ECE 329 Fields and Waves I
Homework 2

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Due September 5, 2023, 11:59 PM

Homework Policy:

- Write your name and NetID on top of every page. This habit will help you in exams in the event of having loose page(s).
- Tag all the questions in Gradescope. Failure to do so results in a 5 points deduction.
- Cheating results in ZERO and 50% reduction in HW average on first offense. A 100% reduction in HW average on second offense.
- Please show detailed process for each problem instead of just an answer. No partial credits would be given otherwise. All answers should include units wherever appropriate.
- No late HW is accepted.
- Regrade requests are available one week following grade release.

You are allowed to work with anyone else, but the work you submit should only belong to you. Note that if you have knowledge of a violation of the Honor Code, then you are obligated to report it. By submitting this homework, you are agreeing to the Honor Code: "I have neither given nor received unauthorized aid on this homework, nor have I concealed any violations of the Honor Code."

Question	Points	Score
1	5	
2	5	
3	5	
4	10	
5	10	
6	5	
7	10	
8	5	
9	5	
10	10	
11	10	
Total:	80	

1. A particle of mass $m = 18\pi \times 10^{-3}\,\mathrm{kg}$ and charge Q is inserted at time $t = 0$ at a distance $d = 19.6\,\mathrm{m}$ above a planar sheet charged uniformly with electric charge density of $\rho_s = 9.8\,\frac{\mu\mathrm{C}}{\mathrm{m}^2}$. The distance d is much smaller than the dimensions of the sheet so that, for all practical purposes the sheet can be assumed to be of infinite extent surrounded by free space.

   (a) (2 points) What should the charge Q of the particle be in order for it to levitate motionless at the position where it was placed at $t = 0$? Assume that the Coulomb electric field generated by charge Q is insignificant relative to the electric field generated by the charged sheet (i.e., treat Q as a test charge).

   (b) (3 points) Consider, next, the case where $Q = 2\,\mathrm{C}$. Describe the motion of the particle for $t > 0$ by calculating its acceleration, $a(t)$, its velocity, $v(t)$, and its distance from the charged sheet. For your calculations, use as your reference coordinate system one with its plane $z = 0$ taken to coincide with the plane of the charged sheet.

2. Gauss' Law for electric field $\mathbf{E}$ states that $\oint_S \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\epsilon_0} \int_V \rho\, dV$ over any closed surface S enclosing a volume V in which electric charge density is specified by $\rho(x, y, z)\ \frac{\mathrm{C}}{\mathrm{m}^3}$.

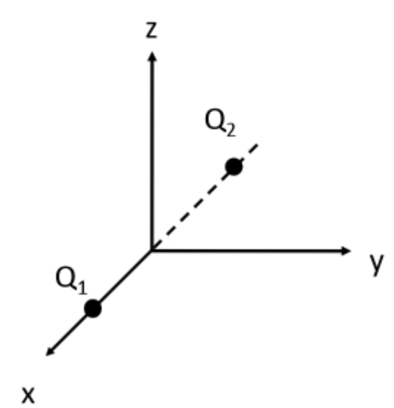
   (a) (1 point) What is the *electric flux* $\oint_S \mathbf{E} \cdot \mathbf{dS}$ over the surface of a cube of volume $V = L^3$ centered on the origin, if $\rho(x, y, z) = 4\ \mathrm{C/m}^3$ within V and $L = 1\,\mathrm{mm}$?

   (b) (2 points) Repeat (a) for $\rho(x, y, z) = x^2 + y^2 + z^2\ \frac{\mathrm{C}}{\mathrm{m}^3}$.

   (c) (1 point) What is the *displacement flux* $\oint_S \mathbf{D} \cdot d\mathbf{S}$ over the surface S in part (b)?

   (d) (1 point) What is the displacement flux in part (b) for any one of the square surfaces of volume V?

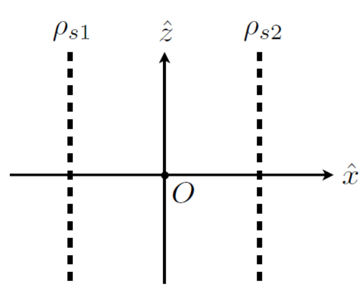
3. (5 points) Two unknown charges, Q_1 and Q_2 are located at $(x, y, z) = (1, 0, 0)$ and $(-1, 0, 0)$, respectively, as shown below. The displacement flux $\int_{yz-plane} \mathbf{D} \cdot \hat{x}\, dy\, dz$ through the entire $yz-$plane (i.e., the $x = 0$ plane) in the $+\hat{x}$ direction is $-2\,\mathrm{C}$. The displacement flux through the $y = 1$ plane in the $-\hat{y}$ direction is $3\,\mathrm{C}$. Determine Q_1 and Q_2 after writing a pair of algebraic equations relating the above displacement fluxes to Q_1 and Q_2. **Hint:** What is the contribution of Q_1 to the flux $\int_{yz-plane} \mathbf{D} \cdot \hat{x}\, dy\, dz$? See Example 5 in Lecture 3.



4. An infinitely long, cylindrical wire of radius $R = 1$ cm is centered along the $\hat{z}$ axis and carries a uniform volumetric charge density ρ_0. As with a uniform, infinitesimally thin line charge along $\hat{z}$ (see Example 2 in Lecture 2 online notes), the electric field associated with this charge distribution is everywhere oriented radially (with no azimuthal or $\hat{z}$ components), owing to the cylindrical symmetry of the charge distribution.
   (a) (3 points) Given that $E_r = -\frac{1}{3\epsilon_0}$ V/m at a distance $r = 3$ cm, use $\mathbf{D} = \epsilon_0\mathbf{E}$ and Gauss' Law with a cylindrical surface of radius $r = 3$ cm and length L to determine the charge density ρ_0 of the wire.
   (b) (3 points) Calculate the strength of the electric field $\mathbf{E}$ inside the wire, where $r = \frac{1}{3}$ cm. **Hint:** How does the total charge enclosed in a cylindrical Gaussian surface of radius $r < R$ vary with r?
   (c) (4 points) Show that the electric field strength is continuous at the surface of the wire, where $r = R$, and sketch the magnitude of the electric field as a function of r for $0 < r < 3R$.

5. Consider two infinite parallel slabs having equal widths W and equal charge densities. Slab 1 is parallel to the xy-plane and extends from $z = -2W$ to $z = -W$ while slab 2 extends from $z = W$ to $z = 2W$. Both slabs have a negative uniform charge density $-\rho_0\,\mathrm{C/m^3}$. The charge density is zero everywhere else.
   (a) (7 points) Determine and sketch the electric field component $\mathbf{E}_z$ in terms of ρ_0 over the region $-3W < z < 3W$ by using shifted and scaled versions of the static field configuration of a single charged slab (See Example 4 in Lecture 3 online). Be sure to label both axes of your plot and mark the field value at each break point.
   (b) (3 points) Verify that your field from part (a) satisfies Gauss' Law in differential form, $\nabla \cdot \mathbf{D} = \rho$ in each slab as well as in the free space regions.

6. (5 points) An infinite sheet that is uniformly charged with a density $\rho_s \,\frac{\mathrm{C}}{\mathrm{m}^2}$ produces an electrostatic field $\mathbf{E}$ which has a magnitude $\frac{\rho_s}{2\epsilon_0}$ and points away from the sheet on both sides. Using superposition, determine the charge density ρ_{s2} that is required to produce a displacement field $\mathbf{D} \equiv \epsilon_0\mathbf{E} = 6\hat{x}\,\frac{\mathrm{C}}{\mathrm{m}^2}$ at the origin (labeled O on the figure below), if the charge density ρ_{s1} is given as follows:



   (a) $\rho_{s1} = 8\,\frac{\mathrm{C}}{\mathrm{m}^2}$
   (b) $\rho_{s1} = -\rho_{s2}$

7. **Curl** and **divergence** exercises:
   (a) (3 points) On a 25-point graph consisting of x and y coordinates each having integer values -2, -1, 0, 1, 2, sketch the vector field $\mathbf{F} = x\hat{x} + y\hat{y}$ and calculate $\nabla \times \mathbf{F}$ (curl of $\mathbf{F}$) and $\nabla \cdot \mathbf{F}$ (divergence of $\mathbf{F}$).
   (b) (3 points) Repeat (a) for $\mathbf{F} = y\hat{x} + 2x\hat{y}$.
   (c) (2 points) (Select one): If $\nabla \times \mathbf{F} \neq 0$, the field strength varies **along** or **across** the direction of the field.
   (d) (2 points) (Select one): If $\nabla \cdot \mathbf{F} \neq 0$, the field strength varies **along** or **across** the direction of the field.

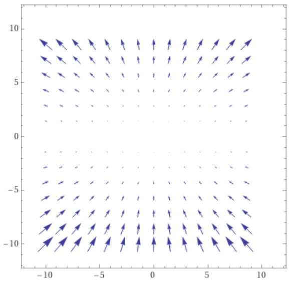
8. Given that $\mathbf{E} = \sin(y)\hat{x} + 2\cos(x)\hat{y}$ V/m in free space.
   (a) (3 points) Determine $\nabla \times \mathbf{E}$ and $\nabla \times \nabla \times \mathbf{E}$.
   (b) (2 points) Determine ρ such that Gauss' Law is satisfied.

9. Given an electrostatic potential $V(x, y, z) = 2x^2 - 3$ V in a certain region of free space, determine the corresponding:
   (a) (2 points) Electrostatic field $\mathbf{E} = -\nabla V$.
   (b) (1 point) Curl $\nabla \times \mathbf{E}$.
   (c) (1 point) Divergence $\nabla \cdot \mathbf{E}$.
   (d) (1 point) Charge density ρ in the region.

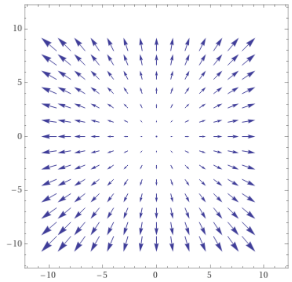
10. **Conceptual Question:**
   (a) (4 points) You have learned about Gauss' Law for electric field $\mathbf{E}$: $\oint_S \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\epsilon_0} \int_V \rho \, dV$ over any closed surface S enclosing a volume V in which electric charge density is specified by $\rho(x, y, z)$ $\frac{\mathrm{C}}{\mathrm{m}^3}$. In differential form, $\nabla \cdot \mathbf{D} = \rho$. Consider another type of field $\mathbf{F}$ with modified Gauss's law $\oint_S \mathbf{F} \cdot \mathbf{dS} = 0$. What would be the differential form for this field's Gauss's law?

   (i) $\nabla \cdot \mathbf{F} = 0$
   (ii) $\nabla \times \mathbf{F} = 0$
   (iii) $\nabla \times \mathbf{F} = \rho$
   (iv) $\nabla \mathbf{F} = 0$

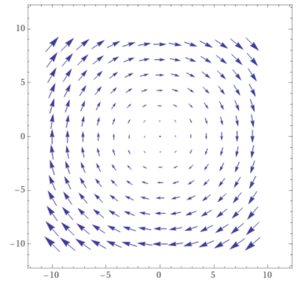
(b) (3 points) Which of the following fields has zero divergence?



(i)

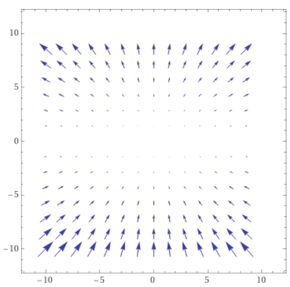


(ii)

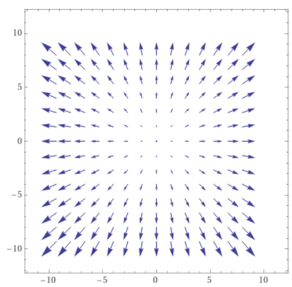


(iii)

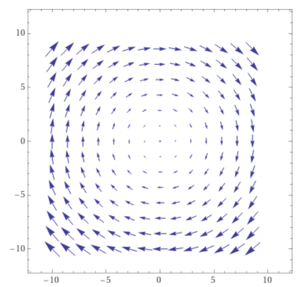
(c) (3 points) Which of the following fields has zero curl?



(i)



(ii)



(iii)

11. **Bonus Problem:**

(a) (4 points) On a 25-point graph consisting of x and y coordinates each having integer values -2, -1, 0, 1, 2, sketch the vector field, $\mathbf{D} = \frac{1}{r^2}\hat{r}$ where r is the distance from the origin. You can exclude the origin from your sketch.

(b) (2 points) Judging from your sketch and your understanding of divergence, do you expect $\nabla \cdot \mathbf{D}$ to be positive, negative, or zero?

(c) (4 points) Calculate the divergence $\nabla \cdot \mathbf{D}$. You can use the coordinate transformations $r = \sqrt{x^2 + y^2 + z^2}$ and $\hat{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$ OR you may apply the divergence in spherical units. Is the result consistent with your predictions from part (b)? If not, explain why.