

20 Poynting theorem and monochromatic waves

- The magnitude of **Poynting vector**

pointing $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ W/m^2 $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\mathbf{J}} = 0$

$\vec{\mathbf{E}} \times \vec{\mathbf{H}} = \vec{\mathbf{S}}$

 $\vec{\mathbf{E}}$ V/m $\vec{\mathbf{H}}$ A/m $\vec{\mathbf{S}}$ W/m^2

$\frac{\text{V}}{\text{m}} \cdot \frac{\text{A}}{\text{m}} = \frac{\text{VA}}{\text{m}^2} \rightarrow \frac{\text{W}}{\text{m}^2}$

represents the amount of power transported — often called energy flux — by electromagnetic fields \mathbf{E} and \mathbf{H} over a unit area transverse to the $\mathbf{E} \times \mathbf{H}$ direction.

This interpretation of the Poynting vector is obtained from a conservation law extracted from Maxwell's equations (see margin) as follows:

- Dot multiply Faraday's law by \mathbf{H} , dot multiply Ampere's law by \mathbf{E} ,

$(\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}) \cdot \mathbf{H}$ ✓
 $(\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot \mathbf{E}$ ✓

$\nabla \cdot \mathbf{D} = \rho$
 $\nabla \cdot \mathbf{B} = 0$
 $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
 $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

and take their difference:

$\underbrace{\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}}_{\nabla \cdot (\mathbf{E} \times \mathbf{H})} = \underbrace{-\frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{H}}_{-\frac{\partial}{\partial t}(\frac{1}{2}\epsilon \mathbf{E} \cdot \mathbf{E} + \frac{1}{2}\mu \mathbf{H} \cdot \mathbf{H})} - \mathbf{J} \cdot \mathbf{E}$

$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$
 $\vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$

- After re-arrangements shown above, the result can be written as

$$\checkmark \quad \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon \mathbf{E} \cdot \mathbf{E} + \frac{1}{2} \mu \mathbf{H} \cdot \mathbf{H} \right) + \underbrace{\nabla \cdot (\mathbf{E} \times \mathbf{H})}_{\text{energy transportation}} + \underbrace{\mathbf{J} \cdot \mathbf{E}}_{\text{power dissipation}} = 0.$$

storage.
energy transportation
power dissipation.

- **Poynting theorem** derived above is a *conservation law* just like the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$:

Poynting theorem

- The first term on the left, charge transporter

$$\mathbf{E} \rightarrow \mathbf{J}$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon \mathbf{E} \cdot \mathbf{E} + \frac{1}{2} \mu \mathbf{H} \cdot \mathbf{H} \right),$$

is time rate of change of total electric and magnetic **energy** density.

Hence, **Poynting theorem is the conservation law for electromagnetic energy**, just like continuity equation is the conservation law for electric charge.

- The second term

$$\nabla \cdot (\mathbf{E} \times \mathbf{H})$$

accounts for energy transport in Poynting theorem, just like $\nabla \cdot \mathbf{J}$ accounts for charge transport in the continuity equation. Therefore

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$$

is **energy flux per unit area** measured in

$$\frac{\text{V A}}{\text{m m}} = \frac{\text{W}}{\text{m}^2} = \frac{\text{J/s}}{\text{m}^2}$$

units, just like \mathbf{J} is charge flux per unit area in $\frac{\text{C/s}}{\text{m}^2} = \frac{\text{A}}{\text{m}^2}$ units.

- Finally, the last term in Poynting theorem (repeated in the margin),

$$\mathbf{J} \cdot \mathbf{E}$$

is called **Joule heating**, and it represents power absorbed per unit volume (which can only be non-zero in the presence of \mathbf{J}).

If $\mathbf{J} \cdot \mathbf{E}$ is negative in any region, then \mathbf{J} in that region is acting as a source of electromagnetic energy, just like any circuit component with negative vi is acting as an energy source in the electrical circuit.

Note that $\mathbf{J} \cdot \mathbf{E}$ had a negative value on the current sheet radiator examined in last lecture. We return to the current sheet radiator in the next example.

Poynting thm:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon \mathbf{E} \cdot \mathbf{E} + \frac{1}{2} \mu \mathbf{H} \cdot \mathbf{H} \right) + \nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{J} \cdot \mathbf{E} = 0$$

Example 1: On $z = 0$ plane we have a time-harmonic surface current specified as

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \frac{\text{A}}{\text{m}}$$

where ω is some frequency of oscillation.

- (a) Determine the radiated TEM wave fields $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ in the regions $z \gtrless 0$,
- (b) The associated Poynting vectors $\mathbf{E} \times \mathbf{H}$, and
- (c) $\mathbf{J}_s \cdot \mathbf{E}$ on the current sheet.

Solution:

- (a) With reference to the solution of the current sheet radiator depicted in the margin (from last lecture), we that an x -polarized surface current $f(t)$ produces the wave fields

$$E_x = -\frac{\eta}{2} f(t \mp \frac{z}{v}) \quad \text{and} \quad H_y = \mp \frac{1}{2} f(t \mp \frac{z}{v})$$

in the surrounding regions propagating away from the current sheet on both sides. Given that $f(t) = 2 \cos(\omega t)$, this implies that

$$E_x = -\eta \cos(\omega t \mp \beta z) \quad \text{and} \quad H_y = \mp \cos(\omega t \mp \beta z)$$

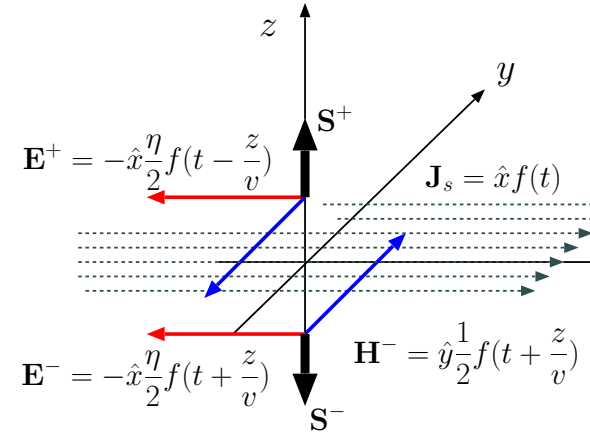
where

$$\beta = \frac{\omega}{c} \quad \text{and} \quad \eta = \eta_o \approx 120\pi \Omega$$

since the current sheet is surrounded by vacuum. Hence in vector form we have

$$\mathbf{E}(z, t) = -\eta \cos(\omega t \mp \beta z) \hat{x} \frac{\text{V}}{\text{m}} \quad \text{and} \quad \mathbf{H}(z, t) = \mp \cos(\omega t \mp \beta z) \hat{y} \frac{\text{A}}{\text{m}},$$

where the upper signs are for $z > 0$, and lower signs for $z < 0$.



(b) The associated Poynting vectors are

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \pm \eta \cos^2(\omega t \mp \beta z) \hat{z} \frac{W}{m^2}.$$

Note that the time-average value of vector \mathbf{S} points in the direction of wave propagation on both sides of the current sheet.

(c) Since on $z = 0$ surface of the current sheet the electric field vector is

$$\mathbf{E}(0, t) = -\eta \cos(\omega t) \hat{x} \frac{V}{m},$$

it follows that $\mathbf{J}_s \cdot \mathbf{E}$ on the same surface is

$$\mathbf{J}_s(t) \cdot \mathbf{E}(0, t) = (\hat{x} 2 \cos(\omega t) \frac{A}{m}) \cdot (-\eta \cos(\omega t) \hat{x} \frac{V}{m}) = -2\eta \cos^2(\omega t) \frac{W}{m^2}.$$

- In the above example, a time-harmonic source current oscillating at some frequency ω produced “monochromatic waves” of radiated fields propagating away from the current sheet on both sides.

- The calculations showed time-varying Poynting vectors $\mathbf{E} \times \mathbf{H}$. The time-averaged values of these time-varying vectors can be easily determined by making use of the trig identity

$$\cos^2(\omega t + \phi) = \frac{1}{2}[1 + \cos(2\omega t + 2\phi)].$$

Since the time-average of the second term on the right is zero, we

can express the time-average of this identity as

$$\langle \cos^2(\omega t + \phi) \rangle = \langle \frac{1}{2}[1 + \cos(2\omega t + 2\phi)] \rangle = \frac{1}{2},$$

where the angular brackets denote the time-averaging procedure.

- Consequently, the result

$$\mathbf{E} \times \mathbf{H} = \pm \eta \cos^2(\omega t \mp \beta z) \hat{z} \frac{W}{m^2}$$

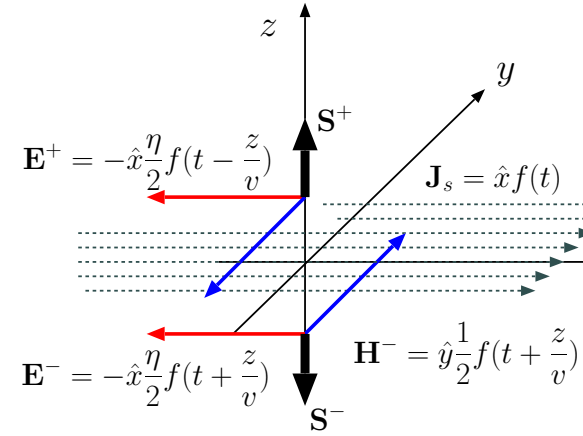
from Example 1 implies that

$$\langle \mathbf{E} \times \mathbf{H} \rangle = \pm \eta \frac{1}{2} \hat{z} \frac{W}{m^2} = \pm 60\pi \hat{z} \frac{W}{m^2},$$

which represent the time-average power per unit area transported by the waves radiated by the current sheet.

- In Poynting theorem the Joule heating term $\mathbf{J} \cdot \mathbf{E}$ is **power absorbed per unit volume**, and, accordingly, $-\mathbf{J} \cdot \mathbf{E}$ is **power injected per unit volume**.
 - Likewise, $\pm \mathbf{J}_s \cdot \mathbf{E}$ can be interpreted as **power absorbed/injected per unit area** on a surface.

In Example 1 above we calculated an instantaneous injected power density of



$$-\mathbf{J}_s \cdot \mathbf{E} = 2\eta \cos^2(\omega t) \frac{W}{m^2}.$$

Clearly, its time-average works out as

$$\langle -\mathbf{J}_s \cdot \mathbf{E} \rangle = \eta \frac{W}{m^2} = 120\pi \frac{W}{m^2}.$$

- Note that $\langle -\mathbf{J}_s \cdot \mathbf{E} \rangle$ exactly matches the sum of $|\langle \mathbf{E} \times \mathbf{H} \rangle|$ calculated on both sides of the current sheet, in conformity with energy conservation principle (Poynting theorem).