields and their phasors.



Field	Phasor	Comment
$\mathbf{E} = \cos(\omega t + \beta y)\, \hat{z}$	$\tilde{\mathbf{E}} = e^{j\beta y} \hat{z}$	**z-polarized wave propagating in $-y$ direction**
	$\tilde{\mathbf{H}} = -\frac{e^{j\beta y}}{\eta} \hat{x}$	**magnetic phasor that accompanies $\tilde{\mathbf{E}}$ above**
$\mathbf{H} = \sin(\omega t - \beta z)\, \hat{y}$	$\tilde{\mathbf{H}} = -je^{-j\beta z}\, \hat{y}$	**wave propagating in $+z$ direction**
$\mathbf{E} = \eta \sin(\omega t - \beta z)\, \hat{x}$	$\tilde{\mathbf{E}} = -j\eta e^{-j\beta z}\, \hat{x}$	**electric field phasor of $\tilde{\mathbf{H}}$ above which is an x-polarized field (see the right column)**

Example 2: Given that

$$\mathbf{H} = \hat{x}H^{+}\cos(\omega t - \beta z) + \hat{y}H^{-}\sin(\omega t + \beta z)$$

representing the sum of wave fields propagating in opposite directions, the corresponding phasor

$$\tilde{\mathbf{H}} = \hat{x}H^{+}e^{-j\beta z} - j\hat{y}H^{-}e^{j\beta z}.$$

The corresponding **E**-field phasor is

$$\tilde{\mathbf{E}} = -\hat{y}\eta H^{+}e^{-j\beta z} + j\hat{x}\eta H^{-}e^{j\beta z},$$

from which

$$\mathbf{E} = -\hat{y}\eta H^{+}\cos(\omega t - \beta z) - \hat{x}\eta H^{-}\sin(\omega t + \beta z).$$

Make sure to check that all the signs make sense, and if you think you have caught an error, let us know.

- In general, we transform between plane TEM wave phasors $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ as follows:

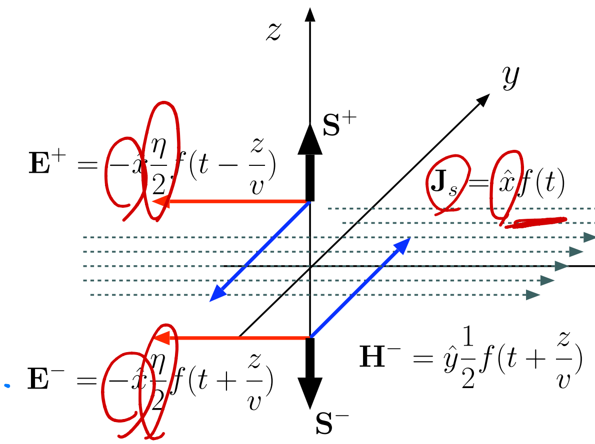
$|\vec{H}| = \frac{|\vec{E}|}{\eta}$

1. **To obtain $\tilde{\mathbf{H}}$ from $\tilde{\mathbf{E}}$:** divide $\tilde{\mathbf{E}}$ by η and rotate the vector direction so that vector $\tilde{\mathbf{S}} \equiv \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^{*}$ points in the propagation direction of the wave — more on complex vector $\tilde{\mathbf{S}}$ later on.

2. **To obtain $\tilde{\mathbf{E}}$ from $\tilde{\mathbf{H}}$:** multiply $\tilde{\mathbf{H}}$ by η and rotate the vector direction so that vector $\tilde{\mathbf{S}} \equiv \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^{*}$ points in the propagation direction of the

$|\vec{E}| = \eta|\vec{H}|$

wave.



Example 3: On $z = 0$ plane we have a monochromatic surface current specified as

$$\mathbf{J}_s = \hat{x}f(t) = \hat{x}2\cos(\omega t)\,\frac{\mathrm{A}}{\mathrm{m}} = \mathrm{Re}\{\hat{x}2\,e^{j\omega t}\}.$$

Determine wave field phasors $\tilde{\mathbf{E}}^{\pm}$ and $\tilde{\mathbf{H}}^{\pm}$ for plane TEM waves propagating away from the $z = 0$ surface on both sides (assumed vacuum).

Solution: We know that an x-polarized surface current $f(t)$ produces

$$E_x = -\frac{\eta}{2}f(t \mp \frac{z}{v}) \quad \text{and} \quad H_y = \mp\frac{1}{2}f(t \mp \frac{z}{v})$$

in surrounding regions. Given that $f(t) = 2\cos(\omega t)$, this implies

$$E_x = -\eta\cos(\omega t \mp \beta z) \quad \text{and} \quad H_y = \mp\cos(\omega t \mp \beta z)$$

where

$$\beta = \frac{\omega}{c} \quad \text{and} \quad \eta = \eta_o \approx 120\pi\,\Omega$$

since the current sheet is surrounded by vacuum. Converting these into phasors, we find

$$\tilde{\mathbf{E}}^{\pm} = -\eta e^{\mp j\beta z}\hat{x} \quad \text{and} \quad \tilde{\mathbf{H}}^{\pm} = \mp e^{\mp j\beta z}\hat{y}.$$

- In the last lecture we calculated the time-average $\mathbf{E}\times\mathbf{H}$ and $\mathbf{J}_s\cdot\mathbf{E}$ of the fields examined in Example 3 using a time-domain approach. The same calculations can be carried out in terms of phasors $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$, and $\tilde{\mathbf{J}}_s$ as follows:

$$\langle\mathbf{E}\times\mathbf{H}\rangle = \frac{1}{2}\text{Re}\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^*\} \quad \text{and} \quad \langle\mathbf{J}_s\cdot\mathbf{E}\rangle = \frac{1}{2}\text{Re}\{\tilde{\mathbf{J}}_s\cdot\tilde{\mathbf{E}}^*\}$$

where $\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^* \equiv \tilde{\mathbf{S}}$ is called complex Poynting vector.

  – The proof of these are analogous to the proof of

$$\langle p(t)\rangle = \frac{1}{2}\text{Re}\{VI^*\}$$

for the average power of a circuit component in terms of voltage and current phasors V and I (see margin).

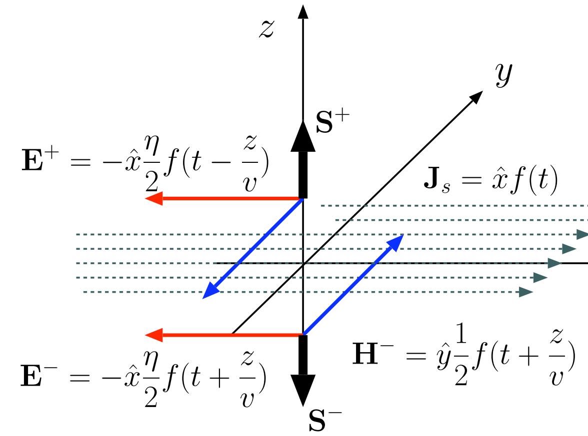
For, instance, given that

$$\tilde{\mathbf{J}}_s = 2\hat{x}\frac{\text{A}}{\text{m}} \quad \text{and} \quad \tilde{\mathbf{E}}^{\pm}(z) = -\eta e^{\mp j\beta z}\hat{x}\,\frac{\text{V}}{\text{m}}$$

in Example 3, it follows that

$$\langle -\mathbf{J}_s(t)\cdot\mathbf{E}(0,t)\rangle = \frac{1}{2}\text{Re}\{-\tilde{\mathbf{J}}_s\cdot\tilde{\mathbf{E}}^*(0)\} = \eta \approx 120\pi\,\frac{\text{W}}{\text{m}^2},$$

in conformity with the result from last lecture.



Instantaneous power

$$p(t) = v(t)i(t)$$

with time-harmonic signals is

$$v(t)i(t) = \left(\frac{Ve^{j\omega t}+\text{cc}}{2}\right)\left(\frac{Ie^{j\omega t}+\text{cc}}{2}\right)$$

where V and I are phasors of $v(t)$ and $i(t)$ and cc indicates the conjugate of the term to the left of + sign.
This can be expanded as

$$v(t)i(t) = \frac{VI^*+\text{cc}}{4} + \frac{VIe^{j2\omega t}+\text{cc}}{4}.$$

The second term has a zero time average. It follows that *time-average power*

$$\langle v(t)i(t)\rangle = \frac{VI^*+\text{cc}}{4} = \frac{1}{2}\text{Re}\{VI^*\}$$

since

$$VI^*+\text{cc} = VI^*+V^*I = 2\text{Re}\{VI^*\}.$$

(Also see ECE 210 text.)