2 Static electric fields — Coulomb's and Gauss's laws

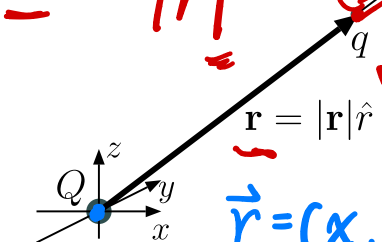
Static electric fields $\mathbf{E}(\mathbf{r})$ are produced by static (non-time-varying) distribution of charges and obey and the electrostatic laws shown in the margin where $\rho(\mathbf{r})$ denotes the net charge density in 3D volume. Over the next few lectures we will find out how these laws emerge from *Coulomb's law.*

At the most elementary level, each stationary point charge (electron or proton) Q is surrounded by its radially directed electrostatic field $\mathbf{E}$ given by Coulomb's law, and in the presence of multiple charges the field vectors of all the charges are added vectorially (linear superposition holds) to obtain a superposition field $\mathbf{E}$.

Laws of electrostatics:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_o$$
$$\nabla \times \mathbf{E} = 0$$

$k = \frac{1}{4\pi\varepsilon_o}$ $\quad \vec{F} = \frac{Qqk}{|r|^2}\hat{r}$ $\quad \hat{r} = \frac{\vec{r}}{|r|}$



$\vec{r} = (x, y, z)$

Force exerted by Q on q:

$$\mathbf{F} = q\mathbf{E}$$

with electric field

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o|\mathbf{r}|^2}\hat{r}$$

With multiple Q's superpose multiple E's

- **Coulomb's law** specifies the electric field of a stationary charge Q at the origin as

  at origin. $\quad Q \to r_o$.

  $$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_o r^2}\hat{r}$$

  as a function of position vector $\mathbf{r} = (x, y, z)$, where $\epsilon_o \approx \frac{1}{36\pi \times 10^9}$ F/m is a scaling constant known as *permittivity of free space*,

  $$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

  $r^2 = x^2 + y^2 + z^2$

  is radial distance from the charge, and $\hat{r} = \frac{\mathbf{r}}{r}$ radial unit vector pointing away from the charge.

  – This Coulomb field $\mathbf{E}(\mathbf{r})$ will exert a force $\mathbf{F} = q\mathbf{E}(\mathbf{r})$ on any stationary "test charge" q brought within distance r of Q (see figure in the margin).

If $qQ > 0$, force $\mathbf{F}$ is repulsive (directed along $\hat{r}$), if $qQ < 0$ it is attractive — *like charges repel, unlike charges attract.*

The existence of a Coulomb field accompanying each charge carrier in its rest frame[1] is taken to be a fundamental property of charge carriers (established by measurements).

- When multiple static charges Q_n are present in a region, the force on a stationary test charge q can be described as $q\mathbf{E}$ in terms of a superposition field

$$\mathbf{E} = \sum_n \frac{Q_n}{4\pi\epsilon_o r_n^2}\hat{r}_n$$

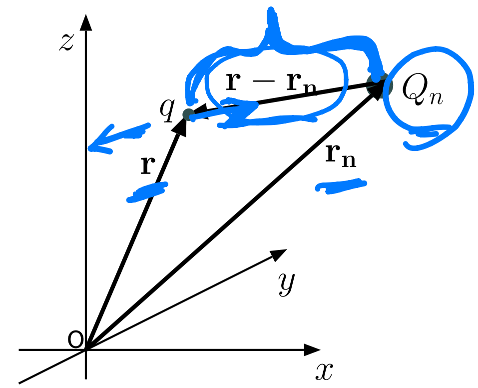
written in terms of the magnitudes and directions of vectors $\mathbf{r}_n$ pointing from each Q_n to q.

  – Equivalently, we can write

unit vector

$$\mathbf{E}(\mathbf{r}) = \sum_n \frac{Q_n}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_n|^2}\frac{\mathbf{r}-\mathbf{r}_n}{|\mathbf{r}-\mathbf{r}_n|},$$

where $\mathbf{r}$ and $\mathbf{r}_n$ now denote the locations of q and Q_n with respect to a common origin — this form is more convenient when static electric field $\mathbf{E}$ is to be calculated for an arbitrary location $\mathbf{r}$ (independent of the test charge notion).



Position vectors of charges are referenced with respect to a common origin O

$$\frac{Q_n}{4\pi\varepsilon_o|\vec{r}-\vec{r}_n|^2}\cdot\underbrace{\frac{\vec{r}-\vec{r}_n}{|\vec{r}-\vec{r}_n|}}_{\text{unit vector.}}$$

$Q_n \rightarrow q.$

[1] In non-inertial rest frames charge carriers will also produce an *additional* field proportional to the *acceleration* of free particles observed in such frames (e.g., Boyer, *Am. J. Phys.*, **47**, 129, 1979; Gupta and Padmanabhan, *Phys. Rev. D*, **57**, 7241, 1998).

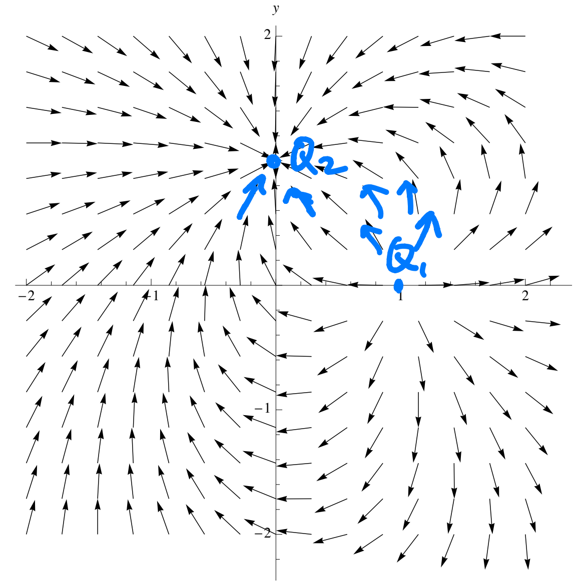
Example 1: Charges $Q_1 = 4\pi\epsilon_o$ and $Q_2 = -2Q_1$ are located at coordinates $\mathbf{r}_1 = (1,0,0) = \hat{x}$ and $\mathbf{r}_2 = (0,1,0) = \hat{y}$, respectively. What is the expression for $\mathbf{E}(\mathbf{r})$ and what is the explicit value of vector $\mathbf{E}(0)$?

Solution: Field $\mathbf{E}$ due to Q_1 and Q_2 at an arbitrary point $\mathbf{r}$ can be obtained as

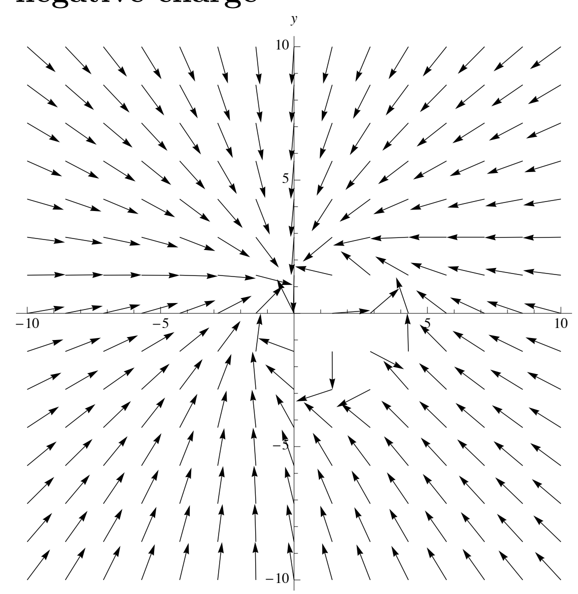
$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \frac{Q_1(\mathbf{r}-\mathbf{r}_1)}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_1|^3} + \frac{Q_2(\mathbf{r}-\mathbf{r}_2)}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_2|^3} \\ &= \frac{(\mathbf{r}-\hat{x})}{|\mathbf{r}-\hat{x}|^3} - \frac{2(\mathbf{r}-\hat{y})}{|\mathbf{r}-\hat{y}|^3} = \frac{(x-1,y,z)}{|(x-1,y,z)|^3} - \frac{2(x,y-1,z)}{|(x,y-1,z)|^3}\,\mathrm{V/m}.\end{aligned}$$

At the origin where $\mathbf{r} = (0,0,0)$, this result gives

$$\mathbf{E}(0,0,0) = \frac{(-1,0,0)}{|(-1,0,0)|^3} - \frac{2(0,-1,0)}{|(0,-1,0)|^3} = -\hat{x} + 2\hat{y}\,\mathrm{V/m}.$$



Field map of a dipole plus a negative charge



- The **vector map** shown in the margin depicts samples of unit vectors $\hat{E}(\mathbf{r}) \equiv \frac{\mathbf{E}(\mathbf{r})}{|\mathbf{E}(\mathbf{r})|}$ for the field $\mathbf{E}(\mathbf{r})$ obtained in Example 1 on a suitable grid established on xy-plane — such plots are useful or visualization purposes. Note that arrows emanate out of the positive charge at $(x,y) = (1,0)$ and converge upon the negative charge at $(x,y) = (0,1)$.
  - Electrostatic fields can be alternatively visualized in terms of so-called **field lines** or **flux lines**, continuous curves which are drawn tangential to unit vectors $\hat{E}(\mathbf{r})$ at every position $\mathbf{r}$. Try tracing out the flux lines over the vector map shown in the margin!

- According to Coulomb's law, electrostatic field of a charge Q placed at the origin points out in the radial direction $\hat{r}$ away from the origin and has a magnitude

$$E_r = \frac{Q}{4\pi\epsilon_o r^2}$$

$\vec{E} = E_r \hat{r}$

that depends on radial distance r, but it does not depend on direction $\hat{r}$. The product of E_r with ϵ_o and the surface area of a sphere at radius r, namely, $S = 4\pi r^2$, yields

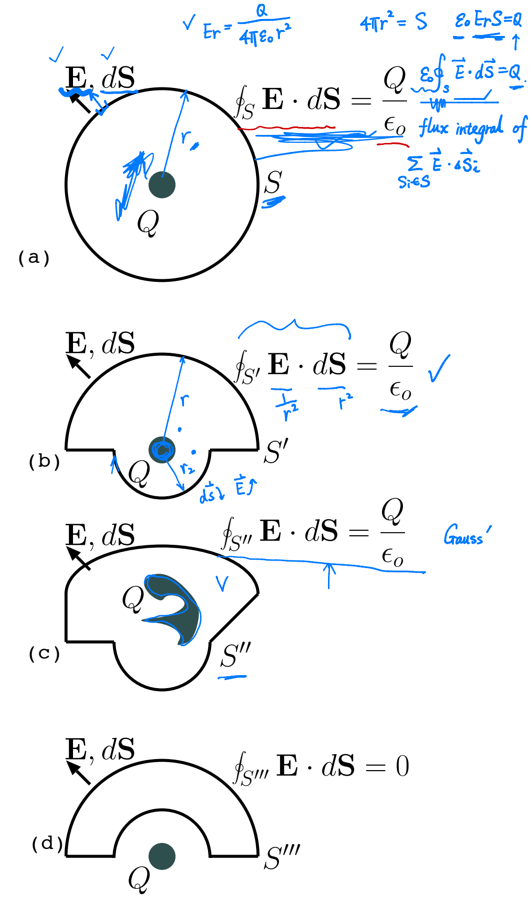
$$\epsilon_o E_r S = Q$$

independent of the radius of the sphere. Let's re-write the same result as

$$\epsilon_o \oint_S \mathbf{E} \cdot d\mathbf{S} = Q,$$

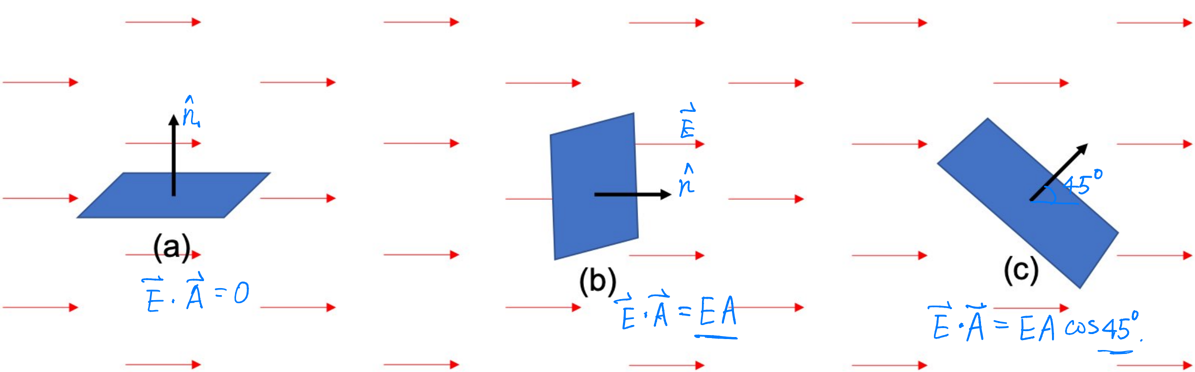
where

  - the "closed surface integral" $\oint_S \mathbf{E} \cdot d\mathbf{S}$ is called the **flux** of $\mathbf{E}$ over surface S bounding the volume $V = \frac{4\pi}{3} r^3$,
  - which in turn denotes the limiting value of the sum of dot products $\mathbf{E}_j \cdot \Delta\mathbf{S}_j$ computed over all surface elements of S having incremental areas $|\Delta\mathbf{S}_j|$ and unit vectors $\Delta\mathbf{S}_j/|\Delta\mathbf{S}_j|$ pointing away from volume V — the limiting value is obtained as all $|\Delta\mathbf{S}_j|$ approach zero (i.e., with increasingly finer subdivision of S into $|\Delta\mathbf{S}_j|$ elements).



Surface integral depends only on the net amount of charge contained within the surface --- charges outside the surface don't matter; surface shape doesn't matter; also charge motion within the surface does not matter.

Compare $\vec{E} \cdot \vec{A}$



$\vec{E} \cdot \vec{A} = 0$

$\vec{E} \cdot \vec{A} = \underline{EA}$

$\vec{E} \cdot \vec{A} = EA \cos \underline{45^\circ}.$

Although we obtained the equality $\epsilon_o \oint_S \mathbf{E} \cdot d\mathbf{S} = Q$ above only for a spherical surface S centered about charge Q, we can easily convince ourselves — see the sketches on the right — that the equality should hold even when we distort the shape of surface S and/or displace Q away from the center so long as we do not move Q outside of S. All such variations are permitted because of inverse r-square dependence of the Coulomb's law and additive nature of fields, and if Q is moved outside the surface then the surface integral (flux) simply goes to zero.

$\vec{E}$ Q $\vec{r}$

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_o r^2}\hat{r}$$

- Hence, given an arbitrary shaped volume V enclosed by an arbitrary shaped surface S and including a net electrical charge Q_V, and defining a **displacement** field

  $$\mathbf{D} \equiv \epsilon_o \mathbf{E},$$

  constitutive relation

  m^2 C $D = C/m^2$

  we obtain

  $$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V, \quad \text{Gauss's law}$$

  a constraint known as **Gauss's law**. At this stage, the introduction of $\mathbf{D}$ is simply a notational convenience.

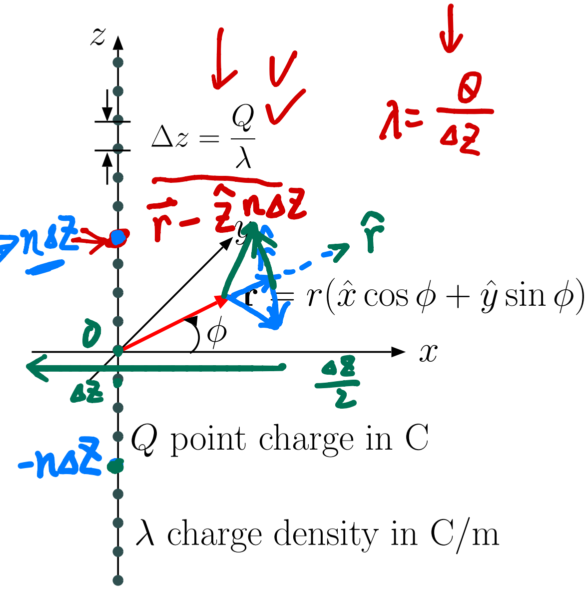
Displacement
$\mathbf{D} = \epsilon_o \mathbf{E} [=] \frac{\mathrm{F}}{\mathrm{m}} \frac{\mathrm{V}}{\mathrm{m}} = \frac{\mathrm{C}}{\mathrm{m}^2}$

Gauss's law

Gauss's law offers an alternative to *implementing an explicit sum of Coulomb fields* for calculating static field distributions $\mathbf{E}$ or $\mathbf{D} = \epsilon_o \mathbf{E}$ — the alternative method can be used when charge distributions have simplifying symmetry properties as will be illustrated in the next set of examples.

Also, later on we will learn that Gauss's law is valid even when charges Q_V within volume V are non-static (i.e., in motion), a condition under which Coulomb's law is no longer valid.

Example 2: Charged particles Q are located uniformly along the z-axis with an average line density of λ C/m extending from $z = -\infty$ to $+\infty$. We will compute the electrostatic field $\mathbf{E}$ of this charge distribution at a distance r from z-axis.



$\lambda = \frac{Q}{\Delta z}$

Having an average charge density of λ C/m implies that individual charges Q are spaced from one another by a distance $\Delta z = \frac{Q}{\lambda}$ along the z-axis. Assuming that charge locations are $z = n\Delta z$, where n is any integer, and using Coulomb's law, we find that

$$\mathbf{E}(\mathbf{r}) = \sum_{n=-\infty}^{\infty} \frac{Q}{4\pi\epsilon_o|\mathbf{r} - \hat{z}n\Delta z|^2}\frac{\mathbf{r} - \hat{z}n\Delta z}{|\mathbf{r} - \hat{z}n\Delta z|} = \sum_{n=-\infty}^{\infty} \frac{\lambda\Delta z(\mathbf{r} - \hat{z}n\Delta z)}{4\pi\epsilon_o|\mathbf{r} - \hat{z}n\Delta z|^3},$$

$\Delta z \ll r$

line charge.

which, for position $\mathbf{r} = r(\hat{x}\cos\phi + \hat{y}\sin\phi)$ on xy-plane, at a distance r to the z-axis, reduces to

$$\mathbf{E} = \sum_{n=-\infty}^{\infty} \frac{\lambda r(\hat{x}\cos\phi + \hat{y}\sin\phi)}{4\pi\epsilon_o(r^2 + n^2\Delta z^2)^{3/2}}\Delta z \quad \text{(microscopic field)}$$ exact

because the $\hat{z}$ component of $\mathbf{E}$ proportional to $n\Delta z$ cancels out (as a result of summation) due to symmetry in n. This field is "purely radial" in the direction

$$\hat{r} \equiv \hat{x}\cos\phi + \hat{y}\sin\phi$$ in xy plane

$\vec{E}(r) = \frac{Q}{4\pi\epsilon_o r^2}\hat{r}\ \frac{V}{m}$

perpendicular to z-axis, and it can be evaluated, for $r \gg \Delta z$, as an integral (remember that sums of infinitesimals are *in effect* definite integrals) C/m.

$$\hat{r}\int_{-\infty}^{\infty} \frac{\lambda r}{4\pi\epsilon_o(r^2 + z^2)^{3/2}}dz = \hat{r}\frac{\lambda r}{4\pi\epsilon_o}\underbrace{\int_{-\infty}^{\infty} \frac{dz}{(r^2 + z^2)^{3/2}}}_{2/r^2} = \hat{r}\frac{\lambda}{2\pi\epsilon_o r}. \quad \text{(macroscopic field)}$$

$\propto \frac{1}{r^2}\ \frac{Q}{r^2} - C/m^2$

$\propto \frac{1}{r}$ $\quad \lambda/r - C/m^2$

- The result
$$\mathbf{E} = \hat{r}\frac{\lambda}{2\pi\epsilon_o r}$$
obtained above, valid for $r \gg \Delta z$, and labelled as ***macroscopic field***, also represents at any r (and z) the *space average* of the ***microscopic field*** taken over small volumes having dimensions of many Δz's (inter-particle separations).

  - In such a spatial average the *rapidly varying* structure of microscopic field (in particular at small r, caused by the discrete nature of charge distribution) is smoothed out as if electrical charge were distributed in space with a continuous density of λ C/m.
  - In realistic applications involving colossal numbers of charge carriers (of the order of 10^{23} in macroscopic chunks of solids) it is practical (and desirable) to focus our attention on macroscopic rather than microscopic fields.

We next illustrate how to obtain the macroscopic field $\mathbf{E} = \hat{r}\frac{\lambda}{2\pi\epsilon_o r}$ directly by using Gauss's law.

Solution using Gauss's law:

We first notice that macroscopic electric field of a charge distribution along the z-axis having an average charge density of λ C/m should be pointing in radial direction $\hat{r}$ away from the z-axis (why?).

Also its magnitude E_r should be independent of azimuth angle ϕ by symmetry.

As a consequence, we can apply Gauss's law $\epsilon_o E_r \hat{r}$

$$\Delta \qquad \oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

as

S

$$0 + 0 + \epsilon_o E_r 2\pi r \cancel{L} = \lambda \cancel{L}$$

over the surface S of a cylindrical volume V of some length L and radius r centered about the z-axis as shown in the margin — notice that our "clever" choice of surface S in this problem resulted in the evaluation of the flux integral in Gauss's law without doing any calculus.

Clearly, this leads to (as obtained before using a line integral)

$$E_r = \frac{\lambda}{2\pi\epsilon_o r} \quad \text{and} \quad \mathbf{E} = \frac{\lambda}{2\pi\epsilon_o r}\hat{r}.$$