

1. Consider the 3D vectors

$$\begin{aligned}\mathbf{A} &= 3\hat{x} + \hat{y} - 2\hat{z}, \\ \mathbf{B} &= \hat{x} + \hat{y} - \hat{z}, \\ \mathbf{C} &= \hat{x} - 2\hat{y} + 3\hat{z},\end{aligned}$$

(a) (1 point) The vector

$$\mathbf{D} = \mathbf{A} + \mathbf{B} = 4\hat{x} + 2\hat{y} - 3\hat{z}.$$

(b) (1 point) The vector

$$\mathbf{A} + \mathbf{B} - 4\mathbf{C} = 4\hat{x} + 2\hat{y} - 3\hat{z} - 4(\hat{x} - 2\hat{y} + 3\hat{z}) = 10\hat{y} - 15\hat{z}.$$

(c) (2 points) The vector *magnitude*

$$|\mathbf{A} + \mathbf{B} - 4\mathbf{C}| = \sqrt{10^2 + 15^2} = 18.03.$$

(d) (2 points) The unit vector \hat{u} along vector

$$\mathbf{A} + 2\mathbf{B} - \mathbf{C} = (3\hat{x} + \hat{y} - 2\hat{z}) + 2(\hat{x} + \hat{y} - \hat{z}) - (\hat{x} - 2\hat{y} + 3\hat{z}) = 4\hat{x} + 5\hat{y} - 7\hat{z}$$

$$|\mathbf{A} + 2\mathbf{B} - \mathbf{C}| = \sqrt{4^2 + 5^2 + 7^2} = 9.49$$

$$\hat{u} = \frac{\mathbf{A} + 2\mathbf{B} - \mathbf{C}}{|\mathbf{A} + 2\mathbf{B} - \mathbf{C}|} = \frac{4\hat{x} + 5\hat{y} - 7\hat{z}}{9.49} = 0.42\hat{x} + 0.53\hat{y} - 0.74\hat{z}.$$

(e) (2 points) The *dot product*

$$\mathbf{A} \cdot \mathbf{B} = (3\hat{x} + \hat{y} - 2\hat{z}) \cdot (\hat{x} + \hat{y} - \hat{z}) = 3 \times 1 + 1 \times 1 + (-2) \times (-1) = 6.$$

(f) (2 points) The *cross product*

$$\begin{aligned}\mathbf{B} \times \mathbf{C} &= (\hat{x} + \hat{y} - \hat{z}) \times (\hat{x} - 2\hat{y} + 3\hat{z}) \\ &= \hat{x} \times (\hat{x} - 2\hat{y} + 3\hat{z}) + \hat{y} \times (\hat{x} - 2\hat{y} + 3\hat{z}) - \hat{z} \times (\hat{x} - 2\hat{y} + 3\hat{z}) \\ &= (-2\hat{z} - 3\hat{y}) + (-\hat{z} + 3\hat{x}) - (\hat{y} + 2\hat{x}) \\ &= \hat{x} - 4\hat{y} - 3\hat{z}.\end{aligned}$$

2. (7 points) In the three cases, we have the same \mathbf{E} and \mathbf{B} at the origin, but different \mathbf{v} . Thus, we have different \mathbf{F} . In each case, \mathbf{v} and \mathbf{F} must satisfy $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Therefore, we have the following three equations

$$\begin{cases} 3\hat{z} & = \mathbf{E} \\ \hat{z} & = \mathbf{E} + \hat{y} \times \mathbf{B} \\ 3\hat{z} + 4\hat{y} & = \mathbf{E} + 2\hat{z} \times \mathbf{B}, \end{cases}$$

from which we obtain

$$\begin{cases} \mathbf{E} & = 3\hat{z} \\ \hat{y} \times \mathbf{B} & = -2\hat{z} \\ \hat{z} \times \mathbf{B} & = 2\hat{y}. \end{cases}$$

If we assume $\mathbf{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$, then

$$\begin{cases} \hat{y} \times (\hat{x}B_x + \hat{y}B_y + \hat{z}B_z) & = -2\hat{z} \\ \hat{z} \times (\hat{x}B_x + \hat{y}B_y + \hat{z}B_z) & = 2\hat{y}. \end{cases}$$

Further simplify into

$$\begin{cases} -\hat{z}B_x + \hat{x}B_z & = -2\hat{z} \\ \hat{y}B_x - \hat{x}B_y & = 2\hat{y}. \end{cases}$$

Therefore, we have

$$\begin{cases} B_x & = 2 \\ B_y & = B_z = 0. \end{cases}$$

In summary, $\mathbf{E} = 3\hat{z}$ [V/m] and $\mathbf{B} = 2\hat{x}$ [Wb/m²].

3. (13 points) Let us use S_1, S_2, S_3, S_4, S_5 and S_6 to denote the surfaces $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$, respectively. We consider S_1 and S_2 first. The unit vector on S_1 pointing away from the volume is along $-\hat{x}$ direction, so we have

$$\int_{S_1} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]_{x=0} \cdot (-\hat{x}) dydz = -\frac{1}{3} [\text{A}].$$

For S_2 , the unit vector pointing away from the volume is along $+\hat{x}$ direction, therefore

$$\int_{S_2} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]_{x=1} \cdot (+\hat{x}) dydz = \frac{1}{3} [\text{A}].$$

Similarly, for S_3, S_4, S_5 and S_6 , we can obtain

$$\int_{S_3} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]_{y=0} \cdot (-\hat{y}) dx dz = -\frac{1}{3} [\text{A}],$$

$$\int_{S_4} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]_{y=1} \cdot (+\hat{y}) dx dz = \frac{1}{3} [\text{A}],$$

$$\int_{S_5} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]_{z=0} \cdot (-\hat{z}) dx dy = 0 [\text{A}],$$

$$\int_{S_6} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 [z^2(\hat{x} + \hat{y} + \hat{z})]_{z=1} \cdot (+\hat{z}) dx dy = 1 [\text{A}].$$

Therefore,

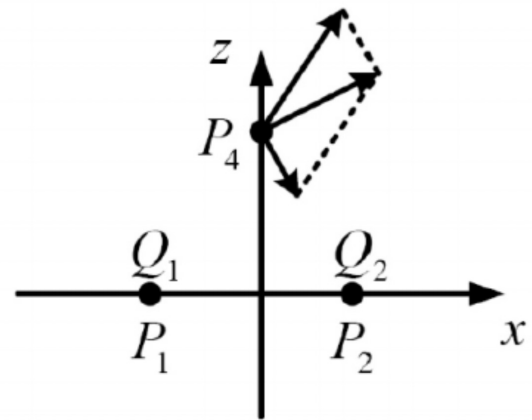
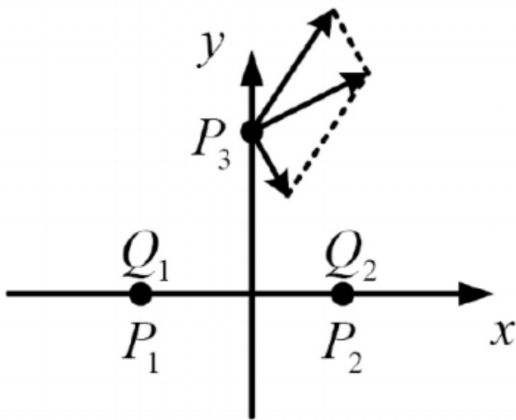
$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \sum_{i=1}^6 \int_{S_i} \mathbf{J} \cdot d\mathbf{S} = 1 [\text{A}].$$

4. (10 points) We know $Q_1 = 8\pi\epsilon_0$ C, so $Q_2 = -Q_1/2 = 4\pi\epsilon_0$ C. The electric field at the point P_3 is the superposition of those induced by Q_1 and Q_2 , namely

$$\begin{aligned} \mathbf{E}_3 &= \sum_{i=1}^2 \frac{Q_i}{4\pi\epsilon_0|r_3 - r_i|^2} \cdot \frac{r_3 - r_i}{|r_3 - r_i|} \\ &= \frac{2}{|\hat{x} + \hat{y}|^3}(\hat{x} + \hat{y}) - \frac{1}{|-\hat{x} + \hat{y}|^3}(-\hat{x} + \hat{y}) \\ &= \frac{3\hat{x} + \hat{y}}{2\sqrt{2}} \text{ [V/m]}. \end{aligned}$$

The electric field at the point P_4 can be obtained in a similar way

$$\begin{aligned} \mathbf{E}_4 &= \sum_{i=1}^2 \frac{Q_i}{4\pi\epsilon_0|r_4 - r_i|^2} \cdot \frac{r_4 - r_i}{|r_4 - r_i|} \\ &= \frac{2}{|\hat{x} + \hat{z}|^3}(\hat{x} + \hat{z}) - \frac{1}{|-\hat{x} + \hat{z}|^3}(-\hat{x} + \hat{z}) \\ &= \frac{3\hat{x} + \hat{z}}{2\sqrt{2}} \text{ [V/m]}. \end{aligned}$$



5. (10 points) The solution to (i) is (e) the charge moves at an angle along y and z: $v_y \neq 0, v_z \neq 0, v_x = 0$. By the Lorentz force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Since the charge is positive, it will experience a force in the z direction due to the electric field. Thus, for the net force to be in the y direction, the $\mathbf{v} \times \mathbf{B}$ term must point both in the opposite z direction and the y direction. By the right hand rule, this is possible only if the particle travels along the yz plane.

The solution to (ii) is (a) $a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$. We have

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_4\mathbf{e}_4) \cdot (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_4\mathbf{e}_4) \\ &= \sum_{n,m=1}^4 a_nb_m\mathbf{e}_n \cdot \mathbf{e}_m \\ &= \sum_{n=1}^4 a_nb_n. \end{aligned}$$

The solution to (iii) is (b) Towards q_2 . Recall $E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$. So the r^2 term dominates in this example as $\frac{1}{1} > \frac{2}{4}$. Since q_3 has the same sign as q_1 and q_2 , it is repelled more by q_1 than by q_2 .

6. (10 points) Firstly, recall that divergence theorem states:

$$\int_V \nabla \cdot \mathbf{F} dV = \int_{\partial V} \mathbf{F} \cdot \mathbf{n} dS$$

We let $\mathbf{F} = \psi \nabla \phi$ and apply divergence theorem:

$$\int_V \nabla \cdot (\psi \nabla \phi) dV = \int_{\partial V} \psi \nabla \phi \cdot \mathbf{n} dS$$

The expression can be simplified using a vector identity: (This is identity (7) in the “List of Vector Identities” sheet found on ECE 329 web site)

$$\nabla \cdot (f \mathbf{A}) = f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

by letting $f = \psi$ and $\mathbf{A} = \nabla \phi$:

$$\int_V (\psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi) dV = \int_{\partial V} \psi (\nabla \phi \cdot \mathbf{n}) dS.$$