Lecture 21

1 The wave equation

In source-free and homogeneous regions, we have the d'Alembert solutions wave equation $\frac{\partial^2 \vec{E}}{\partial z^2} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$ as $\vec{E}, \vec{H} \propto f\left(t \mp \frac{z}{v}\right)$, where the velocity $v = \frac{1}{\sqrt{\mu\epsilon}}$.
These waves are classified as uniform plane - *Transverse ElectroMagnetic* (TEM) waves.

Since the region is source-free, $\rho = \vec{J} = 0$, so Gauss's law tells us $\nabla \cdot \vec{E} = \nabla \cdot \vec{H} = 0$, $\vec{E}$ and $\vec{H}$ are both divergence free.

TEM wave:

- $\vec{E}$ and $\vec{H}$ vectors are transverse to the direction of propagation, which coincides with the direction of the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$
- $\vec{E}$ and $\vec{H}$ are orthogonal to each other.

d'Alembert wave solutions of 1-D scale wave equation $\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}$, which are $\vec{E} = \hat{x}\, f\left(t \mp \frac{z}{v}\right)$ and $H = \pm\hat{y}\, \frac{f\left(t \mp \frac{z}{v}\right)}{\eta}$, are called *uniform plane waves*.

2 Example: From E to B

Let $\vec{E} = \hat{x}\, \Delta\left(\frac{t - \frac{y}{c}}{\tau}\right)$ be a wave solution in free space where $\Delta\left(\frac{t}{\tau}\right)$ is a triangular waveform of duration τ peaking at $t = 0$ (defined in ECE 210). We will next provide two different solutions demonstrating how the wave field $\vec{B}$ accompanying $\vec{E}$ can be found.

Solution 1: Identify the $\vec{E}$ field to be a *TEM wave*, $\vec{E}$ is transverse to propagation direction y, and has a constant phase plane $t - \frac{y}{c} = const.$

a) Find the direction of $\vec{\mathrm{H}}$ based on right hand rule

- Place the fingers of your right hand in the direction of the electric field component.
- Direct your thumb toward the direction of wave travel (power flow).
- Rotate your fingers 90° in a direction so that a right-hand screw is formed.
- The new direction of your fingers is the direction of the magnetic field component.

b) Find the magnitude of $|\vec{\mathrm{H}}| = |\vec{\mathrm{H}}|/\eta$, Divide the electric field component by the wave impedance to obtain the corresponding magnetic field component.

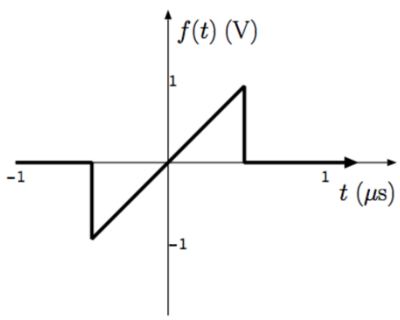
c) $\vec{\mathrm{B}} = \mu\vec{\mathrm{H}}$

Solution 2: *General* solution, doesn't require it to be a TEM wave.

a) From Faraday's law, $\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$

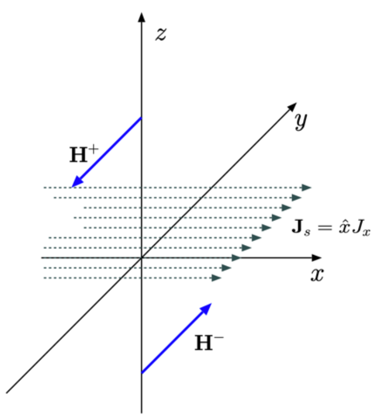
b) Calculate $\vec{B}$ by Finding the time-dependent anti derivative of $\frac{\partial \vec{B}}{\partial t}$

3 Example: Plots



Consider an $\hat{x}$-polarized plane TEM wave field in free space $\vec{E}(z,t) = \hat{x} f\left(t - \frac{z}{c}\right)$, where $f(t) = 2t\ \mathrm{rect}\left(\frac{t}{1us}\right)$, draw the time plot and position plot of $\vec{E}$ field.

4 Infinite current sheet as source

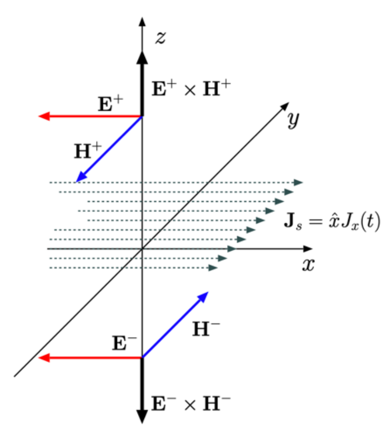


1. DC case

Consider first a static and **constant** surface current density $\vec{J_s} = \hat{x}\, J_x\ A/m$ flowing on $z = 0$ surface, this infinite surface current will produce a static magnetic field $\vec{\mathrm{H}}(z) = \mp\hat{y}\frac{J_x}{2}\ A/m$.

Q: Will it generate a static electric field?

A: No! Because there's no static charge carriers, only currents.



2. AC case

Suppose the current J_x varies with time $\vec{J_s} = \hat{x}\, J_x(t)\ A/m$, it will produce a time varying magnetic field of

$$\vec{\mathrm{H}}(z,t) = \mp\hat{y}\frac{J_x(t \mp \frac{z}{v})}{2}$$

According to d'Alembert solutions in homogeneous and source free regions.

Since $\vec{H}(z,t)$ is now time-varying, it will also induce an electric field.

$$\vec{E}(z,t) = -\hat{x}\frac{\eta}{2}J_x(t \mp \frac{z}{v})$$

Notice that magnetic field $\vec{\mathrm{H}}$ is pointing in different directions $\mp\hat{y}$, and it has a jump in the $z = 0$ plane, because of the boundary condition $n \times (\vec{H}^+ - \vec{H}^-) = \vec{J}_s$; while electric field $\vec{E}$ is always pointing in $-\hat{x}$ direction, because of the boundary condition $n \times (\vec{E}^+ - \vec{E}^-) = 0$.

Exercise: Check the $\vec{E}$, $\vec{\mathrm{H}}$, and the propagation direction $\vec{E} \times \vec{\mathrm{H}}$ on the upper and lower half space.

5 Poynting theorem

Poynting vector is defined as $\vec{S} = \vec{E} \times \vec{\mathrm{H}}$, its magnitude indicates amount of power transported by electromagnetic wave per unit area, and its direction is the wave propagation direction. Its unit is W/m^2.

Start from the Faraday's law and Ampere's law, $\nabla \times \vec{E} = -\mu\frac{\partial \vec{H}}{\partial t}$ and $\nabla \times \vec{H} = \vec{\mathrm{J}} + \epsilon\frac{\partial \vec{E}}{\partial t}$, dot multiply Faraday's law by $\vec{H}$, dot multiply Ampere's law by $\vec{E}$, we get

$\left(\nabla \times \vec{E} = -\mu\frac{\partial \vec{H}}{\partial t}\right) \cdot \vec{H}$ and $\left(\nabla \times \vec{H} = \vec{\mathrm{J}} + \epsilon\frac{\partial \vec{E}}{\partial t}\right) \cdot \vec{E}$

Subtract the 2nd equation from the 1st equation

$$\vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = -\mu\frac{\partial \vec{H}}{\partial t} \cdot \vec{H} - \vec{\mathrm{J}} \cdot \vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t} \cdot \vec{E}$$

The left hand is equal to ____________________ according to vector identity

The right hand can be rearranged into $-\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon\vec{E} \cdot \vec{E} + \frac{1}{2}\mu\vec{H} \cdot \vec{H}\right) - \vec{\mathrm{J}} \cdot \vec{E}$

Poynting theorem is a conservation law for electromagnetic energy as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon\vec{E} \cdot \vec{E} + \frac{1}{2}\mu\vec{H} \cdot \vec{H}\right) + \nabla \cdot (\vec{E} \times \vec{H}) + \vec{\mathrm{J}} \cdot \vec{E} = 0$$

just like continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$ is the conservation law for electric charge.

- First term $\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon\vec{E} \cdot \vec{E} + \frac{1}{2}\mu\vec{H} \cdot \vec{H}\right)$ is time rate of change of total electric and magnetic energy density.
- Second term $\nabla \cdot (\vec{E} \times \vec{H})$ is energy transport in Poynting theorem.
- Last term $\vec{\mathrm{J}} \cdot \vec{E}$ is called Joule heating. It accounts for power absorbed per unit volume.