

1. (10 points) Verifying vector identity

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = \nabla \cdot (\mathbf{E} \times \mathbf{H})$$

for $\mathbf{E} = 5\hat{z}e^{-\alpha y}$ and $\mathbf{H} = 10\hat{x}e^{-\alpha y}$. The left-hand side of the identity gives

$$\begin{aligned} \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} &= (10\hat{x}e^{-\alpha y}) \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 5e^{-\alpha y} \end{vmatrix} - (5\hat{z}e^{-\alpha y}) \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 10e^{-\alpha y} & 0 & 0 \end{vmatrix} \\ &= (10\hat{x}e^{-\alpha y}) \cdot (-5\alpha\hat{x}e^{-\alpha y}) - (5\hat{z}e^{-\alpha y}) \cdot (10\alpha\hat{z}e^{-\alpha y}) \\ &= -50\alpha e^{-2\alpha y} - 50\alpha e^{-2\alpha y} \\ &= -100\alpha e^{-2\alpha y}, \end{aligned}$$

and the right-hand side gives

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \nabla \cdot (5\hat{z}e^{-\alpha y} \times 10\hat{x}e^{-\alpha y}) \\ &= \nabla \cdot (50\hat{y}e^{-2\alpha y}) \\ &= \frac{\partial}{\partial y} (50e^{-2\alpha y}) \\ &= -100\alpha e^{-2\alpha y}. \end{aligned}$$

Thus, the identity is verified.

2. (a) (6 points) The closed surface S can be decomposed into 6 surfaces:

$$S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6,$$

where

$$\begin{aligned} S_1 : & \quad x = -1.5 \\ S_2 : & \quad x = 1.5 \\ S_3 : & \quad y = -1.5 \\ S_4 : & \quad y = 1.5 \\ S_5 : & \quad z = -1.5 \\ S_6 : & \quad z = 1.5 \end{aligned}$$

Therefore,

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{S_1+S_2+S_3+S_4+S_5+S_6} \mathbf{J} \cdot d\mathbf{S},$$

where

$$\begin{aligned} \int_{S_1+S_2} \mathbf{J} \cdot d\mathbf{S} &= \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} [4x(y-1)^2\hat{x} + 6y\hat{y} + 8x^2y^2\hat{z}]|_{x=-1.5} \cdot (-\hat{x}) dydz \\ &\quad + \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} [4x(y-1)^2\hat{x} + 6y\hat{y} + 8x^2y^2\hat{z}]|_{x=1.5} \cdot \hat{x} dydz \\ &= \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} 6(y-1)^2 dydz + \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} 6(y-1)^2 dydz \\ &= 189 \text{ [A]}. \end{aligned}$$

$$\begin{aligned}
\int_{S_3+S_4} \mathbf{J} \cdot d\mathbf{S} &= \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} [4x(y-1)^2\hat{x} + 6y\hat{y} + 8x^2y^2\hat{z}]|_{y=-1.5} \cdot (-\hat{y}) dx dz \\
&\quad + \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} [4x(y-1)^2\hat{x} + 6y\hat{y} + 8x^2y^2\hat{z}]|_{y=1.5} \cdot \hat{y} dx dz \\
&= \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} 9 dx dz + \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} 9 dx dz \\
&= 162 \text{ [A]}.
\end{aligned}$$

$$\begin{aligned}
\int_{S_5+S_6} \mathbf{J} \cdot d\mathbf{S} &= \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} [4x(y-1)^2\hat{x} + 6y\hat{y} + 8x^2y^2\hat{z}]|_{z=-1.5} \cdot (-\hat{z}) dx dy \\
&\quad + \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} [4x(y-1)^2\hat{x} + 6y\hat{y} + 8x^2y^2\hat{z}]|_{z=1.5} \cdot \hat{z} dx dy \\
&= \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} (-8x^2y^2) dx dy + \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} (8x^2y^2) dx dy \\
&= 0 \text{ [A]}.
\end{aligned}$$

Therefore, $\oint_S \mathbf{J} \cdot d\mathbf{S} = 351 \text{ [A]}$.

(b) (3 points) We know that the unit of J_x, J_y, J_z is $[\text{A}/\text{m}^2]$, and x, y, z has unit $[\text{m}]$, thus

$$\begin{aligned}
[J_x] &= [4x(y-1)^2] = [\text{A}/\text{m}^2] \rightarrow [5] = [\text{A}/\text{m}^5], \\
[J_y] &= [6y] = [\text{A}/\text{m}^2] \rightarrow [6] = [\text{A}/\text{m}^3], \\
[J_z] &= [8x^2y^2] = [\text{A}/\text{m}^2] \rightarrow [8] = [\text{A}/\text{m}^6].
\end{aligned}$$

(c) (2 points) Based on the integral form of charge conservation

$$\frac{dQ}{dt} = \frac{d}{dt} \int_V \rho dV = - \oint_S \mathbf{J} \cdot d\mathbf{S} = -351 \text{ [A]},$$

which means the total charge contained in the cube is decreasing.

(d) (4 points) Taking the divergence of $\mathbf{J} = 4x(y-1)^2\hat{x} + 6y\hat{y} + 8x^2y^2\hat{z}$ $[\text{A}/\text{m}^2]$, we get

$$\nabla \cdot \mathbf{J} = \frac{\partial}{\partial x} (4x(y-1)^2) + \frac{\partial}{\partial y} (6y) + \frac{\partial}{\partial z} (8xy^2) = 4(y-1)^2 + 6.$$

According to the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$, we have

$$\rho(0,0,0,t) - \rho(0,0,0,0) = \int_0^t \frac{\partial \rho}{\partial t'}|_{(0,0,0)} dt' = - \int_0^t \nabla \cdot \mathbf{J}|_{(0,0,0)} dt' = - \int_0^t 10 dt' = -10t \text{ [C}/\text{m}^3].$$

Since $\rho(0,0,0,0) = 0$, we have

$$\rho(0,0,0,t) = -10t \text{ [C}/\text{m}^3].$$

3. (a) (2 points)

$$\therefore \quad \nabla \cdot \mathbf{J} = \nabla \cdot (\sigma \mathbf{E}) = \sigma \nabla \cdot \mathbf{E} = \sigma \nabla \cdot \left(\frac{\mathbf{D}}{\epsilon_0} \right) = \frac{\sigma}{\epsilon_0} \rho, \quad \text{Gauss' Law}$$

$$\text{and} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad \text{Continuity Equation}$$

$$\therefore \quad \frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_0} \rho = 0.$$

(b) (3 points) From $\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_0} \rho = 0$ we have $\frac{d\rho}{\rho} = -\frac{\sigma}{\epsilon_0} dt$, or

$$d(\ln \rho) = -\frac{\sigma}{\epsilon_0} dt.$$

By integrating both sides, we obtain $\ln \rho \big|_0^t = -\frac{\sigma}{\epsilon_0} t$, or

$$\ln \frac{\rho(x, y, z, t)}{\rho(x, y, z, 0)} = -\frac{\sigma}{\epsilon_0} t.$$

Therefore,

$$\rho(x, y, z, t) = \rho(x, y, z, 0) e^{-\frac{\sigma}{\epsilon_0} t} = 2\rho_0 \sin(100z) e^{-\frac{\sigma}{\epsilon_0} t} \text{ (C/m}^3\text{)}.$$

(c) (2 points) From $2\rho_0 \sin(100z) e^{-\frac{\sigma}{\epsilon_0} t} = \frac{2\rho_0}{e} \sin(100z)$, we obtain

$$e^{-\frac{\sigma}{\epsilon_0} t} = \frac{1}{e},$$

or

$$t = \frac{\epsilon_0}{\sigma} = \frac{8.854 \times 10^{-12}}{24\pi \times 10^5} \approx 1.18 \times 10^{-18} \text{ (s)}.$$

(d) (3 points) Because ρ is a function of z and t only, we have

$$\nabla \cdot \mathbf{E} = \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0} = \frac{2\rho_0}{\epsilon_0} \sin(100z) e^{-\frac{\sigma}{\epsilon_0} t}.$$

Therefore,

$$E_z = -\frac{\rho_0}{50\epsilon_0} \cos(100z) e^{-\frac{\sigma}{\epsilon_0} t}.$$

Furthermore, we get

$$\mathbf{J}(x, y, z, t) = \sigma \mathbf{E} = -\hat{z} \frac{\sigma \rho_0}{50\epsilon_0} \cos(100z) e^{-\frac{\sigma}{\epsilon_0} t}.$$

4. (a) (1 point) Gauss' Law: $\nabla \cdot \mathbf{D} = \rho$ ($= 0$ for this problem). Since $\mathbf{E} = \hat{x}E_x(z, t)$, which is independent of x and y coordinates, we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_0 \mathbf{E}) = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = 0.$$

- (b) (2 points) From Faraday's Law (Equation (2) in the problem), we have

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = - \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = - \left(\hat{x} \cdot 0 + \hat{y} \frac{\partial E_x}{\partial z} + \hat{z} \cdot 0 \right) = -\hat{y} \frac{\partial E_x}{\partial z}.$$

- (c) (2 points) Take one more curl of the result of Part (b)

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = - \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{\partial E_x}{\partial z} & 0 \end{vmatrix} = -\hat{x} \frac{\partial^2 E_x}{\partial z^2} + \hat{y} \cdot 0 + \hat{z} \cdot 0 = -\hat{x} \frac{\partial^2 E_x}{\partial z^2}.$$

- (d) (3 points) Again, from Faraday's Law (Equation (2) in the problem), after taking curl of both sides, we have

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}).$$

By plugging in Ampere's Law (Equation (4) in the problem), we obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\hat{x} \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}.$$

Combining the above equation with the result of Part (c) yields

$$-\hat{x} \frac{\partial^2 E_x}{\partial z^2} = -\hat{x} \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}, \text{ or } \frac{\partial^2 E_x}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0.$$

- (e) (2 points) The solution to the 1D scalar wave equation (propagating in the \hat{z} direction) can be written as

$$E_x(z, t) = \cos [\omega (t - \sqrt{\mu_0 \epsilon_0} z)] = \cos \left[\omega \left(t - \frac{z}{v} \right) \right].$$

Thus, the propagation speed is

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

5. (a) (2 points)

$$\frac{\partial^2 E_x}{\partial z^2} = -(\mp\beta)^2 E_0 \cos(\omega t \mp \beta z + \phi) = -\beta^2 E_0 \cos(\omega t \mp \beta z + \phi).$$

Since $\beta = \omega\sqrt{\mu_0\epsilon_0}$, the above equation becomes

$$\frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu_0 \epsilon_0 E_0 \cos(\omega t \mp \beta z + \phi).$$

On the other hand,

$$\mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = \mu_0 \epsilon_0 \cdot (-\omega^2) E_0 \cos(\omega t \mp \beta z + \phi) = -\omega^2 \mu_0 \epsilon_0 E_0 \cos(\omega t \mp \beta z + \phi).$$

Therefore,

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0.$$

(b) (6 points) Integrating

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

from 0 to t , we have

$$\begin{aligned} \mathbf{B}(z, t) - \mathbf{B}(z, 0) &= \pm \frac{E_0 \beta}{\omega} \hat{y} [\cos(\omega t \mp \beta z + \phi) - \cos(\pm \beta z + \phi)] \\ &= \pm \hat{y} \left[\frac{E_x}{c} - \frac{E_0}{c} \cos(\pm \beta z + \phi) \right] \\ &= \pm \hat{y} \left[\frac{E_x(z, t)}{c} - \frac{E_x(z, 0)}{c} \right]. \end{aligned}$$

If evaluated at $z = 0$, the above equation becomes

$$\mathbf{B}(0, t) - \mathbf{B}(0, 0) = \pm \hat{y} \left[\frac{E_x(0, t)}{c} - \frac{E_x(0, 0)}{c} \right].$$

Therefore,

$$\begin{aligned} \mathbf{B}(0, t) &= \pm \frac{E_x(0, t)}{c} \hat{y} \mp \frac{E_x(0, 0)}{c} \hat{y} + \mathbf{B}(0, 0) \\ &= \pm \frac{E_x(0, t)}{c} \hat{y}, \end{aligned}$$

which means \mathbf{B} and \mathbf{E} line up at $z = 0$ for all time t . If consider arbitrary z , we obtain

$$\begin{aligned} \mathbf{B}(z, t) &= f\left(t \mp \frac{z}{v_p}\right) = \mathbf{B}\left(0, t \mp \frac{z}{v_p}\right) \\ &= \pm \frac{E_x\left(0, t \mp \frac{z}{v_p}\right)}{c} \hat{y} = \pm \frac{E_x(z, t)}{c} \hat{y}, \end{aligned}$$

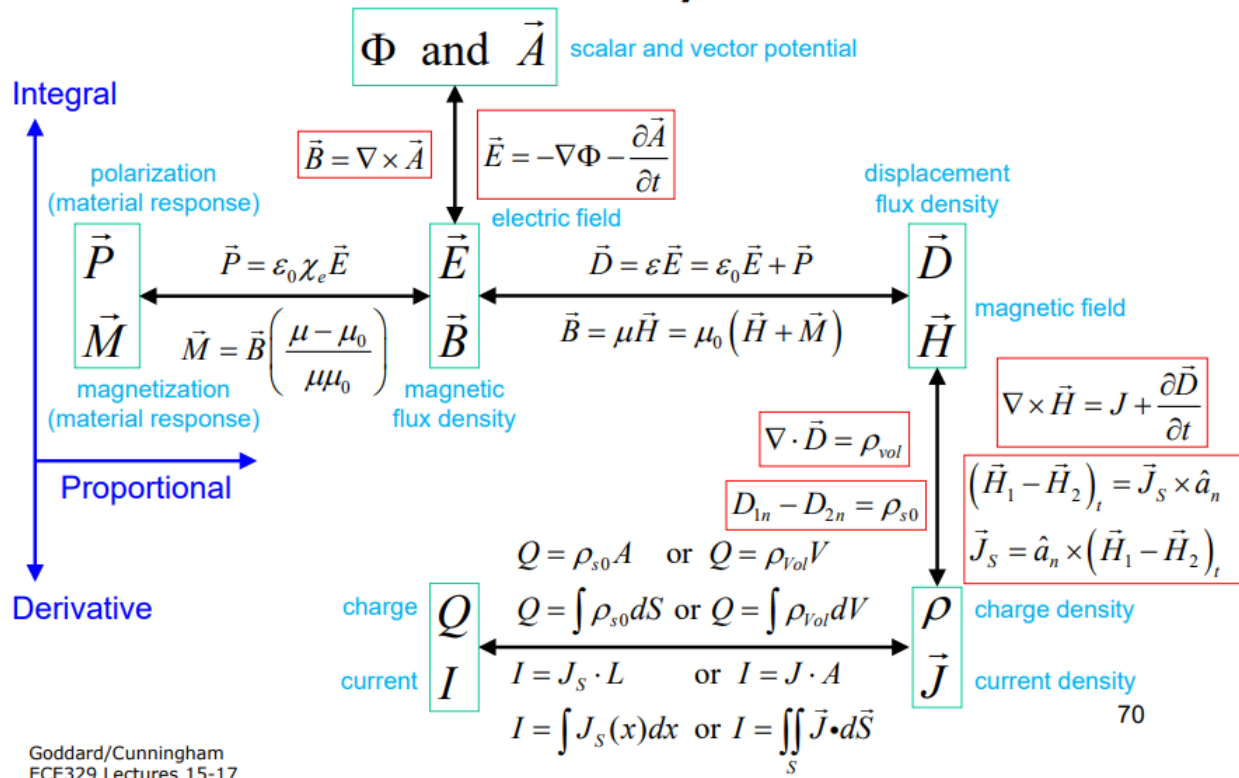
which means \mathbf{B} and \mathbf{E} line up for all z and t . As the phase shift, ϕ could be arbitrary value.

(c) (2 points) By comparing $\mathbf{B} = \pm \frac{E_x}{c} \hat{y}$ and $\mathbf{B} = \mu_0 \mathbf{H}$, we obtain

$$\mathbf{H} = \pm \frac{E_x}{\mu_0 c} \hat{y} = \pm \frac{E_x}{\mu_0} \sqrt{\mu_0 \epsilon_0} \hat{y} = \pm \frac{E_x}{\sqrt{\mu_0 / \epsilon_0}} \hat{y}.$$

If we define $\eta_0 = \sqrt{\mu_0 / \epsilon_0}$, then

$$\mathbf{H} = \pm \frac{E_x}{\eta_0} \hat{y}.$$



6. (5 points) Please refer to the above picture:
7. (10 points) (a) (2 points) The solution is (ii) from the continuity equation.
- (b) (4 points) The solution is (iii), (iv).
- (c) (4 points) The solution is (i), (ii), (v).