5 Curl-free fields and electrostatic potential

- Mathematically, we can generate a curl-free vector field $\mathbf{E}(x, y, z)$ as

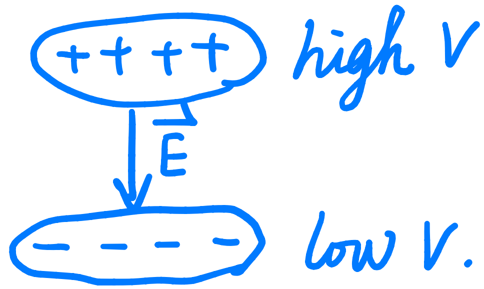
$$\mathbf{E} = -(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}),$$

by taking the **gradient** of any scalar function $V(\mathbf{r}) = V(x, y, z)$. The gradient of $V(x, y, z)$ is defined to be the vector

$$\nabla V \equiv (\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}),$$

pointing in the direction of increasing V; in abbreviated notation, curl-free fields $\mathbf{E}$ can be indicated as

$$\mathbf{E} = -\nabla V.$$



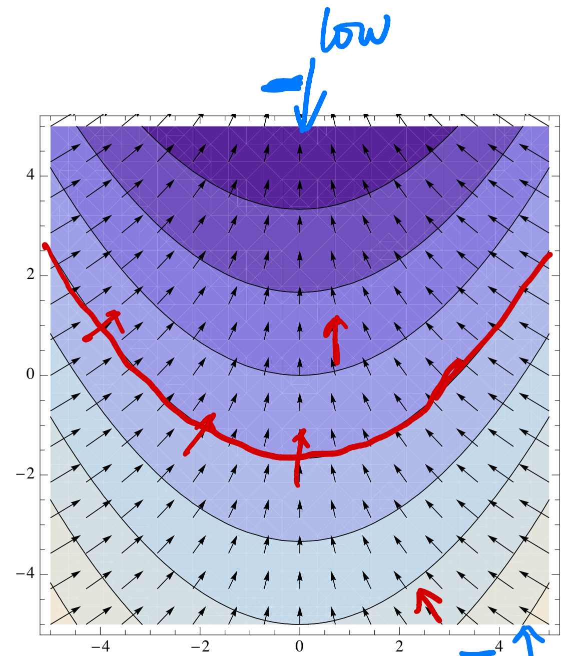
  - **Verification:** Curl of vector ∇V is

$$\nabla \times (\nabla V) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} = \hat{x}0 - \hat{y}0 - \hat{z}0 = 0.$$

  - If $\mathbf{E} = -\nabla V$ represents an **electrostatic field**, then V is called the **electrostatic potential**.
    - Simple dimensional analysis indicates that units of electrostatic potential must be volts (V).

– The prescription $\mathbf{E} = -\nabla V$, including the minus sign (optional, but taken by convention in electrostatics), ensures that electrostatic field $\mathbf{E}$ points from regions of "high potential" to "low potential" as illustrated in the next example.

Electrostatic fields E point from regions of "high V" to "low V"



Light colors indicate "high V" dark colors "low V"

Example 1: Given an electrostatic potential

$$V(x, y, z) = x^2 - 6y \text{ V}$$

in a certain region of space, determine the corresponding electrostatic field $\mathbf{E} = -\nabla V$ in the same region.

Solution: The electrostatic field is

$$\mathbf{E} = -\nabla(x^2 - 6y) = -(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(x^2 - 6y) = (-2x, 6, 0) = -\hat{x}\,2x + \hat{y}6 \text{ V/m}.$$

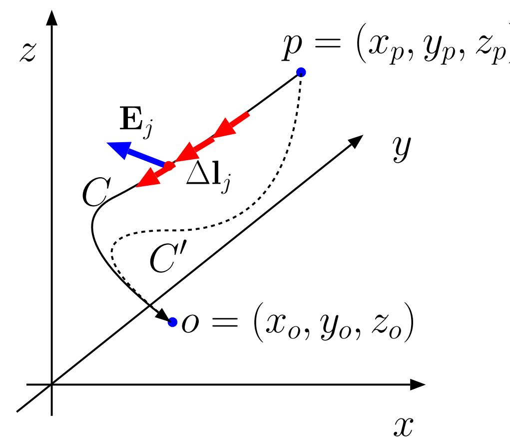
Note that this field is directed from regions of high potential to low potential. Also note that electric field vectors are perpendicular everywhere to "equipotential" contours.

Given an electrostatic potential $V(x, y, z)$, finding the corresponding electrostatic field $\mathbf{E}(x, y, z)$ is a straightforward procedure (taking the negative gradient) as already illustrated in Example 1.

The reverse operation of finding $V(x, y, z)$ from a given $\mathbf{E}(x, y, z)$ can be accomplished by performing a **vector line integral**

$$\int_p^o \mathbf{E} \cdot d\mathbf{l}$$

in 3D space, since, as shown below, such integrals are "path independent" for curl-free fields $\mathbf{E} = -\nabla V$.



- The **vector line integral**

$$\int_p^o \mathbf{E} \cdot d\mathbf{l}$$

  over an integration path C extending from a point $p = (x_p, y_p, z_p)$ in 3D space to some other point $o = (x_o, y_o, z_o)$ is *defined to be*

  – the limiting value of the sum of dot products $\mathbf{E}_j \cdot \Delta\mathbf{l}_j$ computed over all sub-elements of path C having incremental lengths $|\Delta\mathbf{l}_j|$ and unit vectors $\Delta\mathbf{l}_j/|\Delta\mathbf{l}_j|$ directed from p towards o — the limiting value is obtained as all $|\Delta\mathbf{l}_j|$ approach zero (i.e., with increasingly finer subdivision of C into $|\Delta\mathbf{l}_j|$ elements).

- Computation of the integral (see example below) involves the use of infinitesimal displacement vectors

$$d\mathbf{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz = (dx, dy, dz)$$

  and vector dot product

$$\mathbf{E} \cdot d\mathbf{l} = (E_x, E_y, E_z) \cdot (dx, dy, dz) = E_x dx + E_y dy + E_z dz.$$

  The integral

$$\int_p^o \mathbf{E} \cdot d\mathbf{l} = \int_p^o (E_x dx + E_y dy + E_z dz)$$

will in general be *path dependent* except for when $\mathbf{E}$ is curl-free.

Example 2: The field $\mathbf{E} = \hat{x}y \pm \hat{y}x$ is curl-free with the $+$ sign, but not with $-$ as verified below by computing $\nabla \times \mathbf{E}$. Calculate the line integral of $\mathbf{E}$ (for both signs, $\pm$) from a point $o = (0,0,0)$ to point $p = (1,1,0)$ for two different paths C going through points $u = (0,1,0)$ and $l = (1,0,0)$, respectively (see margin).

Solution: First we note that

$$\nabla \times (\hat{x}y \pm \hat{y}x) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & \pm x & 0 \end{vmatrix} = \hat{z}(\pm 1 - 1)$$

which confirms that $\mathbf{E} = \hat{x}y \pm \hat{y}x$ is curl-free with with $+$ sign, but not with $-$. In either case, the integral to be performed is

$$\int_o^p \mathbf{E} \cdot d\mathbf{l} = \int_o^p (E_x dx + E_y dy + E_z dz) = \int_o^p (y\,dx \pm x\,dy).$$

For the first path C_u going through $u = (0,1,0)$, we have

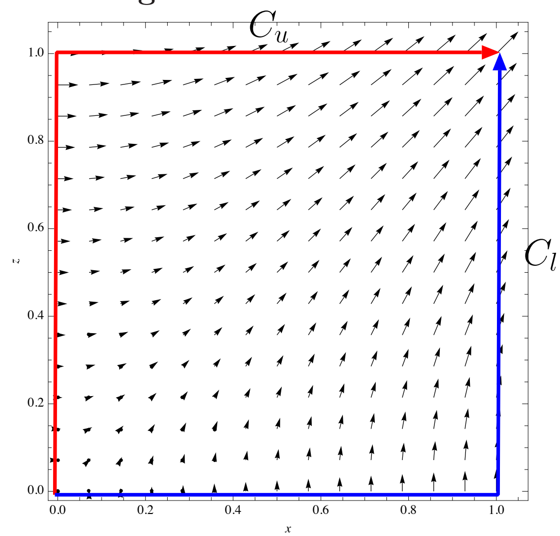
$$\int_o^p (y\,dx \pm x\,dy) = \int_{y=0}^1 (\pm x)\,dy_{|x=0} + \int_{x=0}^1 y\,dx_{|y=1} = 0 + 1 = 1.$$

For the second path C_l going through $l = (1,0,0)$, we have

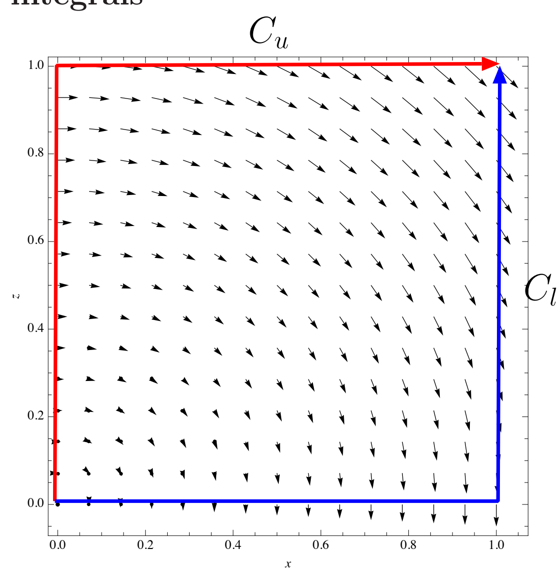
$$\int_o^p (y\,dx \pm x\,dy) = \int_{x=0}^1 y\,dx_{|y=0} \pm \int_{y=0}^1 x\,dy_{|x=1} = 0 \pm 1 = \pm 1.$$

Clearly, the result shows that the line integral $\int_o^p \mathbf{E} \cdot d\mathbf{l}$ is *path independent* for $\mathbf{E} = \hat{x}y + \hat{y}x$ which is curl-free, and path dependent for $\mathbf{E} = \hat{x}y - \hat{y}x$ in which case $\nabla \times \mathbf{E} \neq 0$.

Curl-free: path-independent line integrals



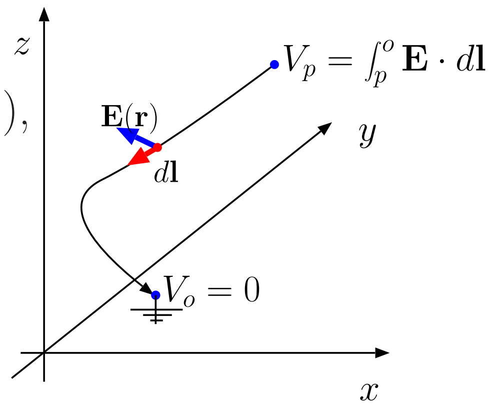
"Curly": path-dependent line integrals



- The **mathematical reason** why curl-free fields have path-independent line integrals is because in those occasions the integrals can be written in terms of **exact differentials**:

  – for curl-free $\mathbf{E} = \hat{x}y + \hat{y}x$ we have $\mathbf{E} \cdot d\mathbf{l}$ as an *exact differential* $ydx + xdy = d(xy)$ of the function xy, in which case $\int_o^p \mathbf{E} \cdot d\mathbf{l} = xy|_o^p = (1 \cdot 1 - 0 \cdot 0) = 1$ over all paths.

  – for $\mathbf{E} = \hat{x}y - \hat{y}x$ with $\nabla \times \mathbf{E} = -2\hat{z} \neq 0$, on the other hand, $\mathbf{E} \cdot d\mathbf{l} = ydx - xdy$ does not form an exact differential $-dV$, and thus there is no path-independent integral $-V|_o^p$, nor an underlying potential function V.



$\mathbf{E} \cdot d\mathbf{l}$ is *guaranteed* to be an exact differential if $\mathbf{E} = -\nabla V = (-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z})$, since in that case the differential of $V(x, y, z)$, namely

$$dV \equiv \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz, \text{ is precisely } -E_x dx - E_y dy - E_z dz = -\mathbf{E} \cdot d\mathbf{l}.$$

  – In that case

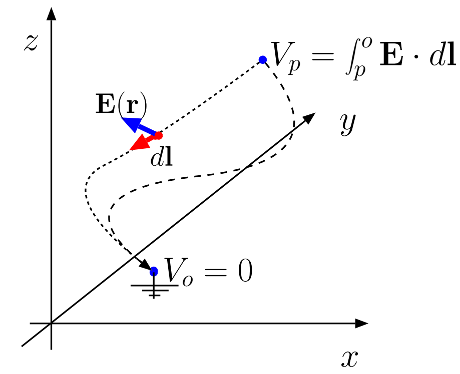
$$\int_p^o \mathbf{E} \cdot d\mathbf{l} = -\int_p^o dV = \int_o^p dV = V_p - V_o$$

  is independent of integration path; thus, if we we call o the "ground", and set $V_o = 0$, then

$$V_p = \int_p^o \mathbf{E} \cdot d\mathbf{l}$$

  denotes the potential drop from (any) point p to ground o.

- The **physical reason** why this integral formula for potential V_p works with *any* integration path is the principle of **energy conservation**:
  - integral $\int_p^o \mathbf{E} \cdot d\mathbf{l}$ represents the **work done** by field **E per unit charge** moved from p to o, so if the line integral were path-dependent there would be ways of creating net energy by making a charge q follow special paths within the electrostatic field **E**, in violation of the general principle of energy conservation (that permits energy conversion but not creation or destruction).



As long as E is curl-free, line integral is path-independent and produces the voltage drop from point p to "ground" o.

Example 3: Given that $V_o = V(0,0,0) = 0$ and

$$\mathbf{E} = 2x\hat{x} + 3z\hat{y} + 3(y+1)\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}},$$

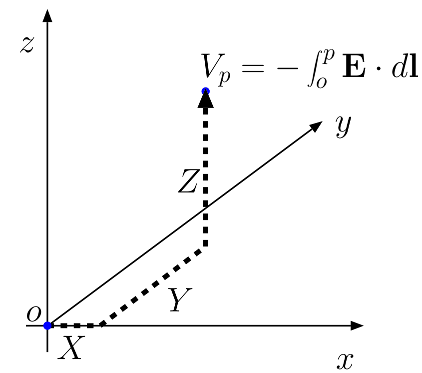
determine the electrostatic potential $V_p = V(X,Y,Z)$ at point $p = (X,Y,Z)$ in volts.

Solution: Assuming that the field is curl-free (it is), so that any integration path can be used, we find that

$$\begin{aligned} V_p &= \int_p^o \mathbf{E} \cdot d\mathbf{l} = -\int_o^p \mathbf{E} \cdot d\mathbf{l} = -\int_o^p (2x\,dx + 3z\,dy + 3(y+1)\,dz) \\ &= -\int_0^X 2x\,dx_{|y,z=0} - \int_0^Y 3z\,dy_{|x=X,z=0} - \int_0^Z 3(y+1)\,dz_{|x=X,y=Y} \\ &= -X^2 - 0 - 3(Y+1)Z. \end{aligned}$$

This implies

$$V(x,y,x) = -x^2 - 3(y+1)z\,\mathrm{V}.$$



Note that

$$
\begin{aligned}
-\nabla(-x^2 - 3(y+1)z) &= \nabla(x^2 + 3(y+1)z) \\
&= \hat{x}2x + \hat{y}3z + \hat{z}3(y+1)
\end{aligned}
$$

yields the original field $\mathbf{E}$, which is an indication that $\mathbf{E}$ is indeed curl-free.

Example 5: According to Coulomb's law electrostatic field of a proton with charge $Q = e$ (where $-e$ is electronic charge) located at the origin is given as

$$\mathbf{E} = \frac{e}{4\pi\epsilon_o r^2}\hat{r},$$

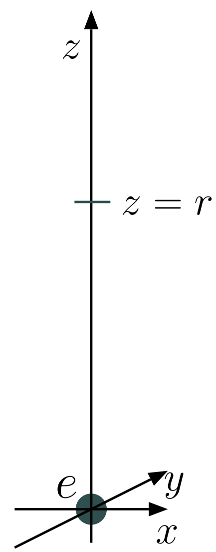
where

$$r = \sqrt{x^2 + y^2 + z^2} \text{ and } \hat{r} = \frac{(x, y, z)}{r}.$$

Determine the electrostatic potential field V established by charge $Q = e$ with the provision that $V \to 0$ as $r \to \infty$ (i.e., ground at infinity).

Solution: Field $\mathbf{E}$ and its potential V will exhibit spherical symmetry in this problem. Therefore, with no loss of generality, we can calculate the line integral from a point p at a distance r from the origin to a point o at ∞ (the specified ground) along, say, the z-axis. Approaching the problem that way, the potential drop from r to ∞ is

$$
\begin{aligned}
V(r) &= \int_{z=r}^{\infty} \frac{e}{4\pi\epsilon_o z^2}\hat{z}\cdot\hat{z}dz \\
&= -\frac{e}{4\pi\epsilon_o z}\Big|_r^{\infty} = \frac{e}{4\pi\epsilon_o r}.
\end{aligned}
$$

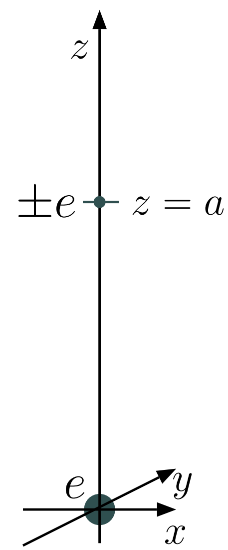


- To convert electrostatic potential V_p (in volts) at any point p to potential energy of a charge q brought to the same point, it is sufficient to multiply V_p with q (or just the sign of q, depending on which energy units we want to use — see the next example).

Example 6: In view of Example 5, what are the potential energies of a proton e and an electron $-e$ placed at distance $r = a$ away from the proton at the origin, where distance

$$a \equiv \frac{4\pi\epsilon_o}{e^2}\frac{\hbar^2}{m_e} = 0.529 \times 10^{-10}\,\mathrm{m}$$

stands for *Bohr radius* — it is the mean distance of the ground state electron in a hydrogen atom from the center of the atom. Recall that $e = 1.602 \times 10^{-19}$ C and $\epsilon_o \approx 10^{-9}/36\pi$ F/m.



Solution: Let's first evaluate the potential $V(r)$ at $r = a$:

$$V(a) = \frac{e}{4\pi\epsilon_o a} \approx \frac{(1.6 \times 10^{-19})36\pi \times 10^9}{4\pi \times 0.53 \times 10^{-10}} = \frac{9 \times 1.6}{0.53} = 27.2\,\mathrm{V}.$$

For the proton, potential energy in Joules is calculated by multiplying $V(a) = 27.2$ V with $q = e = 1.602 \times 10^{-19}$ C. However, by referring to 1.602×10^{-19} J of energy as 1 eV (electron-volt), it is more convenient to refer to potential energy $eV(a)$ of the proton at $r = a$ as

$$eV(a) = 27.2\,\mathrm{eV}.$$

Likewise, for a particle with charge $q = -e$, i.e., an electron, potential energy at the same location is

$$-eV(a) = -27.2\,\mathrm{eV}.$$