Lecture 23

1 Maxwell's equations in time harmonic form

When the fields and the sources in Maxwell's equations are all monochromatic functions of time expressed in terms of their phasors, Maxwell's equations can be transformed into the phasor domain by using $\frac{d}{dt} \to j\omega$

	Time domain		**Frequency domain**
Gauss's law	$\nabla \cdot \vec{D} = \rho$		$\nabla \cdot \tilde{\vec{D}} = \tilde{\rho}$
	$\nabla \cdot \vec{B} = 0$		$\nabla \cdot \tilde{\vec{B}} = 0$
Faraday's law	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\frac{d}{dt} \longleftrightarrow j\omega$	$\nabla \times \tilde{\vec{E}} = -j\omega \tilde{\vec{B}}$
Ampere's law	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$		$\nabla \times \tilde{\vec{H}} = \tilde{\vec{J}} + j\omega \tilde{\vec{D}}$

Also we have $\tilde{\vec{D}} = \epsilon \tilde{\vec{E}}$, $\tilde{\vec{B}} = \mu \tilde{\vec{H}}$, and $\tilde{\vec{J}} = \sigma \tilde{\vec{E}}$.

2 Wave equation in phasor form

In a source-free lossless medium, we have $\tilde{\rho} = \tilde{\vec{J}} = 0$.

We take the Faraday's law and take its curl on both side of the equation,

$$\nabla \times \left(\nabla \times \tilde{\vec{E}}\right) = -j\omega \nabla \times \tilde{\vec{B}} = -j\omega\mu \nabla \times \tilde{\vec{H}}$$

On the left side, we use vector identity and Gauss's law $\nabla \times \nabla \times \tilde{\vec{E}} = \cancel{\nabla\left(\nabla \cdot \tilde{\vec{E}}\right)} - \nabla^2 \tilde{\vec{E}} = -\nabla^2 \tilde{\vec{E}}$

On the right side, we substitute Faraday's law $\nabla \times \tilde{\vec{H}} = j\omega\epsilon \tilde{\vec{E}}$ and get $-j\omega\mu \nabla \times \tilde{\vec{H}} = -j\omega\mu \times j\omega\epsilon \tilde{\vec{E}} = \omega^2 \mu\epsilon \tilde{\vec{E}}$

So we get the wave equation in phasor form as $\nabla^2 \tilde{\vec{E}} + \omega^2 \mu\epsilon \tilde{\vec{E}} = 0$

3 Wave solution in phasor form

In the source-free lossless media, we want to find a simple solution to the wave solution.

Assume the electric field is x-polarized monochromatic wave with phasors $\tilde{\vec{E}} = \hat{x} \tilde{E}_x(z)$, which is only a function of z position.

The vector wave solution can be simplified into $\frac{\partial^2}{dz^2}\tilde{E}_x + \omega^2\mu\epsilon\tilde{E}_x = 0$

Its solution is of the type $\tilde{E}_x(z) = e^{-\gamma z}$ or $e^{\gamma z}$, where $\gamma^2 = -\omega^2\mu\epsilon$

So we get $\gamma = \mp j\beta$, with $\beta = \omega\sqrt{\mu\epsilon}$

And the electric field phasor and magnetic field phasor are

$\tilde{E}_x(z) = e^{\mp j\beta z}$ and $\tilde{H}_y(z) = \frac{\pm e^{\mp j\beta z}}{\eta}$, with intrinsic impedance $\eta = \sqrt{\frac{\mu}{\epsilon}}$

Note: How do we derive the $\tilde{H}$ from $\tilde{E}$ for TEM waves?

4 Wave solution in lossy media

In a lossy media, we have $\sigma \neq 0$, thus only Faraday's law is changed from lossless case, now it still contains the $\tilde{\bar{J}}$ term: $\nabla \times \tilde{\bar{H}} = \tilde{\bar{J}} + j\omega\tilde{\bar{D}}$

Since $\tilde{\bar{J}} = \sigma\tilde{\bar{E}}$, Faraday's law becomes $\nabla \times \tilde{\bar{H}} = \sigma\tilde{\bar{E}} + j\omega\epsilon\tilde{\bar{E}} = (\sigma + j\omega\epsilon)\tilde{\bar{E}}$

The Maxwell's equation in lossy media is the same as the ones in lossless media, except that $j\omega\varepsilon$ has been replaced by $\sigma + j\omega\varepsilon$.

→ The solutions in lossy media also are similar with the ones in lossless media, except we need to replace $j\omega\varepsilon$ by $\sigma + j\omega\varepsilon$.

Lossless media		Lossy media
$\gamma^2 = -\omega^2\mu\epsilon$	$\sigma \neq 0$ →	$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)}$
$\eta = \sqrt{\frac{\mu}{\epsilon}}$	$\sigma \neq 0$ →	$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$

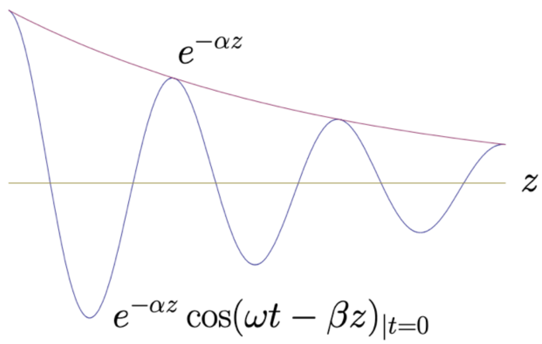
5 Plane waves propagating in lossy media

For example, in the lossy media, we can have x-polarized plane wave propagating in z direction in terms of phasors as

$\tilde{\bar{E}}(z) = \hat{x}E_o e^{-\gamma z}$ and $\tilde{\bar{H}}(z) = \hat{y}\frac{E_o}{\eta}e^{-j\beta z}$

When we translate the $\tilde{\tilde{E}}(z)$ into its time varying form, we get

$$\vec{E}(z,t) = Re\left\{\tilde{\tilde{E}}(z)e^{j\omega t}\right\} = Re\left\{\hat{x}E_o e^{-(\alpha+j\beta)z}e^{j\omega t}\right\} = e^{-\alpha z}\cos\left(\omega t - \beta z\right)$$



So it's a damped wave. We plot its snapshot at $t = 0$ together with exponential envelope as a function of position z

$$e^{-\alpha z}\cos(\omega t - \beta z)\,|_{t=0} = e^{-\alpha z}\cos(\beta z)$$