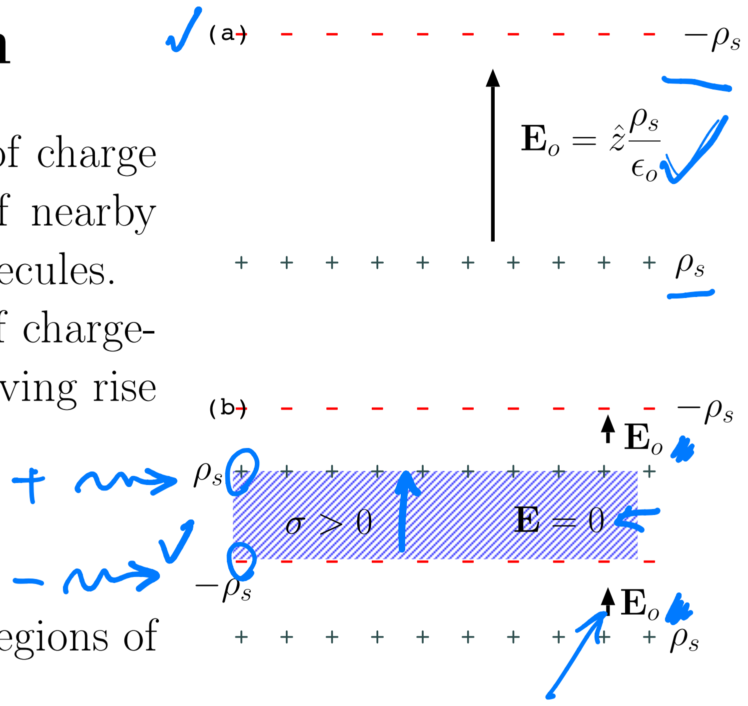
8 Conductors, dielectrics, and polarization

So far in this course we have examined static field configurations of charge distributions assumed to be fixed in free space in the *absence* of nearby materials (solid, liquid, or gas) composed of neutral atoms and molecules.

In the *presence* of material bodies composed of large number of charge-neutral atoms (in fluid or solid states) static charge distributions giving rise to electrostatic fields can be typically[1] found:

1. On exterior surfaces of *conductors* in "steady-state",
2. In crystal lattices occupied by *ionized* atoms, as in depletion regions of semiconductor junctions in diodes and transistors.

In this lecture we will examine these configurations and response of materials to applied electric fields.



A conducting slab inserted into a region with field E_o (as shown in b)develops surface charge which cancels out E_o within the slab.

E_o relates to surface charge as dictated by Gauss's law and superposition principle.

$\sim 10^{-18}$ s

DC: $\vec{E}=0$. $\nabla\cdot\vec{E}=0$.

inside. ρ/ϵ_0.

$\rho=0$.

Conductivity and static charges on conductor surfaces:

- **Conductivity** σ is an *emergent property* of materials bodies containing free charge carriers (e.g., unbound electrons, ionized atoms or molecules) which relates the applied *electric field* $\mathbf{E}$ (V/m) to the *electrical current density* $\mathbf{J}$ (A/m^2) conducted in the material via a linear

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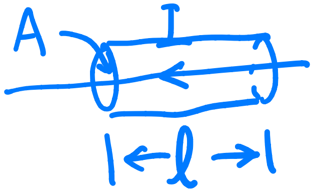
[1] More generally, materials containing charge carriers exhibiting divergence free flows will also exhibit static charge distributions.

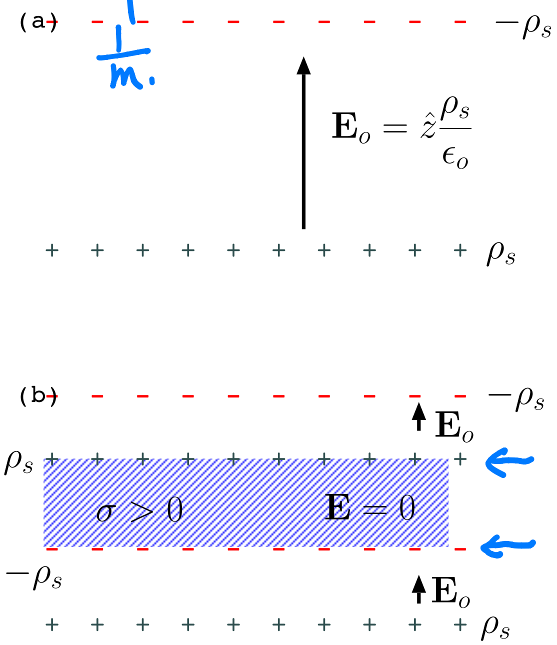
relation[2]

$\sigma \; [=] \; S/m.$ $$\mathbf{J} = \sigma \overset{\downarrow}{\mathbf{E}}. \quad \text{(Ohm's Law)}$$



$\underset{\Omega}{\underset{\uparrow}{R}} = \rho \cdot \dfrac{l}{\underset{\frac{1}{m}}{\underset{\uparrow}{A}}}. \qquad \Omega \cdot m \; [=] \; \rho.$

- Simple physics-based models for σ will be discussed later in Lecture 11. For now it is sufficient to note that:
  - $\sigma \to \infty$ corresponds to a *perfect electrical conductor*[3] (PEC) for which it is necessary that $\mathbf{E} = 0$ (in analogy with $V = 0$ across a short circuit element) independent of $\mathbf{J}$.
  - $\sigma \to 0$ corresponds to a *perfect insulator* for which it is necessary that $\mathbf{J} = 0$ (in analogy with $I = 0$ through an open circuit element) independent of $\mathbf{E}$.



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E_o relates to surface charge as dictated by Gauss's law and superposition principle.

- While (macroscopic) $\mathbf{E} = 0$ in PEC's *unconditionally*, a conductor with a *finite* σ (e.g., copper or sea water) will also have $\mathbf{E} = 0$ in "steady-state" after the decay of transient currents $\mathbf{J}$ that may be initiated within the conductor after applying an external electric field $\mathbf{E}_o$ (see margin).
  - The reason is, mobile free charges (e.g., electrons in metallic conductors) within the conductor will be pulled or pushed by the applied field $\mathbf{E}_o$ to pile up on exterior surfaces of the conductor

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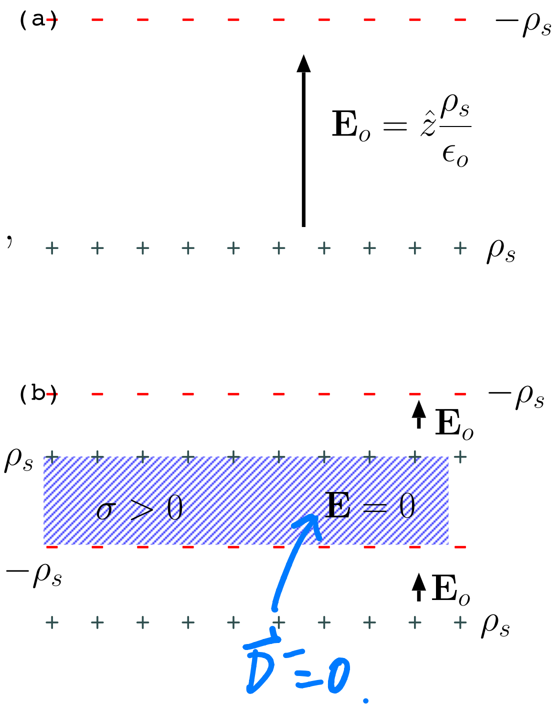
[2] Linear behavior is possible provided charge carriers suffer occasional collisions within the medium.

[3] PEC is an "idealization" that has no real counterpart, even though it is convenient to treat high conductivity materials such as copper as PEC in certain approximate models and calculations. For "superconducting materials" $\sigma \to \infty$ only in the low frequency limit.

until a surface charge density ρ_s that is generated produces a secondary field $-\mathbf{E}_o$ that exactly cancels out the applied $\mathbf{E}_o$ within the interior of the conductor.



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E_o relates to surface charge as dictated by Gauss's law and superposition principle.

- $\mathbf{E} = 0$ in the interior at steady-state implies that potential V =const., as well as $\rho = \nabla\cdot\mathbf{D} = \nabla\cdot\epsilon_o\mathbf{E} = 0$.
- Surface charge density ρ_s and the exterior field on a conductor surface will satisfy the boundary condition equations

external ↓ conductor ↓

$$\hat{n}\cdot(\vec{D}^+ - \cancel{\vec{D}}) = \rho_s$$

$$\hat{n}\cdot(\vec{E}^+ - \cancel{\vec{E}}) = 0$$

$$\rightsquigarrow \quad \hat{n}\cdot\mathbf{D} = \rho_s \quad \text{and} \quad \hat{n}\times\mathbf{E} = 0,$$

with $\hat{n}$ denoting the outward unit normal.

- The transient "time-constant" τ for the *decay* of charge density ρ (and hence $\mathbf{E}$, as claimed above) in a homogeneous[4] conductor (constant σ) can be obtained using the *continuity equation*

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

representing the mathematical statement of **charge conservation** (derived in Lecture 16). Using $\mathbf{J} = \sigma\mathbf{E}$ and $\nabla\cdot\mathbf{E} = \rho/\epsilon_o$, we have

$$\nabla\cdot\mathbf{J} = \sigma\nabla\cdot\mathbf{E} = \frac{\sigma}{\epsilon_o}\rho$$

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[4] See Fisher and Varney, *Am. J. Phys.*, **44**, 464 (1976), for a discussion of contact potential between different metals.

above, from which it follows that

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_o}\rho = 0 \ \text{ with a damped solution } \ \rho(t) = \rho(0)e^{-\frac{\sigma}{\epsilon_o}t}.$$

The decay time-constant

$$\tau = \frac{\epsilon_o}{\sigma}$$

is typically very short ($\sim 10^{-18}$ s) in metallic conductors, which is why such conductors are usually considered to be in steady-state (and have zero interior fields).

- **As a consequence:** in electrostatic[5] problems conducting volumes of materials (e.g., chunks of copper) can be treated as *equipotentials* having zero internal fields and finite surface charge densities $\rho_s = \hat{n} \cdot \mathbf{D}$ expressed in terms of external fields $\mathbf{D}$ normal to the surface.

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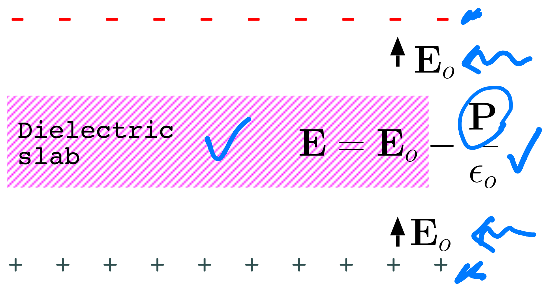
[5] Also applicable *quasi-statically* when externally applied fields $\mathbf{E}_o(t)$ change slowly with time-constants much longer than ϵ_o/σ. The way conductors are treated in high frequency *electromagnetic* problems will be described later on.

Dielectric materials and polarization:

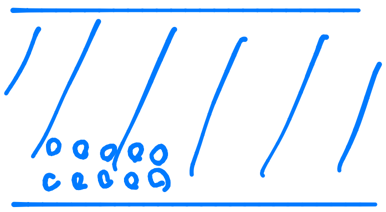
- **Dielectric materials** consist of a large number of charge-neutral atoms or molecules and *ideally* contain no mobile charge carriers (i.e., $\sigma = 0$).

- Electric fields produced by charges located outside or within a dielectric material will **polarize** the dielectric — meaning that its constituent atoms or molecules will be "stretched out" to expose their internal or "bound" charges, electrons and protons — which will in turn cause the electric field inside the dielectric to become *weaker* than (but not zero, as in conductors) what the field would have been in the absence of polarization effect.



`A dielectric slab inserted into a region with an initial field E_o will become polarized.`

`Inside the polarized dielectric the field will be weaker than E_o, but not reduced to zero as in a conductor.`



We will next examine this polarization process and see how Gauss's law can be re-stated to facilitate field calculations in dielectric materials containing **bound charge** carriers, i.e., atomic/molecular electrons and protons which are not free to drift away from one another indefinitely (neglecting possible ionization events).

- Consider a static **free-charge** density $\rho(z)$ that would produce a macroscopic field $\mathbf{E}_o$ satisfying $\rho = \epsilon_o \nabla \cdot \mathbf{E}_o$ in free space, producing, instead, a field $\mathbf{E} = \hat{z} E_z$ inside a dielectric medium composed of an array of neutral atoms or molecules.

Our objective is to relate the field E to $\mathbf{E}_o$ and ρ, and find a way of calculating E when ρ is given.

- In the presence of an electric field $\mathbf{E} = \hat{z}E_z$ in the dielectric each neutral atom of the medium will be in a distorted (but not ripped apart) state forming a $\hat{z}$ oriented **electric dipole**, which can be visualized as a proton-electron pair with a small proton displacement d in z direction with respect to the electron.
  - Consider a regular array of such dipoles

$$\mathbf{p} \equiv ed\hat{z},$$

with Δx, Δy, and Δz spacings between the dipoles (see margin), so that the volumetric dipole density is

$$✓\ N_d \equiv \frac{1}{\Delta x \Delta y \Delta z}\ \mathrm{m}^{-3},$$

Number density

within the array, and, furthermore,

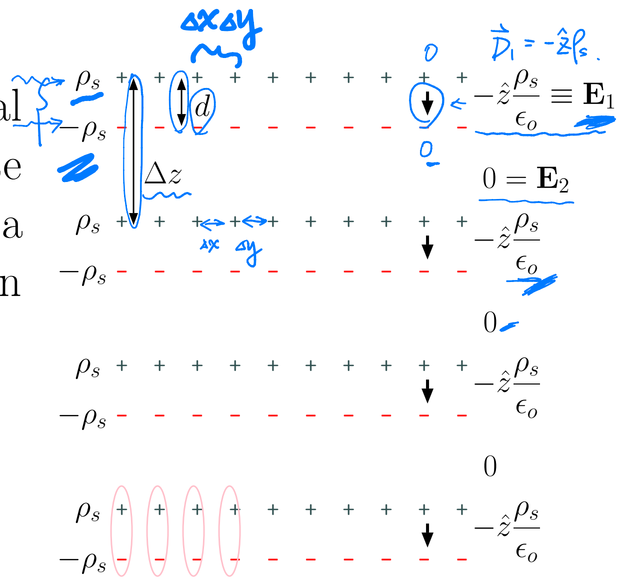
1.6×10^{-19} C.

$$✓✓\ \rho_s = \frac{e}{\Delta x \Delta y}\ \frac{\mathrm{C}}{\mathrm{m}^2}$$

is the magnitude of charge density of the adjacent proton and electron layers (see margin again) formed by arrays of adjacent dipoles displaced in z by intervals Δz.

  - Assuming that the array is infinite in extent in x and y directions, the proton and electron layers with surface charge densities $\pm\rho_s$ will produce interior electric fields

$$✓\ \mathbf{E}_1 = -\hat{z}\frac{\rho_s}{\epsilon_o} = -\hat{z}\frac{e/\epsilon_o}{\Delta x \Delta y}$$



$$\rho_s = \frac{e}{\Delta x \Delta y}$$

(pointing in opposite direction to $\mathbf{E} = \hat{z}E_z$), and exterior fields

$$\mathbf{E}_2 = 0$$

in between the dipole layers. Space averaged macroscopic electric field within the array (with a spatial weighting proportional to the size of regions with the fields $\mathbf{E}_1$ and $\mathbf{E}_2$) produced by the polarized dipoles will then be ✓ $\vec{E}_1 = -\hat{z}\frac{e/\epsilon_o}{\Delta x \Delta y}$

dipole moment

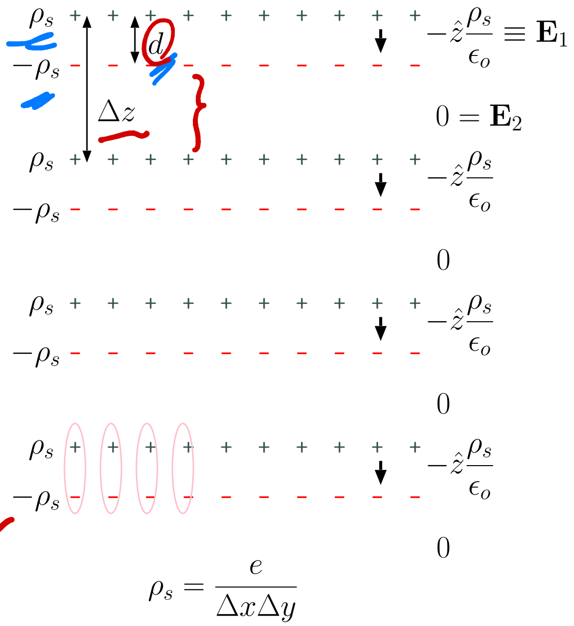
$$✓\ \mathbf{E}_p = \mathbf{E}_1 \frac{d}{\Delta z} + \mathbf{E}_2 \frac{\Delta z - d}{\Delta z} = -\hat{z}\frac{ed/\epsilon_o}{\Delta x \Delta y \Delta z} = -\frac{N_d e d \hat{z}}{\epsilon_o} = -\frac{\mathbf{P}}{\epsilon_o}, ✓$$

$$N_d = \frac{1}{\Delta x \Delta y \Delta z}.$$

where

$$✓\ \mathbf{P} \equiv N_d e d \hat{z} = N_d \mathbf{p}$$

is, by definition, macroscopic **polarization field** of the dielectric, measured in units of C/m^2 (same units as a surface charge density).

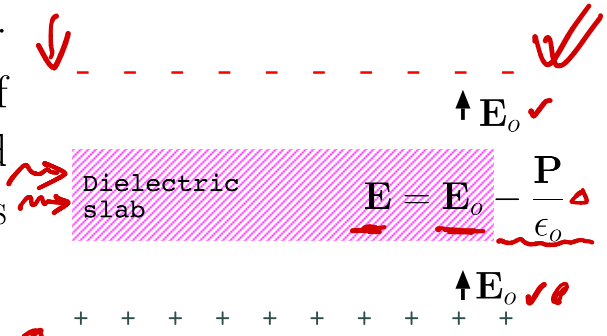


– The total macroscopic field $\mathbf{E}$ in the dielectric is then the sum of field $\mathbf{E}_o$ produced by the free charge density ρ in the region and the polarization field $\mathbf{E}_p = -\frac{\mathbf{P}}{\epsilon_o}$ produced by bound charge carriers of the neutral atoms and/or molecules of the dielectric, i.e.,

$$✓\ \mathbf{E} = \mathbf{E}_o - \frac{\mathbf{P}}{\epsilon_o}, \qquad \vec{P} \propto \epsilon \chi_e \vec{E}$$

$$\nabla\cdot\left(\epsilon_o\vec{E} = \epsilon_o\vec{E}_0 - \vec{P}\right)$$

a result that shows a “reduced field strength” $\mathbf{E}$ (compared to $\mathbf{E}_o$) inside the dielectric since $\mathbf{P}$ and $\mathbf{E}_o$ are colinear.



A dielectric slab inserted into a region with an initial field E_o will become polarized.

Inside the polarized dielectric the field will be weaker than E_o, but not reduced to zero as in a conductor.

$\nabla\cdot\vec{D}=\rho.$ $\nabla\cdot\varepsilon_0\vec{E}=\rho$

- To relate **E** directly to its ultimate cause ρ, we take the divergence of the above relation and use $\rho = \epsilon_o \nabla\cdot\mathbf{E}_o$ to find

$$\nabla\cdot(\epsilon_o\mathbf{E}) = \rho - \nabla\cdot\mathbf{P} \quad \text{[Gauss's law inside material medium (1)]}$$

The equation just obtained will be interpreted as Gauss's law for macroscopic electric field **E** by considering its right side as *total* charge density

$$\rho_t = \rho - \nabla\cdot\mathbf{P}$$

in which ρ denotes the volumetric density of free charge carriers in the region, and, likewise, $-\nabla\cdot\mathbf{P}$ denotes a volumetric density due to bound charges *revealed* as a result of the polarization process.

  - It is furthermore convenient to rearrange Gauss's law as

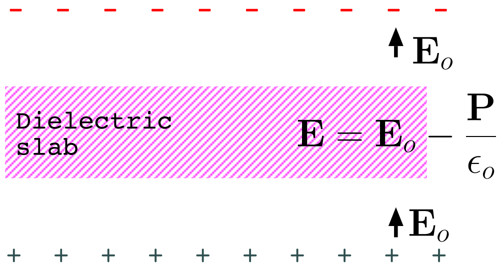
$\nabla\cdot\vec{D}=\rho$ *

$$\nabla\cdot(\underbrace{\epsilon_o\mathbf{E}+\mathbf{P}}_{\vec{D}}) = \rho \quad \text{[Gauss's law inside material medium (2)]}$$

so that only the *free charge density* ρ is retained on the right and the effect of bound charges is lumped on the left side together with $\epsilon_o\mathbf{E}$.

  - It is also convenient to revise the usual definition of electric displacement as

$$\mathbf{D} = \epsilon_o\mathbf{E} + \mathbf{P} \quad \text{[Revised definition of electric displacement]}$$



A dielectric slab inserted into a region with an initial field E_o will become polarized.

Inside the polarized dielectric the field will be weaker than E_o, but not reduced to zero as in a conductor.

so that Gauss's law can be written in its usual form

✓✓ $$\nabla\cdot\underline{\mathbf{D}} = \underline{\rho}, \quad \text{[Gauss's law inside material medium (3)]}$$

but with only the free charge density included on the right and the divergence of (revised) displacement on the left. Also, in integral form we have

✓ $$\oint_S \underline{\mathbf{D}}\cdot d\mathbf{S} = \int_V \rho dV,$$

where the right side denotes the net *free* charge inside volume V.

- In a large class of dielectric materials macroscopic polarization **P** and electric field **E** turn out to be linearly related as

✓ $\vec{P} \equiv \underline{N_d}\,\underline{e d\hat{z}}$

✓ $$\mathbf{P} = \epsilon_o\underline{\chi_e}\mathbf{E},$$

where $\chi_e \geq 0$ is a dimensionless quantity called **electric susceptibility**. For such materials

✓ $$\mathbf{D} = \epsilon_o\mathbf{E} + \mathbf{P} = \epsilon_o\underbrace{(1+\chi_e)}_{\epsilon_r}\mathbf{E} = \underline{\epsilon\mathbf{E}},$$ ←

where

$$\epsilon = \epsilon_o(1+\chi_e) \equiv \epsilon_r\epsilon_o$$

is known as the **permittivity** of the dielectric, and

$$\epsilon_r = 1 + \chi_e$$

its **relative permittivity** or **dielectric constant**.

- Dielectric constant of free space is 1,
  - for air $\epsilon_r \approx 1.0006$, ✓
  - for glass $4 - 10$, ✓
  - dry-to-wet earth $5 - 10$, silicon $11 - 12$, distilled water 81. ✓

  In certain materials χ_e and ϵ are found to be tensors — meaning that **P** and **D** are no longer aligned with **E**. Such materials are said to be **anisotropic**, but they will not be studied in this course. Also, there is an exception to the condition $\chi_e \geq 0$ — in collisionless *plasmas* $\chi_e < 0$, as discussed in ECE 450.

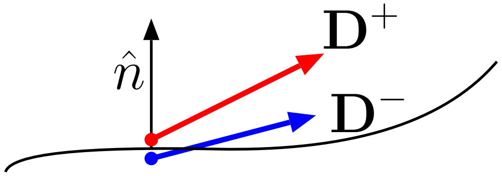
$$\nabla \cdot \vec{D} = \rho .$$

$$\nabla \cdot \underbrace{(\varepsilon_0 \vec{E} + \vec{P})}_{\vec{D}} = \rho .$$

✓ $\oint_s \vec{D} \cdot d\vec{s} = Q_v .$ ✓

★ $\hat{n} \cdot (\vec{D}^+ - \vec{D}^-) = \rho_s$

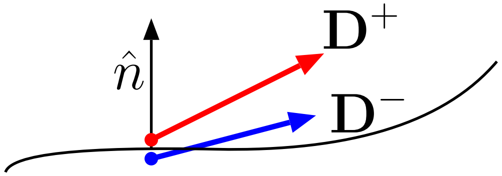
- In Gauss's law applicable in material media ρ denotes the free charge carrier density (after the revisions we have agreed to make). Furthermore, in perfect dielectrics there are no mobile free carriers and Gauss's law typically reduces to $\underline{\nabla \cdot \mathbf{D} = 0}$, while the corresponding **boundary condition** equation for surfaces separating perfect dielectrics becomes



$$✓\ \hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = 0 \quad \Rightarrow \quad D_n^+ = D_n^-,$$

which says that normal component of displacement **D** is continuous on such surfaces. This is accompanied by

$$✓\ \hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) = 0 \quad \Rightarrow \quad E_t^+ = E_t^-$$



stating the continuity of tangential components of **E**, which is universally true as we have seen earlier.