Lecture 20

1 The wave equation

In source-free and homogeneous regions where $\rho = \vec{J} = 0$ and ϵ and μ are constant, we can simplify Maxwell's equations as:

$$\nabla \cdot \vec{E} = 0 \qquad \text{Gauss's law}$$

$$\nabla \cdot \vec{H} = 0$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \text{Faraday's law}$$

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} \qquad \text{Ampere's law}$$

Where we use $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$ to write this set only in terms of $\vec{E}$ and $\vec{H}$.

Remember that we can derive two divergence equations from the two curl equations together with the continuity equation. Now we will combine the two curl equations.

1.1 Wave equation for E

We take the Faraday's law and apply curl operation on both sides, and we get

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H} \tag{1}$$

From the vector identity $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$.

Since the electric field $\vec{E}$ is divergence-free in the absence of sources (Gauss's law, $\nabla \cdot \vec{E} = 0$), thus $\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E}$, which is the left hand side of Equ. 1.

Substituting the Ampere's law to the right hand side of Equ. 1, we obtain the vector wave equation for the electric field $\vec{E}$:

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

1.2 Wave equation for H

We take the Ampere's law and apply curl operation on both sides, and we get

$$\nabla \times \nabla \times \vec{H} = \epsilon \frac{\partial}{\partial t} \nabla \times \vec{E} \tag{2}$$

From the vector identity $\nabla \times \nabla \times \vec{H} = \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H}$.

Since the electric field $\vec{H}$ is always divergence-free due to non-existence of magnetic monopole ($\nabla \cdot \vec{H} = 0$), thus $\nabla \times \nabla \times \vec{H} = -\nabla^2 \vec{H}$, which is the left hand side of Equ. 2.

Substituting the Faraday's law to the right hand side of Equ. 2, we obtain the vector wave equation for the electric field $\vec{H}$:

$$\nabla^2 \vec{H} = \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

The electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a source-free medium or in a vacuum.

The wave equations for $\vec{E}$ and $\vec{H}$ have the same format. Remember, same equation same solution.

2 Wave equation in 1-D space

We want to find a simple, non-trivial solution to the wave equation for $\vec{E}$. Let's assume the solution is x-polarized and only depends on z and t: $\vec{E}(x, y, z, t) = \hat{x} E_x(z, t)$

Then 1D scalar wave equation becomes $\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}$

A simple non-zero solution is $E_x = \cos(\omega(t - \sqrt{\mu\epsilon} z))$ or $E_x = \cos\left(\omega\left(t - \frac{z}{v}\right)\right)$, where the velocity $v = 1/\sqrt{\mu\epsilon}$.

Q: is $E_x = \cos(\omega(t + \sqrt{\mu\epsilon} z))$ also solution to the 1D scalar wave equation? A: <u>Y / N</u>

So a general solution in material media is $E_x = \cos(\omega(t \mp \sqrt{\mu\epsilon} z))$ or $E_x = \cos\left(\omega\left(t \mp \frac{z}{v}\right)\right)$. and a general solution in vacuum is $E_x = \cos\left(\omega\left(t \mp \sqrt{\mu_o \epsilon_o} z\right)\right)$ or $E_x = \cos\left(\omega\left(t \mp \frac{z}{c}\right)\right)$. The – sign corresponds to a wave propagating in $+z$ direction, while the $+$ sign corresponds to a wave propagating in $-z$ direction.

3 Intrinsic Impedance

Let $\vec{E} = \cos(\omega(t \mp \sqrt{\mu\epsilon} z))\hat{x}$, we want to find out the magnetic field intensity H that accompanies this x-polarized electric field wave solution

Using the Faraday's law $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$, we first calculate the left hand side $\nabla \times \vec{E}$,

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Finding the time-dependent anti-derivative (and using $v = 1/\sqrt{\mu\epsilon}$), we obtain

$\vec{H}$=__________________________

The ratio between the magnitude of $\vec{E}$ and $\vec{H}$ is called the intrinsic impedance $\eta = \sqrt{\frac{\mu}{\epsilon}}$

The intrinsic impedance of vacuum is $\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \approx 120\pi \approx 377\ [\Omega]$

4 d'Alembert wave solution

For an arbitrary waveform f(t), solutions of the 1D scalar wave equation are known as d'Alembert solutions

$$\vec{E}, \vec{H} \propto f\left(t \mp \frac{z}{v}\right)$$

Verification

Suppose $E_x = f\left(t \mp \frac{z}{v}\right)$, or $E_x = f(u)$ and $u = t \mp \frac{z}{v}$. Using chain rule, we have

$\frac{\partial E_x}{\partial z} =$	$\frac{\partial E_x}{\partial t} =$
$\frac{\partial^2 E_x}{\partial z^2} =$	$\frac{\partial^2 E_x}{\partial t^2} =$