Lecture 15

1 Magnetostatics formula

1.1 Ampere's Law

Integral form: $\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S}$

Differential form: $\nabla \times \vec{H} = \vec{J}$

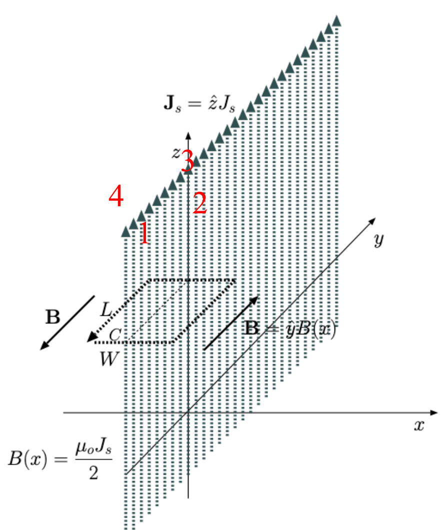
Where $\vec{B} = \mu_o \vec{H}$

1.2 Gauss's law for magnetic field

Differential form: $\nabla \cdot \vec{B} = 0$

2 Ampere's Law Examples

2.1 Infinite current sheet



Consider a uniform surface current density $J_s = J_s\hat{z}\ A/m$ flowing on $x = 0$ plane. The current sheet extends infinitely in y and z directions. Determine $\vec{B}$ and $\vec{H}$.

Similar to the Gaussian pillbox approach for electrostatic problems, this time we will use a rectangular box (dashed line) to calculate circulation integral of $\vec{B}$.

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_o I_C$$

Due to the symmetry of the sheet structure, the magnetic field $\vec{B}$ is pointing in _____ direction for $x > 0$, and $\vec{B}$ is pointing in _____ direction for $x < 0$.

Along rectangular edge 1&3, $\vec{B}$ and $d\vec{l}$ is perpendicular, so $\vec{B} \cdot d\vec{l} = 0$

So expand Ampere's law as $B(x)L + 0 - B(-x)L + 0 = \mu_o J_s L$

And we obtain $\vec{B} = \hat{y}\frac{\mu_o J_s}{2} sgn(x)$ and $\vec{H} = \hat{y}\frac{J_s}{2} sgn(x)$

The magnitude of Magnetic field on both sides is independent of position.

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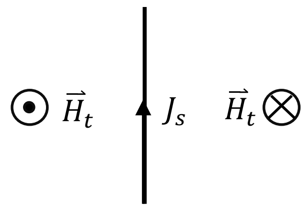
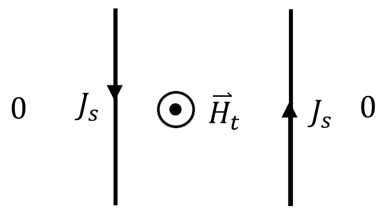
Case 1:



For a surface current sheet of current density J_s going to the top, using the right hand rule, magnetic field H is going out of the paper on right side, and going into the paper on left side.

Tangential component of $\vec{H}$ jumps by the amount of the surface current J_s.

Case 2:

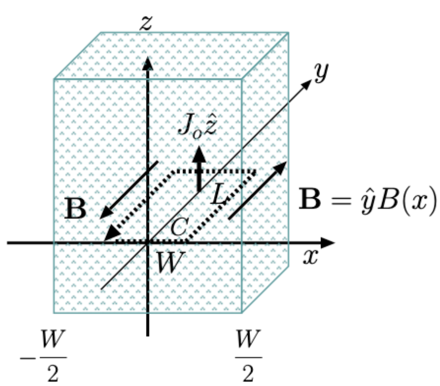


If we have two surface current sheets of the same amount of current density J_s, one going to the top, and the other one going to the bottom.

Using superposition,

- The magnetic field $\vec{H}$ only exist in the interior of the sheets, its magnitude is J_s. (Again, because tangential component of $\vec{H}$ jumps by the amount of the surface current J_s.)
- The magnetic field $\vec{H}$ at the exterior of the sheets is 0.

2.2 Infinite current slab



Consider a slab of thickness W over $-\frac{W}{2} < x < \frac{W}{2}$ which extends infinitely in y and z directions and conducts a uniform current density of $\vec{J} = \hat{z}J_o\ A/m^2$. Determine $\vec{H}$ if the current density is zero outside the slab.

Similar to previous example, we choose a rectangular path C for the circulation integral

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_o I_C.$$

When path C is inside the slab, it expands to

$B(x)L + 0 - B(-x)L + 0 =$ ___________________ → B inside the slab = _________________

When path C is inside the slab, it expands to

$B(x)L + 0 - B(-x)L + 0 =$ ___________________ → B outside the slab = ________________

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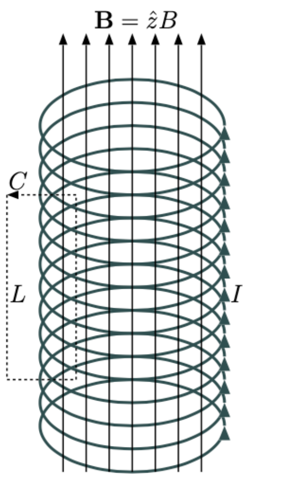
$\vec{H} = \begin{cases} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{cases}$

2.3 Infinite solenoid



An infinite solenoid having N loops per unit length is stacked in z-direction, each loop carrying a current of I A in counter-clockwise direction when viewed from the top. Determine $\vec{H}$.

Infinite solenoid has magnetic field only inside the loops, there's no magnetic field outside the loops.

3 Summary of static electric fields and static magnetic fields

- Static electric fields: *Curl-free* and are governed by $\nabla \times \vec{E} = 0$, $\nabla \cdot \vec{D} = \rho$ where $\vec{D} = \varepsilon\vec{E}$ with $\varepsilon = \varepsilon_r \varepsilon_o$.
- Static magnetic fields: *Divergence-free* and are governed by $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{H} = \vec{J}$ where $\vec{B} = \mu\vec{H}$ with $\mu = \mu_r \mu_o$. μ_r is the relative permeability.

4 Magnetostatic potential

Since magnetostatic field $\vec{B}$ is divergence-free ($\nabla \cdot \vec{B} = 0$). $\vec{B}$ field can be written as the curl of a vector $\vec{A}$, where

$$\vec{B} = \nabla \times \vec{A}$$

$\vec{A}$ is called **magnetostatic potential** or vector potential.

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Together with the chosen Coulomb gauge $\nabla \cdot \vec{A} = 0$, because of the Helmholtz theorem, (Given their curl and divergences, vector fields can be uniquely reconstructed in regions V of 3D space, see Lect 04), we can solve for a unique $\vec{A}$.

Magnetostatic version of **Poisson's equation** $\nabla^2 \vec{A} = -\mu_o \vec{J}$

General approach to solve for magnetic field in magnetostatics:

Given $\vec{J}$, use Poisson's equation $\nabla^2 \vec{A} = -\mu_o \vec{J}$ to solve for $\vec{A}$, then take $\vec{B} = \nabla \times \vec{A}$ and $\vec{B} = \mu \vec{H}$ to get $\vec{B}$ and $\vec{H}$.

General approach to solve for electric field in electrostatics:

Given ρ, use Poisson's equation $\nabla^2 V = -\rho/\varepsilon_o$ to solve for V, then take $\vec{E} = -\nabla V$ and $\vec{D} = \varepsilon \vec{E}$ to get $\vec{E}$ and $\vec{D}$.

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