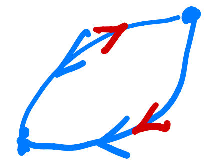
6 Circulation and boundary conditions

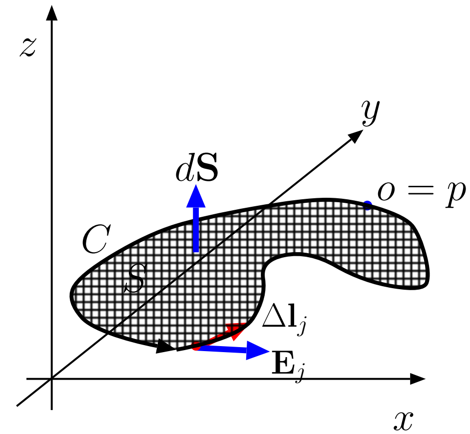
Since curl-free static electric fields have path-independent line integrals, it follows that over closed paths C (when points p and o coincide)

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0,$$

$\nabla \times \vec{E} = 0$



where the $\oint_C \mathbf{E} \cdot d\mathbf{l}$ is called the **circulation** of field $\mathbf{E}$ over closed path C bounding a surface S (see margin).



Closed loop integral over path C enclosing surface S.

Note that the area increment dS of surface S is taken by convention to point in the right-hand-rule direction with respect to "circulation" direction C.

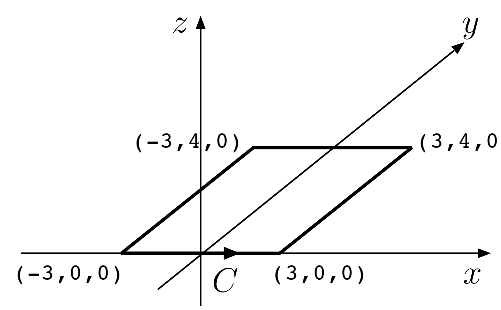
Example 1: Consider the static electric field variation

$$\mathbf{E}(x, y, z) = \hat{x}\frac{\rho x}{\epsilon_o}$$

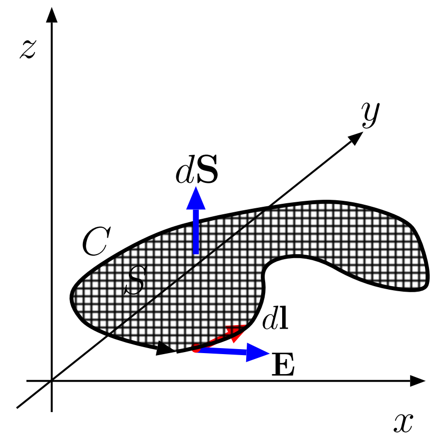
that will be encountered within a uniformly charged slab of an infinite extent in y and z directions and a finite width in x direction centered about $x = 0$. Show that this field $\mathbf{E}$ satisfies the condition $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$ for a rectangular closed path C with vertices at $(x, y, z) = (-3, 0, 0)$, $(3, 0, 0)$, $(3, 4, 0)$, and $(-3, 4, 0)$ traversed in the order of the vertices given.

Solution: Integration path C is shown in the figure in the margin. With the help of the figure we expand the circulation $\oint_C \mathbf{E} \cdot d\mathbf{l}$ as

$$\begin{aligned}
\mathbf{E} &= \int_{x=-3}^{3} \hat{x}\frac{\rho x}{\epsilon_o} \cdot \hat{x}dx + \int_{y=0}^{4} \hat{x}\frac{\rho 3}{\epsilon_o} \cdot \hat{y}dy + \int_{x=3}^{-3} \hat{x}\frac{\rho x}{\epsilon_o} \cdot \hat{x}dx + \int_{y=4}^{0} \hat{x}\frac{\rho(-3)}{\epsilon_o} \cdot \hat{y}dy \\
&= \int_{x=-3}^{3} \frac{\rho x}{\epsilon_o}dx + 0 + \int_{x=3}^{-3} \frac{\rho x}{\epsilon_o}dx + 0 = 0.
\end{aligned}$$



Note that in expanding $\oint_C \mathbf{E}\cdot d\mathbf{l}$ above for the given path C, we took $d\mathbf{l}$ as $\hat{x}dx$ and $\hat{y}dy$ in turns (along horizontal and vertical edges of C, respectively) and ordered the integration limits in x and y to traverse C in a counter-clockwise direction as indicated in the diagram.



STOKE'S THM:
Circulation of E around close path C equals the flux over enclosed surface S of the curl of E taken in direction of dS.

dS points in right-hand-rule direction with respect to "circulation" direction C.

- Vector fields $\mathbf{E}$ having zero circulations over all closed paths C are known as **conservative fields** (for obvious reasons having to do with their use in modeling static fields compatible with conservation theorems).
  - The concepts of *curl-free* and *conservative* fields overlap, that is

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = 0 \quad \Leftrightarrow \quad \nabla\times\mathbf{E} = 0$$

    over all closed paths C and at each $\mathbf{r}$.

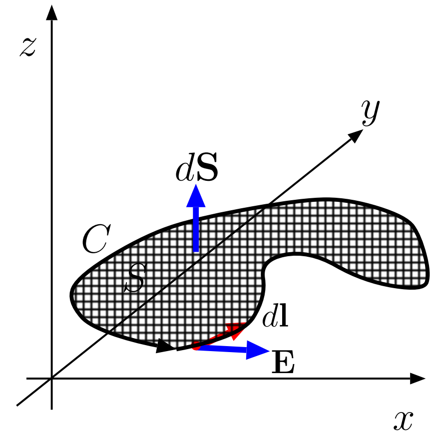
- The above relationship between circulation and curl is also a consequence of **Stoke's theorem** (discussed in MATH 241) which asserts that

Stoke's thm.

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_S \nabla\times\mathbf{E}\cdot d\mathbf{S},$$

where
  - the integration surface S on the right is bounded by the closed integration contour C of the left side, and

- the incremental area element $d\mathbf{S}$ on the right points across area S in the direction indicated by a **right-hand rule** as follows:



STOKE'S THM:
Circulation of E around close path C equals the flux over enclosed surface S of the curl of E taken in direction of dS.

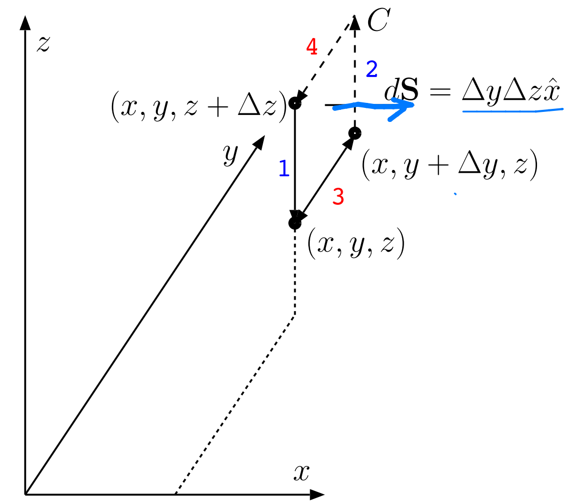
dS points in right-hand-rule direction with respect to "circulation" direction C.

Point your right thumb in chosen circulation direction C; then your right fingers point through surface S in the direction that should be adopted for $d\mathbf{S}$.

- Given Stoke's theorem, $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$ follows immediately for all C, if $\nabla \times \mathbf{E} = 0$ is true over all $\mathbf{r}$.

• Stoke's theorem clearly implies that **curl is circulation per unit area**, just as the divergence theorem showed that **divergence is flux per unit volume**.

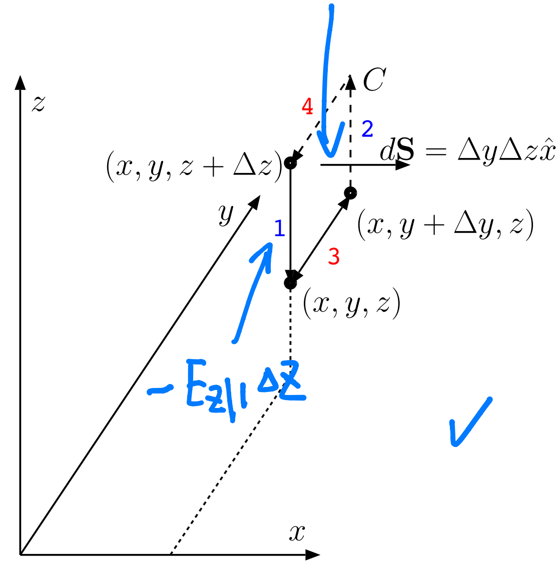
- The only difference is, curl also has a direction, which is the normal unit of the plane that contains the maximal value of circulation per unit area found at that location (over all possible orientations of $d\mathbf{S}$).



• We will verify Stoke's thm after explaining the **circulation per unit area** notion in steps:

- Let us first calculate the circulation of a vector field $\mathbf{E}$ taken about an arbitrary point (x, y, z) on a constant x plane around a square contour with small edge dimensions Δy and Δz parallel to y and z axes as shown in the margin.

- For a small rectangular contour "C_x" on a constant x plane with sufficiently small Δy and Δz dimensions parallel to y and z axes (see figure in the margin), we have



$$\oint_{C_x} \mathbf{E}\cdot d\mathbf{l} \approx E_{z|2}\Delta z - E_{y|4}\Delta y - E_{z|1}\Delta z + E_{y|3}\Delta y$$
$$= (E_{z|2} - E_{z|1})\Delta z - (E_{y|4} - E_{y|3})\Delta y.$$

It follows that

$$\frac{1}{\Delta y \Delta z}\oint_{C_x} \mathbf{E}\cdot d\mathbf{l} \approx (\frac{E_{z|2} - E_{z|1}}{\Delta y} - \frac{E_{y|4} - E_{y|3}}{\Delta z})$$

and

$$\lim_{\Delta y, \Delta z \to 0} \frac{1}{\Delta y \Delta z}\oint_{C_x} \mathbf{E}\cdot d\mathbf{l} = (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) = \hat{x}\cdot \nabla\times\mathbf{E},$$

meaning that the x component of $\nabla\times\mathbf{E}$ is the circulation of $\mathbf{E}$ per unit area on a constant x surface.

- Likewise, y and z components of $\nabla\times\mathbf{E}$ are circulations of $\mathbf{E}$ per unit area on constant y and z surfaces, and in general

$$\nabla\times\mathbf{E} = \lim_{\Delta x, \Delta y, \Delta z \to 0}(\frac{1}{\Delta y \Delta z}\oint_{C_x} \mathbf{E}\cdot d\mathbf{l}, \frac{1}{\Delta x \Delta z}\oint_{C_y} \mathbf{E}\cdot d\mathbf{l}, \frac{1}{\Delta x \Delta y}\oint_{C_z} \mathbf{E}\cdot d\mathbf{l}).$$

- Furthermore, based on the above result, we can recognize that vectors $\nabla\times\mathbf{E}$ point everywhere in directions perpendicular to planes of maximum circulations per unit area in the $\mathbf{E}$ field and have magnitudes corresponding to the maximum values of circulations per unit area at every point.

- Now, to confirm **Stoke's theorem**

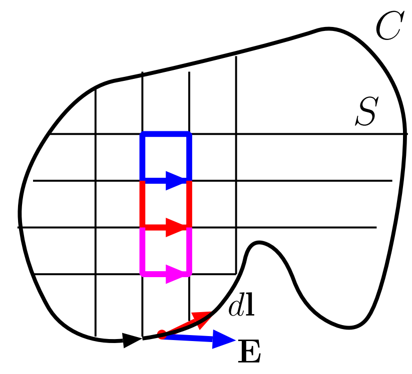
$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_S \nabla\times\mathbf{E}\cdot d\mathbf{S}$$

pertinent for a closed path C and its enclosed surface S, we will make use of the diagram shown in the margin.

  - The circulations over small squares shown in the diagram are approximately equal to the products of their areas and the normal components of $\nabla\times\mathbf{E}$ calculated at the center points (basd on what we learned above).
  - When all such circulations covering surface S are added up, the result is $\int_S \nabla\times\mathbf{E}\cdot d\mathbf{S}$ in the limit of vanishing size for the squares,
  - as well as $\oint_C \mathbf{E}\cdot d\mathbf{l}$ because in the grand sum of the circulations over all the squares, all the contributions mutually cancel out (like the overlapping edges of red and blue squares) except for those calculated along the periphery C!

We can now summarize the general constraints governing static electric fields as

$$\nabla\times\mathbf{E}(\mathbf{r}) = 0, \quad \nabla\cdot\mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}), \quad \text{where } \mathbf{D}(\mathbf{r}) = \epsilon_o\mathbf{E}(\mathbf{r}).$$



Sum of circulations over small squares cancel in the interior edges and only survive around the exterior path C. This way, circulation around C matches the sum of the fluxes of curl E calculated over the small squares.

Laws of electrostatics:

$$\begin{aligned}\nabla\times\mathbf{E} &= 0\\ \nabla\cdot\epsilon_o\mathbf{E} &= \rho\end{aligned}$$

They also apply "quasi-statically" over a region of dimension L when a time-varying field source $\rho(\mathbf{r},t)$ has a *time-constant* τ much longer than the propagation time delay L/c of $\mathbf{E}(\mathbf{r},t)$ field variations across the region (c is the speed of light).

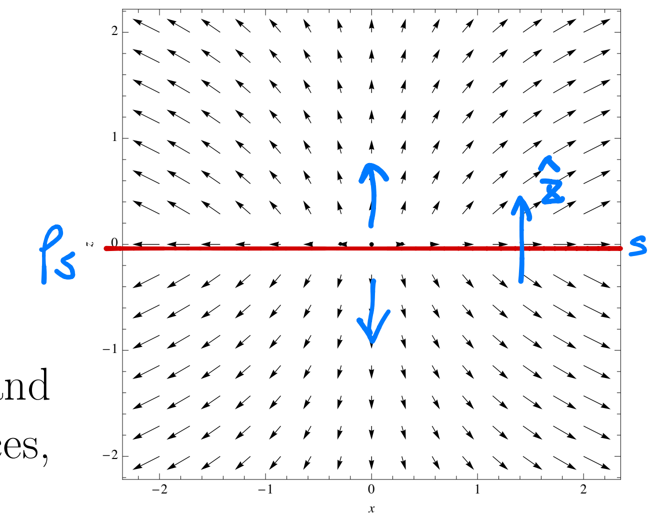
In electro-quasistatics (EQS) $\mathbf{E}(\mathbf{r},t)$ will be accompanied by a slowly varying magnetic field $\mathbf{B}(\mathbf{r},t)$ (to be studied starting in Lecture 12).

- Vector fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$ governed by these equations will in general be continuous functions of position coordinates $\mathbf{r} = (x, y, z)$ *except* at boundary surfaces where charge density function $\rho(\mathbf{r})$ requires a representation in terms of a surface charge density $\rho_s(\mathbf{r})$.

  - For instance, according to our earlier results, static electric field of a charge density (see sketch at the margin)

$$\rho(\mathbf{r}) = \rho_s \delta(z)$$

would be



$$\mathbf{E}(\mathbf{r}) = \hat{z}\frac{\rho_s}{2\epsilon_o}\mathrm{sgn}(z) \quad \Rightarrow \quad \mathbf{D}(\mathbf{r}) = \hat{z}\frac{\rho_s}{2}\mathrm{sgn}(z).$$

    - Consider a superposition of these fields with fields $\mathbf{E}_o(\mathbf{r})$ and $\mathbf{D}_o(\mathbf{r}) = \epsilon_o \mathbf{E}_o(\mathbf{r})$ produced by arbitrary continuous sources, namely (macroscopic) fields

$$\mathbf{E}(\mathbf{r}) = \hat{z}\frac{\rho_s}{2\epsilon_o}\mathrm{sgn}(z) + \mathbf{E}_o(\mathbf{r}) \quad \text{and} \quad \mathbf{D}(\mathbf{r}) = \hat{z}\frac{\rho_s}{2}\mathrm{sgn}(z) + \epsilon_o \mathbf{E}_o(\mathbf{r}).$$

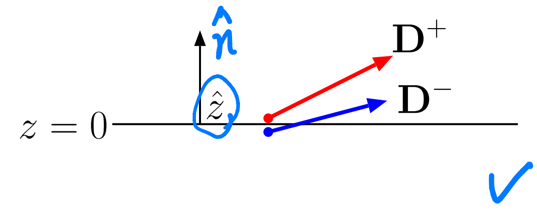
Since fields $\mathbf{E}_o(\mathbf{r})$ and $\mathbf{D}_o(\mathbf{r})$ vary continuously, these field expressions must satisfy

$$\hat{z}\cdot(\mathbf{D}^+ - \mathbf{D}^-) = \rho_s \quad \text{and} \quad \hat{z}\times(\mathbf{E}^+ - \mathbf{E}^-) = 0$$

where

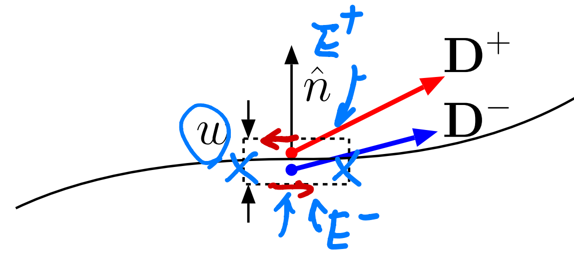
$$\mathbf{E}^+ \equiv \mathbf{E}(x, y, 0^+) \quad \text{and} \quad \mathbf{E}^- \equiv \mathbf{E}(x, y, 0^-)$$

refer to limiting values of $\mathbf{E}$ at $z = 0$ plane from *above* and *below*, respectively, and likewise for



$$\mathbf{D}^+ \equiv \mathbf{D}(x, y, 0^+) \quad \text{and} \quad \mathbf{D}^- \equiv \mathbf{D}(x, y, 0^-).$$

- The above "boundary condition equations" can be written in a more general form (see margin for justification) as



$$\hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = \rho_s \quad \text{and} \quad \hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) = 0$$

where $\hat{n}$ denotes a unit vector normal to any surface of an arbitrary orientation carrying a surface charge density ρ_s, while field vectors with superscripts $+$ and $-$ indicate limiting values of fields measured on either side of the charged surface (with $\hat{n}$ pointing from $-$ to $+$).

  – The equations can be further simplified as

$$D_n^+ - D_n^- = \rho_s \quad \text{and} \quad E_t^+ = E_t^-$$

where D_n and E_t refer to normal component of $\mathbf{D}$ and tangential component of $\mathbf{E}$, respectively. Clearly, these **boundary conditions** say that at any surface S,

  - tangential component of electric field $\mathbf{E}$ needs to be continuous, but
  - normal component of $\mathbf{D}$ can change by an amount equal to the charge density ρ_s carried by the surface.

Constraint

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

around the dotted path yields

$$E_t^+ = E_t^-$$

in $w \to 0$ limit.

Gauss's law

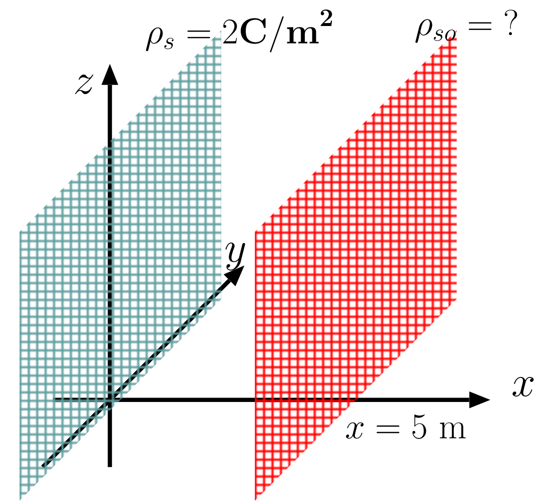
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

applied over the dotted volume (seen in profile) yields

$$D_n^+ - D_n^- = \rho_s$$

in $w \to 0$ limit.

$\mathbf{D} = 0$ for $x < 0$.



Example 2:

Measurements indicate that $\mathbf{D} = 0$ in the region $x < 0$.

Also, $x = 0$ and $x = 5$ m planes contain surface charge densities of $\rho_s = 2$ C/m^2 and ρ_{so}, respectively.

Determine ρ_{so} and $\mathbf{D}$ for $-\infty < x < \infty$ if there are no other charge distributions.

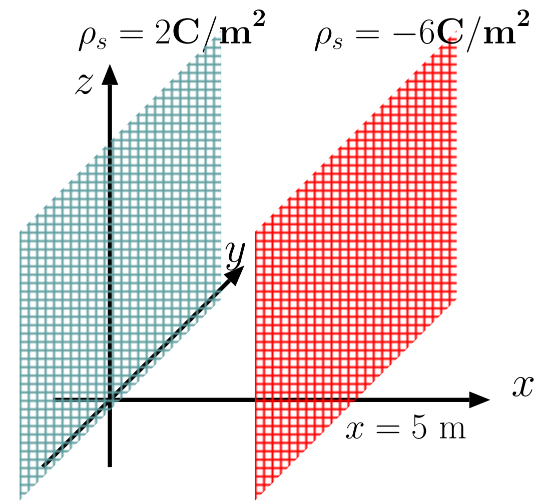
Solution:

Since the normal component of $\mathbf{D}$ must increase by $\rho_s = 2$ C/m^2 when we cross the charged surface $x = 0$, we must have $\mathbf{D} = \hat{x}2$ C/m^2 in the region $0 < x < 5$ m.

Having $\mathbf{D} = 0$ in the region $x < 0$ requires that the field due to surface charge ρ_{so} on $x = 5$ m plane must cancel the field due $\rho_s = 2$ C/m^2 on $x = 0$ plane — this requires that ρ_{so} be -2 C/m^2.

In that case $\mathbf{D} = 0$ in the region $x > 5$ m, because $\mathbf{D}$ must increase by $\rho_{so} = -2$ C/m^2 when we cross the charged surface at $x = 5$ m.

$\mathbf{D} = 3\hat{y}$ for $x < 0$.



Example 3: In the region $x < 0$ measurements indicate a constant displacement field $\mathbf{D} = 3\hat{y}$ C/m^2. Also, $x = 0$ and $x = 5$ m planes contain surface charge densities of $\rho_s = 2$ C/m^2 and $\rho_s = -6$ C/m^2 respectively. Determine $\mathbf{D}$ for $x > 0$ if $\mathbf{D}$ is known to be uniform in the intervals $0 < x < 5$ m and $x > 5$ m.

Solution: First we note that $\mathbf{E} = \frac{\mathbf{D}}{\epsilon_o} = \hat{y}\frac{3}{\epsilon_o}$ V/m is tangential to $x = 0$ and $x = 5$ m surfaces. Since the tangential component of $\mathbf{E}$ cannot change at any boundary, we will have a uniform $E_y = \frac{3}{\epsilon_o}$ in all regions, $-\infty < x < \infty$, implying that $D_y = 3$ C/m^2 throughout (caused by charges at $|y| \to \infty$).

Second, we note that normal component of $\mathbf{D}$ with respect to $x = 0$ and $x = 5$ m surfaces, namely D_x, is zero in $z < 0$. Since the normal component of $\mathbf{D}$ must increase by an amount ρ_s when we cross a charged surface, we must have $D_x = 2$ C/m^2 in the region $0 < x < 5$ m, and $D_x = 2 + (-6) = -4$ C/m^2 in $x > 5$ m.

In summary,

$$\mathbf{D} = \begin{cases} \hat{y}3, & \text{for } x < 0, \\ \hat{x}2 + \hat{y}3, & \text{for } 0 < x < 5 \text{ m} \\ -\hat{x}4 + \hat{y}3, & \text{for } x > 5 \text{ m} \end{cases} \frac{\text{C}}{\text{m}^2}.$$