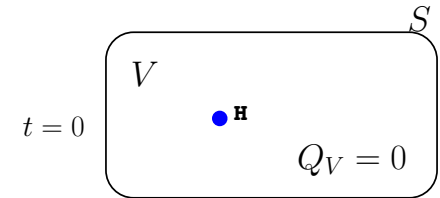


16 Charge conservation, continuity eqn, displacement current, Maxwell's equations

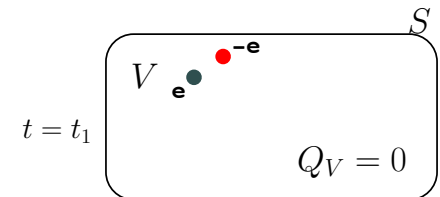
- **Total electric charge is *conserved*** in nature in the following sense: if a process generates (or eliminates) a positive charge, it always does so as accompanied by a negative charge of equal magnitude.
 - Example: **Photoionization** of atoms and molecules can generate free positive ions and free negative electrons in pairs (see margin figure). Photoionization is a process that converts bound charge carriers into free charge carriers.
 - Example: **Recombination** when a positive ion and an electron get together to produce a charge neutral atom or molecule.
 - Example: **Annihilation** of an electron (negative charge) by a positron (positive charge of equal magnitude) and the reverse process of **pair creation**.

As a consequence, if the total electric charge Q_V contained in any finite volume V changes as a function of time, this change must be attributed to a net transport of charge, i.e., electric current, across the bounding surface S of volume V as detailed below.

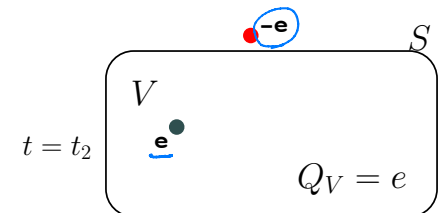
(a) At $t=0$ volume V contains a neutral atom but no net charge



(b) At $t=t_1$ volume V contains a proton and a free electron after the ionization of the hydrogen atom. There is still no net charge in the volume.



(c) At $t=t_2$ volume V now contains only a proton after the exit of free electron through surface S . Now V contains a net charge e .



- Consider two distinct surfaces S_1 and S_2 bounded by the same closed loop C (as shown in the margin) such that a volume V is contained between the two surfaces.

– Let

$$I_1 = - \int_{S_1} \mathbf{J} \cdot d\mathbf{S}_1$$

and

$$I_2 = \int_{S_2} \mathbf{J} \cdot d\mathbf{S}_2$$

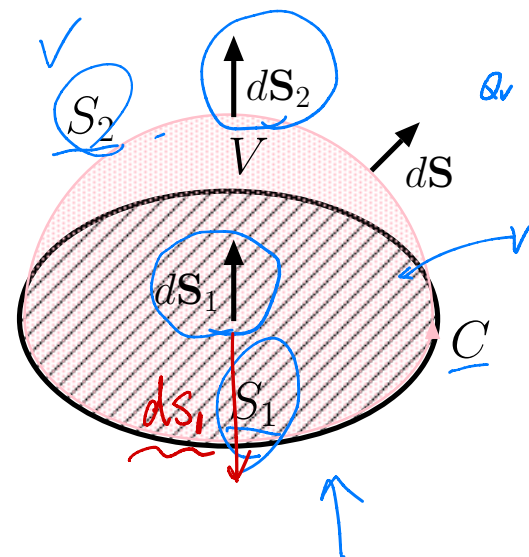
denote currents flowing through surfaces S_1 and S_2 , respectively.

- Note that current I_1 through surface S_1 enters volume V , while current I_2 through surface S_2 exits volume V (with the directions assigned to $d\mathbf{S}_1$ and $d\mathbf{S}_2$).
- If $I_1 \neq I_2$, then **current out** is not matched by the **current in**, and as a result, the net charge Q_V contained in volume V increases with time at a rate $I_1 - I_2$ provided that charge is *conserved* in the sense discussed above. In that case, we have

$$\frac{dQ_V}{dt} = I_1 - I_2.$$

This relationship can be expressed as

$$\frac{d}{dt} \int_V \rho dV = \int_{S_1} \mathbf{J} \cdot d\mathbf{S}_1 - \int_{S_2} \mathbf{J} \cdot d\mathbf{S}_2$$



since, in terms of charge density ρ , charge in volume V is

$$Q_V = \int_V \rho dV.$$

The expression can also be cast as

$$\frac{\partial \rho}{\partial t} \rightsquigarrow \int_V \frac{\partial \rho}{\partial t} dV = - \oint_S \mathbf{J} \cdot d\mathbf{S} \quad - \nabla \cdot \vec{J}$$

where S is the union of surfaces S_1 and S_2 enclosing V , and $d\mathbf{S}$ is an outward area element of S (see margin). This relationship is known as continuity equation. Its *differential form* is

Continuity
equation

$$\rightsquigarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}, \quad \checkmark \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

which follows from the *integral form* above as a consequence of **divergence theorem** (recall Lecture 4).

Continuity equation is a mathematical re-statement of the principle of conservation of charge.

- While **Faraday's law**

$$\checkmark \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \checkmark$$

indicates that time-varying $\mathbf{B}$ induces time-varying electric fields $\mathbf{E}$,
Ampere's law, written as

$$\nabla \cdot \left(\checkmark \nabla \times \mathbf{H} = \mathbf{J}, \right) \checkmark \quad \nabla \cdot \vec{\mathbf{J}} = 0 \quad \times \quad \nabla \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t}$$

makes no such claim about a time-varying $\mathbf{E}$ inducing a time-varying

$$\leadsto \mathbf{B} = \mu_0 \mathbf{H} \quad \nabla \cdot \vec{\mathbf{B}} = 0 \quad \nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

- This “asymmetry” was noted by James Clerk Maxwell who realized that the form of Ampere's law given above must be “incomplete” under time-varying situations.
- Noting the inconsistency of Ampere's law with the continuity equation under time varying conditions, he re-wrote the Ampere's law as

$$\checkmark \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \checkmark$$

in 1861 by adding the term on the right which is now called the “displacement current”.

- Maxwell postulated that the displacement current term is needed in Ampere's law because only then the divergence of Ampere's law avoids falling into conflict with charge conservation (under time varying conditions).

**Revised
Ampere's
law (with
“displacement
current”)**

Verification of Maxwell's claim: Since $\nabla \times \mathbf{H}$ is divergence-free (just like the curl of vector potential $\mathbf{A}$, namely $\mathbf{B}$), it follows that the divergence of *Maxwell's* modified Ampere's law — often called **Ampere-Maxwell equation** — is

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = 0. \quad \nabla \cdot \left(\frac{\partial \mathbf{D}}{\partial t} \right) \quad \nabla \cdot \vec{D} = \rho$$

- In the absence of the second term due to displacement current, this results would be inconsistent with the continuity equation

$$\checkmark \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

unless $\frac{\partial \rho}{\partial t} = 0$ (the static case).

- By contrast, including the second term, the result above is recognized as the continuity equation per se, since by Gauss's law — assuming that it applies with no change under time varying situations —

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = \frac{\partial \rho}{\partial t}.$$

- The modified Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

postulated by Maxwell under the assumption that Gauss's law is also valid under time-varying conditions, leads to some specific predictions about how time-varying fields should behave.

- These predictions — concerning the propagation of electromagnetic waves — were validated experimentally by Heinrich Hertz around 1888.

- The experiments confirmed that time-varying electric and magnetic fields obey collectively (and at microscopic scales) the differential relations

$$\begin{cases}
 \checkmark \nabla \cdot \mathbf{D} = \rho \checkmark & \text{Gauss's law} \\
 \rightsquigarrow \nabla \cdot \mathbf{B} = 0 \\
 \rightsquigarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \text{Faraday's law} \\
 \checkmark \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, & \text{Ampere's law}
 \end{cases}$$

where

$$\checkmark \mathbf{D} = \epsilon_0 \mathbf{E} \text{ and } \mathbf{B} = \mu_0 \mathbf{H}$$

provided that ρ and $\mathbf{J}$ describe the distributions of all charges and currents associated with free and bound charge carriers¹. Alternatively, the same differential relations — known collectively as Maxwell's equations — are also valid for macroscopic fields, provided that ρ and $\mathbf{J}$ describe only the free charge contributions and

$$\rightsquigarrow \mathbf{D} = \epsilon \mathbf{E} \text{ and } \mathbf{B} = \mu \mathbf{H} \checkmark \quad \vec{J} = \sigma \vec{E}$$

$$\begin{aligned}
 -\frac{\partial}{\partial t} \nabla \cdot \vec{B} &= 0. \\
 \nabla \cdot \vec{B} &= 0. \checkmark
 \end{aligned}$$

Maxwell's
equations

$$\begin{aligned}
 \nabla \cdot \vec{J} + \frac{\partial}{\partial t} \nabla \cdot \vec{D} \\
 \nabla \cdot \vec{J} &= -\frac{\partial \rho}{\partial t} \checkmark \\
 -\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t} \nabla \cdot \vec{D} &= 0. \\
 \checkmark \rho &= \nabla \cdot \vec{D}
 \end{aligned}$$

¹In the classical domain, down to scales of about $\hbar/mc$, the Compton wavelength — at shorter scales quantized and generalized versions (known as *electroweak* theory) are needed.

in terms of suitably defined permittivities and permeabilities ϵ and μ — see next Lecture.

- The *unnamed* Maxwell equation

$$\nabla \cdot \mathbf{B} = 0$$

can be viewed to be a consequence of Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and the fact that magnetic monopoles have never been observed.

Explanation: Since $\nabla \times \mathbf{E}$ is divergence-free, taking the divergence of Faraday's law, we get

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0.$$

This constraint requires $\nabla \cdot \mathbf{B}$ to be an invariant scalar at all locations in space. As a consequence, if $\nabla \cdot \mathbf{B} = 0$ at some instant in time, it should remain so at all times. Given that $\nabla \cdot \mathbf{B} = 0$ for static fields, this relationship must also continue to be valid when $\mathbf{B}$ starts changing with time.

The fact that $\nabla \cdot \mathbf{B}$ remains fixed at a zero value everywhere, whereas $\nabla \cdot \mathbf{D}$ varies like ρ , is in fact a consequence of the fact that there appears to be no *magnetic charges* (monopoles) in nature. Had there been “point charges for

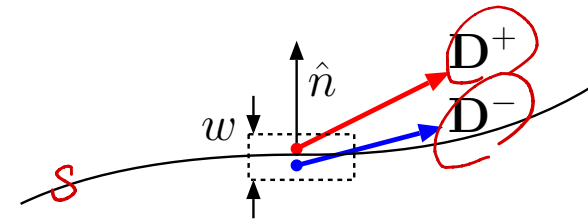
magnetic fields” in nature, $\nabla \cdot \mathbf{B}$ would have equaled the density of those charges, and magnetic field lines would have started and stopped on them (rather than looping into themselves). But no one has observed of any evidence for such magnetic charges *anywhere*, even going back to the very early times in the history of the universe (accessible by making observations of very far astronomical objects). So, $\nabla \cdot \mathbf{B} = 0$.

- Finally, the full set of Maxwell’s *boundary condition equations* concerning any interface with a normal unit vector $\hat{n}$ are

$$\begin{array}{ll}
 \nabla \cdot \vec{D} = \rho & \checkmark \hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = \rho_s \leftarrow \\
 \nabla \cdot \vec{B} = 0 & \Delta \hat{n} \cdot (\mathbf{B}^+ - \mathbf{B}^-) = 0 \\
 \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & \checkmark \hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) = 0 \\
 \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} & \checkmark \Delta \hat{n} \times (\mathbf{H}^+ - \mathbf{H}^-) = \vec{J}_s \leftarrow
 \end{array}$$

$\nabla \rightarrow \hat{n}$
 $\cdot \times \checkmark$
 $\vec{D} \rightarrow \vec{D}^+ - \vec{D}^-$
 $\rho, \vec{J} \rightarrow \rho_s, \vec{J}_s$
 $\frac{\partial}{\partial t} \times$

- We had already seen how the first and third boundary condition equations arise.
- The second boundary condition equation concerning the normal component of $\mathbf{B}$ is another consequence of the absence of magnetic charges (see margin).
- A detailed justification of the last boundary condition concerning tangential $\mathbf{H}$ will be given explicitly during Lecture 19. This equation allows a discontinuous change in the tangential component of $\mathbf{H}$ if the interface contains a non-zero surface current $\mathbf{J}_s$.



Constraint

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot d\mathbf{S}$$

around the dotted path yields

$$\hat{n} \times (\mathbf{H}^+ - \mathbf{H}^-) = \mathbf{J}_s$$

in $w \rightarrow 0$ limit.

Constraint

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

applied over the dotted volume (seen in profile) yields

$$\checkmark B_n^+ - B_n^- = 0$$

in $w \rightarrow 0$ limit.