18 Wave equation and plane TEM waves in source-free media

With this lecture we start our study of the full set of Maxwell's equations shown in the margin by first restricting our attention to *homogeneous* and *non-conducting* media with constant ϵ and μ and zero σ.

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.
\end{aligned}$$

$$0 = \nabla \cdot \vec{J} + \frac{\partial}{\partial t} \underbrace{\nabla \cdot \vec{D}}_{\rho}$$

- Our first objective is to show that non-trivial (i.e., non-zero) time-varying field solutions of these equations can be obtained even in the absence of ρ and $\mathbf{J}$.
  - We already know *static* ρ and $\mathbf{J}$ to be the **source** of *static* electric and magnetic fields.
  - We will come to understand that time varying ρ and $\mathbf{J}$, which necessarily obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

constitute the **source** of *time-varying electromagnetic* fields.

Despite these intimate connections between the sources ρ and $\mathbf{J}$ and the fields

$$\mathbf{D} = \epsilon \mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu \mathbf{H},$$

non-trivial field solutions can exist in source-free media as we will see shortly.

- Such field solutions in fact represent electromagnetic waves, a familiar example of which is **light**.

- Another example is **radiowaves** that we use when we communicate using wireless devices such as radios, cell-phones, WiFi, etc.

- Different types of electromagnetic waves are distinguished by their oscillation frequencies, and include

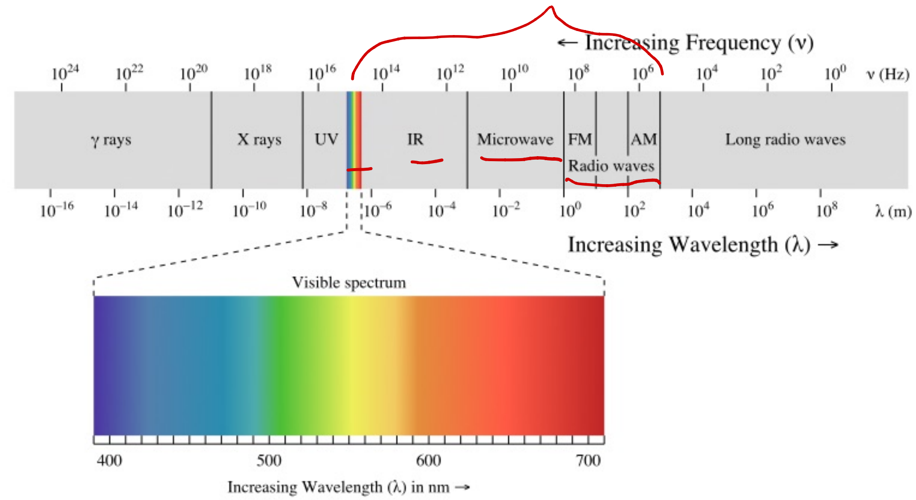
  - radiowaves,
  - microwaves,
  - infrared,
  - light,

  EE

  - ultraviolet,
  - X-rays, and gamma rays,

  physicists

EE.



going across the **electromagnetic spectrum** from low to high frequencies.

We are well aware that these types of electromagnetic waves can travel across empty regions of space — e.g., from sun to Earth — transporting energy and heat as well as momentum.

  - Next, we will discover their general properties by examining Maxwell's equations under the restriction $\rho = \mathbf{J} = 0$.

- In source-free and homogeneous regions where $\rho = \mathbf{J} = 0$ and ϵ and μ are constant, we can simplify Maxwell's equations as shown in the margin.
  - If there are non-trivial solutions of these equations, namely $\mathbf{E}(\mathbf{r}, t) \neq 0$ and $\mathbf{H}(\mathbf{r}, t) \neq 0$, they evidently need to be divergence-free.
  - They also have to be "curly" according to the last two equations: Faraday's and Ampere's laws.

$\nabla \cdot \vec{D} = 0$ $\quad \rho = 0.$

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{H} &= 0 \\
\nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\
\nabla \times \mathbf{H} &= \epsilon \frac{\partial \mathbf{E}}{\partial t}.
\end{aligned}
$$

$\nabla \cdot \vec{B} = 0$ $\quad -\frac{\partial B}{\partial t}$ $\quad \underbrace{\vec{J}}_{=0} + \frac{\partial \vec{D}}{\partial t}.$

$\nabla\times$ $\quad \nabla\times$

$\vec{D} = \varepsilon \vec{E}$

$\vec{B} = \mu \vec{H}.$

- Next we will make use of vector identity

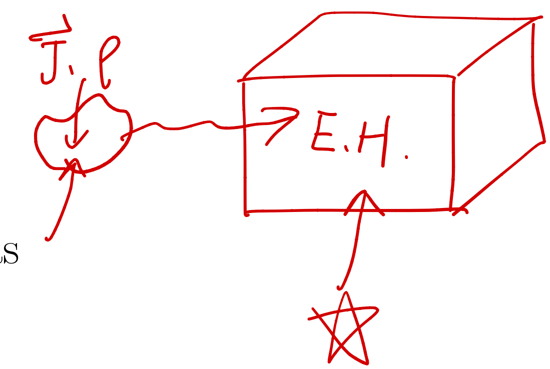
$$
\nabla \times (\nabla \times \mathbf{E}) = \nabla(\cancel{\nabla \cdot \mathbf{E}}) - \nabla^2 \mathbf{E} \qquad (\nabla \cdot \mathbf{E} = 0)
$$

which should be familiar from an earlier homework problem.

  - Since the electric field $\mathbf{E}$ is divergence-free in the absence of sources, this identity simplifies as

$$
\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}
$$

where in the right side $\nabla^2 \mathbf{E}$ is the Laplacian of $\mathbf{E}$.



  - Using this result we can express the curl of Faraday's law as

$$
\nabla \times [\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}] \;\Rightarrow\; -\nabla^2 \mathbf{E} = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H},
$$

$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}.$

which combines with the Ampere's law to produce

$$
\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2},
$$

$\nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 H}{\partial t^2}$ $\quad \nabla \times \nabla \times \vec{H} = \varepsilon \frac{\partial}{\partial t} \nabla \times \vec{E}$

which can be written explicitly as $\vec{E}(x, y, z, t)$.

3D vector wave equation

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(E_x, E_y, E_z)

Recall that our objective is to see whether a non-trivial time-varying solution of Maxwell's equations can exist in source-free media.

Our objective at this stage is not finding a general solution; it is instead identifying a simple example of a non-trivial time-varying $\mathbf{E}(\mathbf{r}, t)$, if we can.

For example, can a field solution

$$\mathbf{E}(\mathbf{r}, t) = \hat{x} E_x(z, t)$$

that only depends on z and t and "polarized" in x-direction exist? If it can exist, what would be the properties of this x-**polarized** solution?

E field direction → polarized

- To find out, we note that with $\mathbf{E} = \hat{x} E_x(z, t)$, the above "wave equation" is reduced to

1D scalar wave equation

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2},$$

$\mu_0 \varepsilon_0 = \frac{1}{c^2}$

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2}}$$

E_x ?

an equation that is known as a **1D scalar wave equation**, as opposed to the **3D *vector* wave equation** above.

  – Now, by substitution, we can easily show that

$$E_x = \cos(\omega(t - \sqrt{\mu\epsilon} z)),$$

$+z$.

$\cos(\omega(t - \frac{z}{c}))$.

$\cos(\omega(t - \frac{z}{v}))$

$\omega = 2\pi f$.

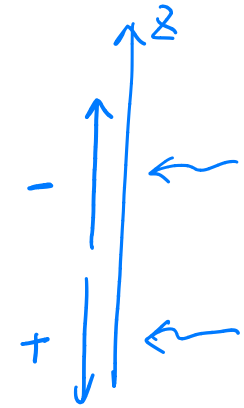
$v = \frac{1}{\sqrt{\mu\varepsilon}}$.

$c = 3\times10^8$ m/s.

satisfies the 1D wave equation and represents an x-polarized time-periodic field solution with an **oscillation frequency** ω.

– 1D wave equation can also be satisfied by

$$E_x = \cos(\omega(t + \sqrt{\mu\epsilon} z)).$$ $-z$ direction



Let us jointly refer to these solutions as

$$E_x = \cos(\omega(t \mp \frac{z}{v})),$$

where

$$v \equiv \frac{1}{\sqrt{\mu\epsilon}}$$

has the dimensions of m/s (i.e., velocity) and the algebraic signs $\mp$ distinguish between the "travel directions" of these *possible* "wave solutions" as elaborated later on.

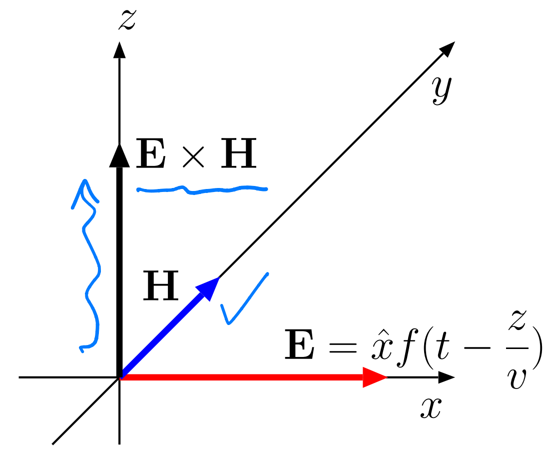
- Let us next find out the magnetic field intensity **H** that accompanies the x-polarized electric field wave solution

$$\checkmark \quad \mathbf{E} = \hat{x} \cos(\omega(t \mp \frac{z}{v})).$$

– Since the curl of **E** is

$$\vec{B} = \mu \vec{H}$$

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{y}\frac{\partial E_x}{\partial z} = \pm \hat{y} \sin(\omega(t \mp \frac{z}{v}))\frac{\omega}{v}, = -\frac{\partial \vec{B}}{\partial t}.$$



Faraday's law

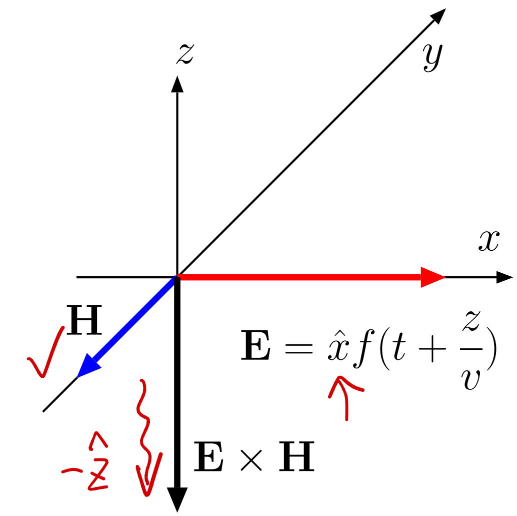
$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

requires that **H** should satisfy

$$-\mu \frac{\partial \mathbf{H}}{\partial t} = \pm \hat{y} \sin(\omega(t \mp \frac{z}{v}))\frac{\omega}{v}.$$

Finding the *time-dependent* anti-derivative (and remembering $v = 1/\sqrt{\mu\epsilon}$), we obtain

$$\mathbf{H} = \pm \hat{y} \sqrt{\frac{\epsilon}{\mu}} \cos(\omega(t \mp \frac{z}{v})).$$



- The results above, namely our x-polarized non-trivial field solutions of Maxwell's equations in source-free homogeneous space, can be represented more compactly as

$$\mathbf{E} = \underbrace{\hat{x} f}_{E_x}(t \mp \frac{z}{v}) \quad \text{and} \quad \mathbf{H} = \pm \hat{y} \frac{f(t \mp \frac{z}{v})}{\eta},$$

where

$$f(t) \equiv \cos(\omega t) = \mathrm{Re}\{e^{j\omega t}\} = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

is the field **waveform**,

$$\eta \equiv \sqrt{\frac{\mu}{\epsilon}} = \frac{|\vec{E}|}{|\vec{H}|} \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\,\Omega.$$

is known as **intrinsic impedance** (and measured in units of ohms).

$$E_x(z,t) = f(t \mp \frac{z}{c}) \rightarrow \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

$f(u)$, $u = t \mp \frac{z}{c}$; $f(t \mp \frac{z}{v})$; $\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 E_x}{\partial t^2}$; $\frac{1}{v^2} = \epsilon\mu$

$$\frac{\partial E_x}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} = f'(t \mp \frac{z}{c})(\mp \frac{1}{c})$$

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial z}\right) = f''(t \mp \frac{z}{c})(\mp \frac{1}{c})^2$$

$$\frac{\partial}{\partial u}\left(\frac{\partial E_x}{\partial z}\right)\frac{\partial u}{\partial z}$$

$$\frac{\partial E_x}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = f'(t \mp \frac{z}{c}) \cdot 1$$

$$\frac{\partial^2 E_x}{\partial t^2} = f''(t \mp \frac{z}{c}) \cdot 1$$

- Since Maxwell's equations with constant μ and ϵ are linear and time-invariant (LTI), the field solutions above can be further generalized by using their weighted and time-shifted superpositions such as

  $$f(t) = \sum_n A_n \cos(\omega_n t + \theta_n)$$

  and

  $$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

  having frequency dependent weighting factors A_n and $F(\omega)$. And since according to Fourier analysis all practical signals $f(t)$ can be synthesized in these forms, it follows that the field solutions above are valid with *arbitrary* waveforms $f(t)$.

d'Alembert wave solutions

Solutions

$f(t)$

$$\mathbf{E}, \mathbf{H} \propto f(t \mp \frac{z}{v})$$

of the 1D scalar wave equation with arbitrary $f(t)$ are known as **d'Alembert wave solutions**.

- d'Alembert solution

  $$\mathbf{E}, \mathbf{H} \propto f(t - \frac{z}{v})$$

  describes electromagnetic waves traveling in $+z$ direction, whereas solution

  $$\mathbf{E}, \mathbf{H} \propto f(t + \frac{z}{v})$$

describes electromagnetic waves traveling in $-z$ direction (see margin).

In each case the travel speed is

$$v = \frac{1}{\sqrt{\mu\epsilon}} \xrightarrow{\text{free space}} \frac{1}{\sqrt{\mu_o\epsilon_o}} \equiv c \approx 3 \times 10^8 \,\mathrm{m/s}.$$

- **H** solution can be obtained from **E** by dividing it with η and rotating it by 90° so that vector $\mathbf{E} \times \mathbf{H}$ points in direction the waves travel.
- **E** can be obtained from **H** by multiplying it with η and rotating it by 90° so that vector $\mathbf{E} \times \mathbf{H}$ — called **Poynting vector** — once again points in direction the waves travel.

In each case the intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \xrightarrow{\text{free space}} \sqrt{\frac{\mu_o}{\epsilon_o}} \equiv \eta_o \approx 120\pi \,\text{ohms}.$$

Transformation rules above also hold for y**-polarized** wave solutions

$$\mathbf{E} = \hat{y} f(t \mp \frac{z}{v}) \quad \text{and} \quad \mathbf{H} = \mp\hat{x}\frac{f(t \mp \frac{z}{v})}{\eta}.$$

Question: What about z**-polarized** waves

$$\mathbf{E} = \hat{z} f(t \mp \frac{z}{v}),$$

can they exist?

Answer: No, z-polarized waves $\hat{z}f(t \mp \frac{z}{v})$ traveling in $\pm z$ direction cannot exist because they would violate the divergence-free condition $\nabla\cdot\mathbf{E} = 0$.