Lecture 8

1 Poisson's and Laplace's equations

1.1 Laplacian

From Gauss's Law $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$ (valid for electrostatic and electromagnetic field), for electrostatic field, we have $\vec{E} = -\nabla \mathrm{V}$, then

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla \mathrm{V}) = \frac{\rho}{\varepsilon_o}$$

We define **Laplacian** $\nabla^2 \mathrm{V} = \nabla \cdot \nabla \mathrm{V}$ as $\nabla^2 \equiv$ ______________________________

1.2 Poisson's equations

For an electrostatic potential,

$$\nabla^2 \mathrm{V} = \frac{\rho}{\varepsilon_o}$$

Solving the Poisson equation amounts to finding the electric potential V for a given charge distribution ρ.

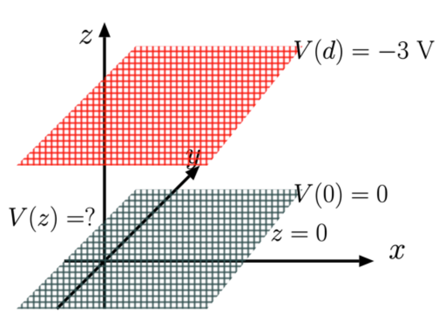
1.3 Laplace's equations

A special case of Poisson's equation for charge free regions where $\rho(x, y, z) = 0$ is the Laplace's equation

$$\nabla^2 \mathrm{V} = 0$$

2 Examples

2.1 Infinite parallel plates



Consider a pair of infinite parallel conducting plates at the z = 0 and z = 2 m planes. The plate at z = 0 is grounded. The plate at z = 2 m is held at a constant potential V = −3 V.

- Due to the symmetry of the geometry, E field is only pointing perpendicular to the plate.
- Relation between $\vec{E}$ and V in 1D case is $V = Ed$, where d is the separation of the plates.

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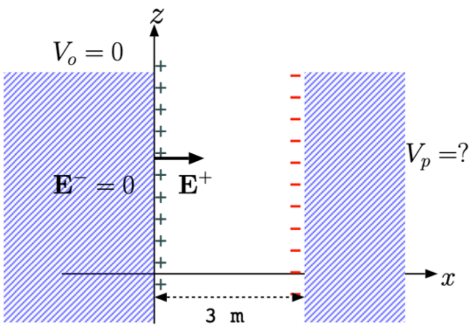
- Solution of 1D case is a linear function $V(z) = Ax + B$.
- The constants A and B can be determined by the potential on two planes.

2.2 Double-slab problem



A pair of copper blocks separated by a distance $d = 3\,m$ in x direction, hold surface charge densities of $\rho_s = \pm 2\frac{C}{m}$ on $x = 0m$ and $x = 3m$ surfaces, respectively. The blocks are assigned constant potentials $V = 0$ and V_p. What is the potential difference V_p?

- The field $\vec{E}$ inside each block is ______________.
- Due to the symmetry of the geometry, $\vec{E}$ field is only pointing in ___________direction.
- Consider the positively charged infinite sheet at $x = 0m$, boundary condition is

$$\hat{x} \cdot \left(\vec{D}^+ - \vec{D}^-\right) = \rho_s$$

- The problem is again 1D problem. Relation between $\vec{E}$ and V in 1D case is $V = Ed$

3 Solution to Poisson's Equation

Recall the electric field $\vec{E}(\vec{r})$ and the electrostatic potential $V(\vec{r})$ from a single charge Q at the origin are:

$\vec{E}(\vec{r})$ = ___________________________________ $\propto 1/r^2$

$V(\vec{r})$ = ___________________________________ $\propto 1/r$

Consider a point charge shifted to location $\vec{r}'$, the potential $V(\vec{r}) = \frac{Q}{4\pi\varepsilon_o|\vec{r}-\vec{r}'|}$

For a small cube located at $\vec{r}'$ inside a charged region with charge density ρ, the total charge inside is $\rho(\vec{r}')d^3\vec{r}'$

Use superposition, for an arbitrary ρ existing over a finite region in space,

$$V(\vec{r}) = \int \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Where $\vec{r}$ is the observation location, and $\vec{r}'$ is the dummy variable for every location inside the charged region.

4 Summary of electrostatic field solutions

4.1 From V to $\vec{E}$

$$\vec{E} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

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4.2 From $\vec{E}$ to V

$V =$ ________________

4.3 From $\vec{E}$ to ρ

$\rho =$ ________________

4.4 From ρ to V

$V =$ ________________

4.5 From V to ρ

$\vec{E} =$ ________________

4.6 From ρ to $\vec{E}$

$\vec{E} =$ ________________

5 Conductor

The field inside a perfect electric conductor (PEC) $\vec{E} =$ ________________

The charge density inside a PEC $\rho =$ ________________

This is because after the conductor is inserted in to the region where $\vec{E}_o$ exists, free charges will pile up on the surface of the conductor, and produce a secondary field that cancels out the original $\vec{E}_o$, and create zero electric field inside the conductor.

$\vec{E} = 0$ inside the conductor implies that the entire conductor region is equipotential.

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