11 Lorentz-Drude models for conductivity and susceptibility and polarization current

In this lecture we will describe simple microscopic models for conductivity σ and electric susceptibility χ_e of material media composed of free and bound charge carriers. The models were first developed by Lorentz and Drude prior to the establishment of quantum mechanics. In these models free charge carriers motions are described using Newtonian dynamics and atoms are represented as electric dipoles $\mathbf{p} = -e\mathbf{r}$ ($\mathbf{r}$ is electron displacement from atomic nucleus) behaving like damped 2nd order systems.

Conductivity: Conducting materials such as copper, sea water, ionized gases (plasmas) contain a finite density N of mobile and free charge carriers at the microscopic level (in addition to neutral atoms and molecules sharing the same macroscopic space) — these elementary mobile carriers can be electrons, positive or negative ions, or positive "holes" (in semi-conductor materials).

- Each elementary charge carrier with a charge q and mass m and subject to a macroscopic electrical force $q\mathbf{E}$ will be modelled by a dynamic equation

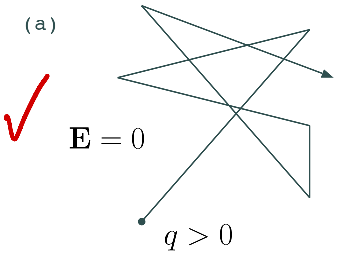
  ✓ $$m\frac{d\mathbf{v}}{dt} = \overbrace{q\mathbf{E} - m\frac{\mathbf{v}}{\tau}}^{\vec{F}},$$

  $$m\frac{d\vec{v}}{dt} + m\frac{\vec{v}}{\tau} = \cancel{q\vec{E}}. \qquad \vec{E}=0$$

  which is effectively Newton's second law — "force equals mass times acceleration" — in which $\mathbf{v}$ denotes the macroscopic velocity[1] of charge

  $$\text{DC:}\quad m\frac{\vec{v}}{\tau} = q\vec{E}.$$

✓



In the absence of an applied electric field E, free charge q exhibits a "random walk" between collisions such that its average velocity v is zero.

By collisions we refer to the inreactions of q with zero-mean miscroscopic electric fields within the conductor due to charges entrapped in the lattice.

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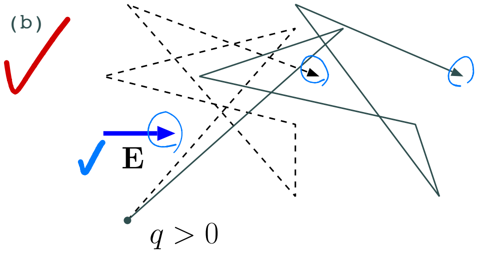

In the presence of an applied electric field E, the mean position of free charge q>0 drifts in the direction of field vector E with some non-zero mean velocity v.

Avg. drift velocity v reperesents a balance between acceleration force due to E and an opposing friction force produced by collisions of q with the lattice at random intervals with some mean value τ .

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[1] Think of *microscopic* velocity of each charge carrier as $\mathbf{v} + \delta\mathbf{v}$, where $\delta\mathbf{v}$ is an independent zero-mean

carriers and $-m\frac{\mathbf{v}}{\tau}$ denotes a macroscopic friction force proportional to $-\mathbf{v}$. Friction is a consequence of "collisions" of charge carriers with the neutral background at a frequency of $\nu = \frac{1}{\tau}$ collisions per unit time, and causes the decay of $\mathbf{v}$ as

$$✓ \quad \mathbf{v}(t) = \mathbf{v}(0)e^{-t/\tau}$$

in the absence of field $\mathbf{E}$. Therefore, when $\mathbf{E} = 0$ the carriers settle down to a steady state with $\mathbf{v} = 0$ (in $t \gg \tau$ limit), meaning that no macroscopic current density $\mathbf{J}$ will be found in the absence of $\mathbf{E}$ in regions with homogeneous charge carrier densities.

- With a constant but non-zero $\mathbf{E}$, steady-state solution of the above equation is

$\frac{1}{\tau} = \nu$

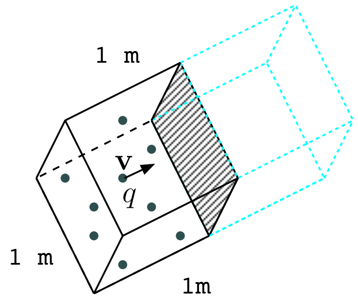
$$\rightsquigarrow ✓ \; \underline{\mathbf{v} = \frac{q\tau}{m}\mathbf{E}}, \quad \text{where } \left|\frac{q\tau}{m}\right| \text{ is known as } \underline{\textbf{mobility}}.$$

  - Assuming N charge carriers per unit volume each moving (on the average) with this steady-state velocity in a given material, we can calculate the average flux density of charge through the region as

$1/\text{m}^3 \quad C \quad \text{m/s}$

$$\rightsquigarrow \; \underline{\mathbf{J} = Nq\mathbf{v} = \frac{Nq^2}{m\nu}\mathbf{E}\,\frac{\text{C/s}}{\text{m}^2}} \sim \text{A/m}^2 \qquad \text{C/s} = \text{A}.$$

which is commonly referred to as **current density** (see margin). If a given material contains several species of carriers with charge,



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A cube of unity volume contains
N charge carriers q each moving
with an average  velocity of
v=1 m/s towards the shaded surface
of the cube of a unity surface area.

In 1 second all N charges in the
volume cross the shaded surface,
transporting Nq Coulombs of charge
per second per unit area.

By contrast, charge transport rate
across the same surface is
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$$Nqv\ \text{C/s/m}^2 = Nqv\ \text{A/m}^2$$

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if the average charge velocity
is an arbitrary v.

Thus, in vector notation, we can
define the current density in the
region as
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$$\mathbf{J} = Nq\mathbf{v}$$

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such that current I (in A) across
any surface S having area elements
dS is given by
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$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

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random variable for each charge carrier whereas *macroscopic* velocity $\mathbf{v}$ corresponds to the statistical average of all $\mathbf{v} + \delta\mathbf{v}$.

$\vec{J} = \underset{\sim}{\sigma}\vec{E}$

mass, collision frequency, and number density of q_s, m_s, ν_s, and N_s, respectively, then current density can be expressed as

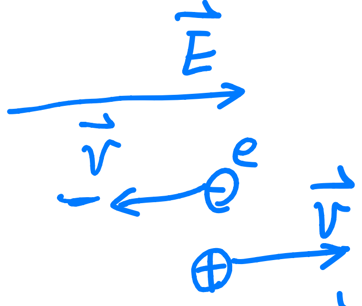
Ohm's Law and DC conductivity

$$\mathbf{J} = \sigma \mathbf{E},$$

with

$$\sigma = \sum_s \sigma_s \quad \text{and} \quad \sigma_s = \frac{N_s q_s^2}{m_s \nu_s}$$

denoting the medium and species conductivities, respectively, under DC conditions.



$I \propto qN(\vec{V}_i - \vec{V}_e)$

- With a time varying field $\mathbf{E}$ the corresponding current density will also be time varying, in which case conductivity σ should be defined in the frequency domain using phasor techniques (remember ECE 210).
  - Briefly, using phasors $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{J}}$ such that

$$\mathbf{E}(t) = \mathrm{Re}\{\tilde{\mathbf{E}}e^{j\omega t}\} \quad \text{and} \quad \mathbf{J}(t) = \mathrm{Re}\{\tilde{\mathbf{J}}e^{j\omega t}\}, \text{ etc.},$$

$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t).$

$\frac{d}{dt} \to j\omega.$

we have a phasor transformed Newton's force balance equation

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} - m\frac{\mathbf{v}}{\tau} \quad \Rightarrow \quad mj\omega\tilde{\mathbf{v}} = q\tilde{\mathbf{E}} - m\frac{\tilde{\mathbf{v}}}{\tau},$$ ✓

$\vec{v} = \mathrm{Re}\{\tilde{v}e^{j\omega t}\}.$

$(mj\omega + \frac{m}{\tau})\tilde{v} = q\tilde{E}$

$m\nu$

$\tilde{v} = \boxed{\frac{q\tilde{E}}{m(j\omega+\nu)}}$

from which it follows that

$$\tilde{\mathbf{J}} = \sigma\tilde{\mathbf{E}},$$

with

$$\sigma = \sum_s \sigma_s \quad \text{and} \quad \sigma_s = \frac{N_s q_s^2}{m_s(\nu_s + j\omega)}. \checkmark \checkmark$$

ω↑ σ↓

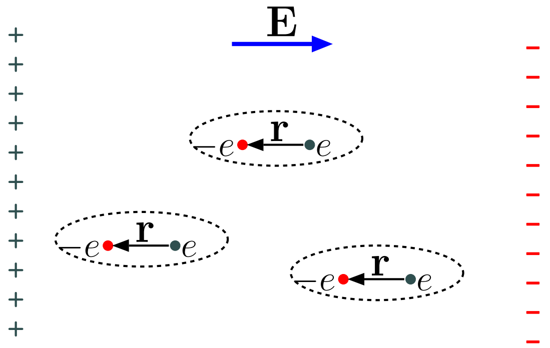
AC conductivity

- Note that the AC conductivity just derived can be approximated by the DC conductivity derived earlier for all AC frequencies ω much smaller than species collision frequencies ν_s.
  - In many cases of practical interest, this condition can be easily met, and we are often well justified to ignore the frequency dependence and complex character of conductivity σ revealed in above derivation.
- More advanced quantum mechanical derivations of σ_s give the same results except with *effective masses* specified by quantum theory replacing the particle masses m_s used in classical models.

- Typical DC conductivities:
  - For silver, copper, gold, $\sigma \sim$ several $\times\ 10^7$ S/m
  - For seawater $\sigma \approx 4$ S/m
  - For intrinsic silicon $\sigma = 1.6 \times 10^{-3}$ S/m
  - For dry earth $\sigma \sim 10^{-5}$ S/m
  - For glass $\sigma \sim 10^{-10} - 10^{-14}$ S/m

Superconductivity occurs in certain materials at low temperatures when the DC conductivity vanishes as a consequence of correlated charge carrier motions which are ignored in the Lorentz-Drude model.

Susceptibility: In perfect dielectrics there are no free charge carriers and so $\sigma = 0$. However, in general such materials are polarizable and therefore they have a non-zero susceptibility χ_e and a dielectric constant $\epsilon_r = 1 + \chi_e > 1$.



- In Lorentz-Drude model, each polarized atom or molecule is considered to be a dipole $\mathbf{p} = -e\mathbf{r}$, with $\mathbf{r}$ representing the displacement vector of an atomic electron from atomic nucleus when the atom is polarized because of an applied electric field.

- If the polarizing force on the atom is removed, observations indicate that the dipole field of the atom $\mathbf{E}_p \propto \mathbf{p} \propto \mathbf{r}$ will decay as a damped co-sinusoid with a decay time constant $\tau_d = \frac{1}{\alpha}$ and a characteristic damped frequency $\sqrt{\omega_o^2 - \alpha^2} \approx \omega_o$ satisfying a condition $\omega_o \gg \alpha = \frac{1}{\tau_d}$ (strongly underdamped).

  - Possible values of ω_o for a given atom can be obtained from the energy levels of bound states of the atom (calculated using standard quantum[2] models like in PHYS 214) and time constants $\tau_d = \frac{1}{\alpha}$ (which are finite because energies $\hbar\omega_o$ radiated away are also finite) are related to observed line widths (2α) in the emission spectra of excited atoms.

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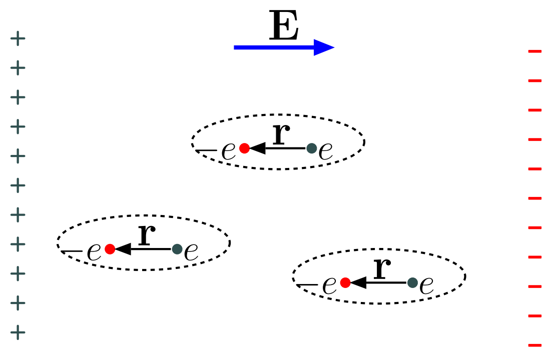
[2] For a quantum mechanical derivation of susceptibility, see, e.g., *Mott*, "Elements of Wave Mechanics", Chapt 4, Sect 10 (1958); *Miller*, "Quantum Mechanics for Scientists and Engineers", Sect 7.3 (2008).

Electron displacement having the inferred damped co-sinusoid form

$$\mathbf{r}(t) = \mathbf{r}_o e^{-t/\tau_d} \cos(\sqrt{\omega_o^2 - \alpha^2} t) \approx \mathbf{r}_o e^{-t/\tau_d} \cos(\omega_o t)$$

is “zero-input response” (remember ECE 210) of a linear second-order ODE that can be constructed using Newton’s second law of classical mechanics:



- If we assume that mass m times acceleration $\frac{d^2\mathbf{r}}{dt^2}$ of a displaced electron equals the sum of a
  - force $-e\mathbf{E}$ exerted by an applied macroscopic electric field $\mathbf{E}$,
  - a spring-like restoring force $-m\omega_o^2\mathbf{r}$ responsible for the binding of the electron to the nucleus, and
  - a friction-like dissipative force $-m2\alpha\frac{d\mathbf{r}}{dt}$,

  we get

$$m\frac{d^2\mathbf{r}}{dt^2} = -e\mathbf{E} - m\omega_o^2\mathbf{r} - m2\alpha\frac{d\mathbf{r}}{dt},$$

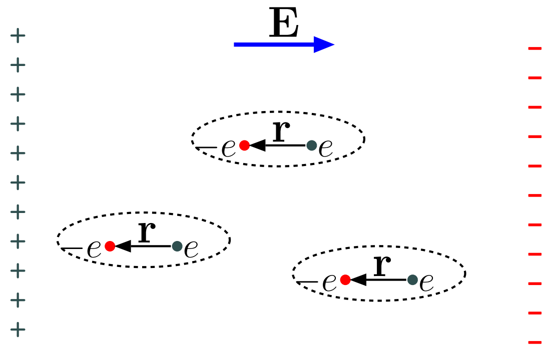
  for which $\mathbf{r}(t)$ given above is the zero-input solution in the absence of $\mathbf{E}$.

- To find the DC susceptibility of a dielectric composed of dipoles constrained by the above equation, we note that steady-state solution of the equation with a non-zero constant field $\mathbf{E}$ is

$$\mathbf{r} = -\frac{e}{m\omega_o^2}\mathbf{E}.$$

Consequently, dipole moment of a single polarized atom is

$$\mathbf{p} = -e\mathbf{r} = \frac{e^2}{m\omega_o^2}\mathbf{E},$$



and polarization field in a dielectric with a dipole density of N_d is

$$\mathbf{P} = N_d\mathbf{p} = \frac{N_d e^2}{m\omega_o^2}\mathbf{E}.$$

This result can also be written as

$$\mathbf{P} = \epsilon_o \chi_e \mathbf{E},$$

where

$$\chi_e \equiv \frac{N_d e^2/m\epsilon_o}{\omega_o^2}$$

DC susceptibility

is DC susceptibility. AC susceptibility can be derived using phasor techniques, but at frequencies $\omega \ll \omega_o$, AC susceptibility is well approximated by the DC susceptibility derived above.

Polarization current: Consider the case of a time varying electric field $\mathbf{E}(t)$ in a dielectric medium at a frequency $\omega \ll \omega_o$ such that the relations

$$\mathbf{r} = -\frac{e}{m\omega_o^2}\mathbf{E} \quad \text{and} \quad \mathbf{P} = \epsilon_o \chi_e \mathbf{E}$$

from above are accurate.

- Time variation of $\mathbf{E}$ will imply the time variation of electron displacement $\mathbf{r}$, so that there will be *in effect* a non-zero electron velocity

  $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\frac{e}{m\omega_o^2}\frac{d\mathbf{E}}{dt}$$

  capable of producing a current.

  - With N_d such electrons per unit volume, each carrying a charge $-e$, we will have a net flux density of charge in the region given by

    $$\mathbf{J}_p = -eN_d\mathbf{v} = \frac{N_d e^2}{m\omega_o^2}\frac{d\mathbf{E}}{dt} = \frac{d\mathbf{P}}{dt}.$$

    This flux is effectively an AC current density carried by bound charges found in a dielectric medium. Even though, a DC current is not possible in a perfect dielectric containing only bound charges, evidently AC currents

    $$\mathbf{J}_p = \frac{d\mathbf{P}}{dt} \quad \text{Polarization current density}$$

    are possible — we call this type of AC current as **polarization current density**.

Polarization current density

- In our studies of time-varying electromagnetic fields we will include the effects of polarization currents $\frac{d\mathbf{P}}{dt}$ along with the effects of conduction currents $\sigma\mathbf{E}$.