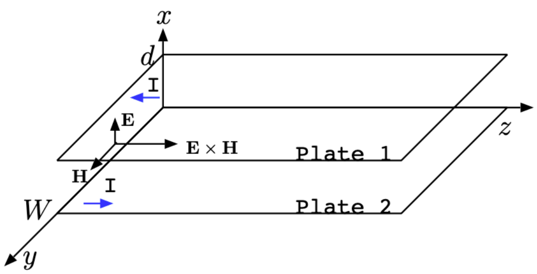
Lecture 29

1 Telegrapher's equations



In a parallel-plate transmission line, we assume that TEM wave fields in between the TL are

$$\vec{E} = \hat{x}E_x(z,t) \quad \text{and} \quad \vec{H} = \hat{y}H_y(z,t)$$

Let's derive Telegrapher's equation now.

Step 1: The $\vec{E}$ and $\vec{H}$ field must satisfy Maxwell's equations. Then from the two curl equations, we have

Faraday's Law $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \Rightarrow \frac{\partial E_x}{\partial t} = -\mu \frac{\partial H_y}{\partial t}$ (we drop $\hat{y}$ on both side of the equation)

Ampere's Law $\nabla \times \vec{H} = \sigma \vec{E} - \epsilon \frac{\partial \vec{E}}{\partial t} \Rightarrow -\frac{\partial H_y}{\partial t} = \sigma E_x + \epsilon \frac{\partial E_x}{\partial t}$ (we drop $\hat{x}$ on both side of the equation)

So two vector curl equation becomes two scalar equations.

Step 2: Now, multiply both equations by d (separation between the plates) and recognize the voltage between the places are $V = E_x d$, then we can use V to replace E_x.

$$\frac{\partial V}{\partial t} = -\mu d \frac{\partial H_y}{\partial t} \text{ and } -d\frac{\partial H_y}{\partial t} = \sigma V + \epsilon \frac{\partial V}{\partial t}$$

Step 3: We need to use current I to replace H_y in the above two equations.

Let's multiply the above two equations with plate width W, and we get

$$W\frac{\partial V}{\partial t} = -\mu W \frac{\partial H_y}{\partial t} \quad \text{and } -d\frac{\partial H_y}{\partial t} = W\sigma V + \epsilon W \frac{\partial V}{\partial t}$$

Notice that tangential magnetic field H_y is jumping across the plate inner surface, while the H field inside the plate is zero. So due to boundary condition, $H_y = J_{sz}$. Then current $I \equiv J_{sz}W = HyW$.

So divide the above equation by W and d, respectively

$$-\frac{\partial V}{\partial t} = \mathcal{L}\frac{\partial I}{\partial t} \quad \text{and} \quad -\frac{\partial I}{\partial t} = \mathcal{C}\frac{\partial V}{\partial t} + \mathcal{G}V$$

Where $\mathcal{L} = \mu \frac{d}{W}$, $\mathcal{C} = \epsilon \frac{W}{d}$, and $\mathcal{G} = \sigma \frac{W}{d}$

Assume the wires is lossless, then $\sigma = 0$. We get the lossless Telegrapher's equation. It is the govern equation of V and I on transmission lines, instead of $\vec{E}$ and $\vec{H}$.

$$-\frac{\partial V}{\partial t} = \mathcal{L}\frac{\partial I}{\partial t} \quad \text{and} \quad -\frac{\partial I}{\partial t} = \mathcal{C}\frac{\partial V}{\partial t}$$

2 Coefficients in Telegrapher's equations

Notice the geometric factor (GF) for parallel plates is $\frac{W}{d}$. Capacitance per unit length is ϵ times geometric factor, conductance per unit length is σ times geometric factor, and inductance per unit length is μ times the inverse of geometric factor.

(And the geometric factor for coaxial cable is $\frac{2\pi}{ln\frac{b}{a}}$. So for coaxial cables, $\mathcal{L} = \mu \frac{ln\frac{b}{a}}{2\pi}$, $\mathcal{C} = \epsilon \frac{2\pi}{ln\frac{b}{a}}$, and $\mathcal{G} = \sigma \frac{2\pi}{ln\frac{b}{a}}$)

3 Wave equation for V and I

If we combine the two Telegrapher's equation, we can obtain a 1D scalar wave equation for V and I.

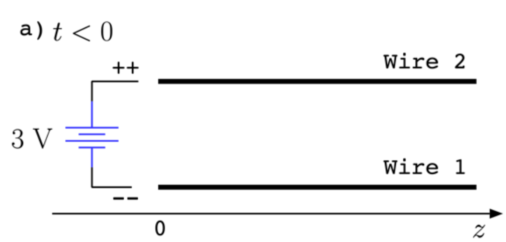
$$\frac{\partial^2 V}{\partial z^2} = \mathcal{L}\mathcal{C}\frac{\partial^2 V}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 I}{\partial z^2} = \mathcal{L}\mathcal{C}\frac{\partial^2 I}{\partial t^2}$$

The solution of the wave equation is d'Alembert wave solutions

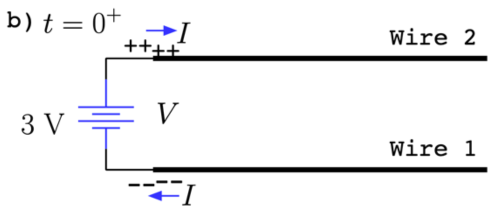
$$V(z,t) = f\left(t \mp \frac{z}{v}\right) \quad \text{where propagation speed } v \equiv \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = \frac{1}{\sqrt{\mu\epsilon}}$$

$$I(z,t) = \pm\frac{f\left(t \mp \frac{z}{v}\right)}{Z_o} \quad \text{where characteristic impedance } Z_o \equiv \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \frac{1}{GF}\sqrt{\frac{\mu}{\epsilon}}$$

4 Physical meaning of wave propagation

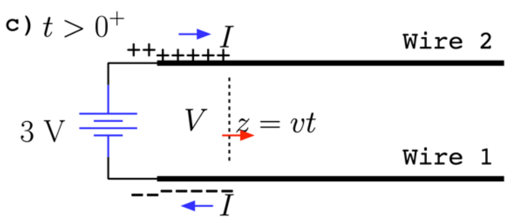


Assume we have an unload TL, such that V (z, t) = I(z, t) = 0 on the TL for t < 0.



At $t = 0$, the + and - terminals of a 3 V battery makes contact with the terminals of a charge neutral TL. the excess + and - charges on battery terminals will "spill onto" the TL terminals as shown in the figure for $t = 0^+$.

Currents I and voltage V marked in the diagram are confined only to location z = 0+ at t = 0+,



At $t > 0^+$, the currents I and voltage V starts to propagate along the TL.