Introduction

Professor Lynford Goddard
Section E (MWF 1-2pm)
Room 2254 Micro and Nanotechnology Lab
lgoddard@illinois.edu
Office hours: TH 10-11AM
Hybrid: In 2254 + Zoom
Opportunity to practice solving problems and discuss concepts
Zoom expectations (spring 2021)

• If we interact in a virtual environment, appropriate classroom behavior is expected
  – Dress appropriately
  – Assume all video and audio is recorded
  – Use your real names – can add pronouns if desired
  – Respectful to classmates and instructors
  – Have out: lecture notes, paper, and pen/pencil to take notes
  – Close unneeded applications

• Turn video on but mute yourself until you want to speak
  – Use Virtual Background – solid color, e.g., light gray
  – Quiet well lit, distraction-free room

• Ask concise clear questions in chat
  – Send private messages to the TA not me (to each other when working in group is fine but this will show up in our records)

• View shared screen, participants on top, chat at side
Other administrative

• Course webpage: https://courses.engr.illinois.edu/ece329/
  – Syllabus
  – Course Calendar
  – Grading Policy
  – Homework assignments
  – Gradescope, Canvas
  – Past Exams
  – Class Notes
  – Recorded Lectures

• Rao’s book (7th ed. or 6th is good also)
How to get an A in ECE329

- **Time Management.** Allocate 10 hrs/wk of *regularly scheduled times* in the week outside of class for 329:
  - 30 min for *reading* of textbook *before* each class
  - 30 min for *studying* Kudeki’s online notes *before* each class
  - 30 min for *studying* these notes *between* classes
  - 75 min for *practicing* problems *at the tutorial session*
  - 4-5 hrs/wk for HWs
    - In a semester, all lectures total only 32.5hrs, which is less than 1 week at a job! It’s up to you to put in the time to learn
    - Get a 1” binder (organize lecture notes/HWs/exams)
    - Start assignments early. **Do all problems by yourself first.** If you get stuck, form study groups to work on problems together but **ALWAYS** write-up and submit **YOUR OWN** solutions. Do not blindly copy.
    - Ask questions and come to office hrs if you get stuck. Don’t let confusion snowball.
How to get an A in ECE329

• **Practice doing problems.** Get comfortable with the math manipulations and associated physical meaning, and you will find exam problems to be easier
  - HW problems
  - Example problems worked in lecture and online class notes
  - Old exam problems
  - Office hours

• **Review your prerequisites.**
  - Vector calculus, line/surface integrals: Math 241
  - Linear circuits, system analysis, phasors: ECE 110/210
  - Electric and magnetic fields: Physics 212

• **Come to class!!**
  - **HW & Participation are a significant part of your grade**
  - I will discuss topics to be emphasized on exams and give hints about how to approach the more difficult homework problems.
Upcoming Schedule

• Lectures 1-6 are a review of PHYS 212 and MATH 241 – covered quickly
  – Read Chapters 1 and 2 of Rao’s text over the next 2 weeks

• i-Clicker polls will be used for challenge problems beginning in Lecture 3

• First day survey for fall 2023: https://forms.illinois.edu/sec/1048481900
ECE 329

Fields and Waves I

by

Lynford L. Goddard and Brian T. Cunningham
**EM Spectrum**

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Wavelength</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Radio</strong></td>
<td>10^9 m</td>
</tr>
<tr>
<td>1 Hz</td>
<td>100</td>
</tr>
<tr>
<td>10 Hz</td>
<td>10</td>
</tr>
<tr>
<td>10^2 Hz</td>
<td>1</td>
</tr>
<tr>
<td><strong>Infrared</strong></td>
<td>10^3 m</td>
</tr>
<tr>
<td>10^3 Hz</td>
<td>100</td>
</tr>
<tr>
<td>10^4 Hz</td>
<td>10</td>
</tr>
<tr>
<td><strong>Visible</strong></td>
<td>10^4 m</td>
</tr>
<tr>
<td>10^4 Hz</td>
<td>100</td>
</tr>
<tr>
<td>10^5 Hz</td>
<td>10</td>
</tr>
<tr>
<td><strong>UV</strong></td>
<td>10^5 m</td>
</tr>
<tr>
<td>10^5 Hz</td>
<td>100</td>
</tr>
<tr>
<td>10^6 Hz</td>
<td>10</td>
</tr>
<tr>
<td><strong>X-Ray</strong></td>
<td>10^6 m</td>
</tr>
<tr>
<td>10^6 Hz</td>
<td>100</td>
</tr>
<tr>
<td>10^7 Hz</td>
<td>10</td>
</tr>
<tr>
<td><strong>Gamma Ray</strong></td>
<td>10^7 m</td>
</tr>
<tr>
<td>10^7 Hz</td>
<td>100</td>
</tr>
<tr>
<td>10^8 Hz</td>
<td>10</td>
</tr>
</tbody>
</table>

\[ \lambda f = c \]
# Speed of Light

<table>
<thead>
<tr>
<th>Distance Travelled</th>
<th>Travel Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Across a virus</td>
<td>100 \times 10^{-18} \text{ sec}</td>
</tr>
<tr>
<td>300 meters</td>
<td>1 \text{ microsecond}</td>
</tr>
<tr>
<td>NY to LA</td>
<td>13 \text{ milliseconds}</td>
</tr>
<tr>
<td>Around earth</td>
<td>0.133 \text{ sec}</td>
</tr>
<tr>
<td>Earth to Moon</td>
<td>1.2 \text{ sec}</td>
</tr>
<tr>
<td>Earth to Sun</td>
<td>8.3 \text{ minutes}</td>
</tr>
<tr>
<td>Earth to Mars</td>
<td>3-21 \text{ minutes}</td>
</tr>
<tr>
<td>Sun to nearest star</td>
<td>4 \text{ years}</td>
</tr>
<tr>
<td>Diameter of our galaxy</td>
<td>100,000 \text{ years}</td>
</tr>
<tr>
<td>Edge of known universe</td>
<td>15 billion \text{ years}</td>
</tr>
</tbody>
</table>

\[ c \equiv 299,792,458 \text{ m/s (used to define the meter!)} \]
Sending/Receiving EM Waves

- Radio telescope
- Cell phone tower
- Cell phone
- Television
- Wireless Internet
- RF ID Tag
- Directional antenna
- Patch Antenna
- Microwave patch antenna
Guiding EM Waves

LONGITUDINAL VIEW

- Battery 2-wire line (dc)
- Generator 2-wire line (ac)
- Coaxial line (dc, ac, rf)
- Axon (animal nerve) line

CROSS SECTION

- Strip line (rf)
- Microstrip line (rf)
- Rectangular waveguide (rf)
- Optical fiber (light)
- Wireless link with antennas

Noiseless, lossless
Brief History of EM

Charles Coulomb in 1785 demonstrated how electric charges repel one another.

Andre Marie Ampere discovered that an electric current produces a magnetic field in 1820.

Michael Faraday in 1831 showed that since an electric current could produce a magnetic field, a changing magnetic field can produce an electric current. "Principle of Induction" used for first electric generators.
James Clerk Maxwell (1831-1879)

• 1855-1868 - formulates field equations for electromagnetism. Predicts existence of EM wave propagation and the speed of light. Shows theoretical possibility of generating electromagnetic radiation

• 1873: publishes *Treatise on Electricity and Magnetism*
# Maxwell’s Equations

## Integral form

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \oint E \cdot dL = - \frac{d}{dt} \int_B \cdot dS ]</td>
<td>Faraday</td>
</tr>
<tr>
<td>[ \oint H \cdot dL = \int J \cdot dS + \frac{d}{dt} \int_D \cdot dS ]</td>
<td>Ampere</td>
</tr>
<tr>
<td>[ \int_D \cdot dS = \int \rho , dv ]</td>
<td>Gauss</td>
</tr>
<tr>
<td>[ \oint B \cdot dS = 0 ]</td>
<td>Gauss</td>
</tr>
</tbody>
</table>

## Differential Form

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \nabla \times E = - \frac{\partial B}{\partial t} ]</td>
<td>Faraday</td>
</tr>
<tr>
<td>[ \nabla \times H = J + \frac{\partial D}{\partial t} ]</td>
<td>Ampere</td>
</tr>
<tr>
<td>[ \nabla \cdot D = \rho ]</td>
<td>Gauss</td>
</tr>
<tr>
<td>[ \nabla \cdot B = 0 ]</td>
<td>Gauss</td>
</tr>
</tbody>
</table>

*\( E \) Electric Field
*\( H \) Magnetic Field
*\( D = \varepsilon E \) “Displacement Flux Density”
*\( B = \mu H \) “Magnetic Flux Density”
*\( J \) Current Density
*\( \rho \) Charge Density
Computers use Differential Form for Complex Objects

From huge objects to the nanoscale, Maxwell’s Equations always work!
Cell Phone Interaction with Human Head

Cut plane through the cellphone

Maps of the E-field and SAR within the cut plane. Relative intensities are shown in dB.


Photonic Bandgap Defect Mode Lasers

Goddard/Cunningham
ECE329 Lecture 1

Sections 1.1-1.2

Vector Algebra
Cartesian Coordinates
Differential Length Vector
Differential Surface Vector

Adapted from Prof. Cunningham's Notes
Today’s Topics

• Review
  – Scalars (numbers) and vectors
  – Unit vectors
  – Vector addition & subtraction
  – Magnitude
  – Dot product
  – Cross product

• New Topics
  – Differential length vector
  – Differential surface vector
Cartesian Coordinate System

Unit vectors
\[ \hat{a}_x \quad \hat{a}_y \quad \hat{a}_z \]

Notation
\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \]

Alternate Notation
\[ \vec{A} = < A_x, A_y, A_z > \]
\[ \vec{A} = A_1 \hat{a}_1 + A_2 \hat{a}_2 + A_3 \hat{a}_3 \]
Vector Math

\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \]
\[ \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \]

Magnitude of \( \vec{A} \)
\[ |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \]

Unit vector in direction of \( \vec{A} \)
\[ \hat{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \]

Multiplication of \( \vec{A} \) by a scalar
\[ m\vec{A} = mA_x \hat{a}_x + mA_y \hat{a}_y + mA_z \hat{a}_z \]
(Review at home)

**Vector Math**

\[
\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \quad \quad \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z
\]

**Addition of A and B**

\[
\vec{A} + \vec{B} = (A_x + B_x) \hat{a}_x + (A_y + B_y) \hat{a}_y + (A_z + B_z) \hat{a}_z
\]

**Subtraction of A and B**

\[
\vec{A} - \vec{B} = (A_x - B_x) \hat{a}_x + (A_y - B_y) \hat{a}_y + (A_z - B_z) \hat{a}_z
\]
Dot Product

\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \]
\[ \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \]

Dot Product of \( \vec{A} \) and \( \vec{B} \)

\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \]

Resulting quantity is a SCALAR

Physical Meaning

(Magnitude of \( \vec{A} \))\( \times \) (Projection of \( \vec{B} \) onto \( \vec{A} \))

or (Magnitude of \( \vec{B} \))\( \times \) (Projection of \( \vec{A} \) onto \( \vec{B} \))

\[ \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \]
Dot Product

\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \]
\[ \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \]

Dot Product of \( \vec{A} \) and \( \vec{B} \)

\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \]

\( \vec{A} \perp \vec{B} \)

\[ \cos \theta = 0 \]
\[ \vec{A} \cdot \vec{B} = 0 \]

\( \vec{A} \parallel \vec{B} \)

\[ \cos \theta = 1 \]
\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \]

\[ \vec{A} \perp \vec{B} \iff \vec{A} \cdot \vec{B} = 0 \]

\[ \vec{A} \parallel \vec{B} \iff \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \]
Dot Product

\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \quad \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \]

\[ \vec{A} \perp \vec{B} \iff \vec{A} \cdot \vec{B} = 0 \]
\[ \vec{A} \parallel \vec{B} \iff \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \]

Dot Product of Unit Vectors

\[ \hat{a}_x \cdot \hat{a}_x = 1 \quad \hat{a}_y \cdot \hat{a}_x = 0 \quad \hat{a}_z \cdot \hat{a}_x = 0 \]
\[ \hat{a}_x \cdot \hat{a}_y = 0 \quad \hat{a}_y \cdot \hat{a}_y = 1 \quad \hat{a}_z \cdot \hat{a}_y = 0 \]
\[ \hat{a}_x \cdot \hat{a}_z = 0 \quad \hat{a}_y \cdot \hat{a}_z = 0 \quad \hat{a}_z \cdot \hat{a}_z = 1 \]

\[ \vec{A} \cdot \vec{B} = \left( A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \right) \cdot \left( B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \right) \]

\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad \text{Scalar} \]
Discussion Problem

• Given \( \mathbf{A} = \langle 3,2,1 \rangle, \ \mathbf{B} = \langle 1,1,-1 \rangle, \ \mathbf{C} = \langle 1,2,3 \rangle \), find:
  a. \( |\mathbf{A} + \mathbf{B} - 4\mathbf{C}| \)
  b. unit vector along \( (\mathbf{A} + 2\mathbf{B} - \mathbf{C}) \)
  c. \( \mathbf{A} \cdot \mathbf{C} \)
Cross Product

\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \]
\[ \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \]

Cross Product of \textbf{A} and \textbf{B}

Resulting quantity is a VECTOR

**Magnitude**

\[ \vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta \]

**Direction**

Perpendicular to BOTH \textbf{A} and \textbf{B}
Two possible vectors satisfy this condition
Determined using the RIGHT HAND RULE

Vector out of page pointing at the classroom
Right Hand Rule

Bus Driver in Kauai Demonstrating the Right Hand Rule
Cross Product

\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \]
\[ \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \]

Cross Product of \( \mathbf{A} \) and \( \mathbf{B} \)

\[ \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \]

Same magnitude
Opposite direction

Vector out of page pointing at the classroom

Vector into page pointing \textit{away} from classroom
Cross Product

\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \]
\[ \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \]

Cross Product of \( \mathbf{A} \) and \( \mathbf{B} \)

\[ \vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin \theta \hat{a}_N \]

\( \vec{A} \perp \vec{B} \)

\[ \sin \theta = 1 \]
\[ \vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\hat{a}_N \]

\( \vec{A} \parallel \vec{B} \)

\[ \sin \theta = 0 \]
\[ \vec{A} \times \vec{B} = 0 \]

\[ \vec{A} \perp \vec{B} \Leftrightarrow \vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\hat{a}_N \]

\[ \vec{A} \parallel \vec{B} \Leftrightarrow \vec{A} \times \vec{B} = 0 \]
Cross Product

\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \]

\[ \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \]

Cross Product of Unit Vectors

\[ \hat{a}_x \times \hat{a}_x = 0 \]

\[ \hat{a}_y \times \hat{a}_x = -\hat{a}_z \]

\[ \hat{a}_z \times \hat{a}_x = \hat{a}_y \]

\[ \hat{a}_x \times \hat{a}_y = \hat{a}_z \]

\[ \hat{a}_y \times \hat{a}_y = 0 \]

\[ \hat{a}_z \times \hat{a}_y = -\hat{a}_x \]

\[ \hat{a}_x \times \hat{a}_z = -\hat{a}_y \]

\[ \hat{a}_y \times \hat{a}_z = \hat{a}_x \]

\[ \hat{a}_z \times \hat{a}_z = 0 \]

Cross Product of \textbf{Unit Vectors}

\[ \bar{A} \perp \bar{B} \iff \bar{A} \times \bar{B} = |\bar{A}| |\bar{B}| \hat{a}_N \]

\[ \bar{A} \parallel \bar{B} \iff \bar{A} \times \bar{B} = 0 \]

\[ \bar{A} \times \bar{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{a}_x + (A_zB_x - A_xB_z)\hat{a}_y + (A_xB_y - A_yB_x)\hat{a}_z \]

Vector

Perpendicular to \textbf{A} & \textbf{B}

Right Hand Rule to Select Direction
Vector From Origin to a Point

\( \mathbf{r}_1 \) to \( \mathbf{P}_1 \): \( (x_1, y_1, z_1) \)

(Review at home)
Vector Between Two Points

\[ \mathbf{r}_1 \]
\[ \mathbf{r}_2 \]

\[ \mathbf{P}_1 : (x_1, y_1, z_1) \]
\[ \mathbf{P}_2 : (x_2, y_2, z_2) \]
Vector Between Two Points

\[ \vec{R}_{12} = (\text{FinalPosition}) - (\text{InitialPosition}) \]

\[ \vec{R}_{12} = (x_2 - x_1)\hat{a}_x + (y_2 - y_1)\hat{a}_y + (z_2 - z_1)\hat{a}_z \]
Differential Length Vector

Always tangent to a curve or surface

Exact vector is different at different places on the curve or surface

What we will plug into Faraday’s law

\[ d\hat{l} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z \]

(in cartesian coordinates only)
Imagine you are walking in a river and the water pushes with a force, \( \mathbf{F} \), determined by the river’s VELOCITY VECTOR FIELD.

How much work does the river do if you walk from \( P_1 \) to \( P_2 \) along the path shown?
Discussion Problem

• Given $\mathbf{A} = \langle 3,2,1 \rangle$, $\mathbf{B} = \langle 1,1,-1 \rangle$, $\mathbf{C} = \langle 1,2,3 \rangle$, find:
  a. $\mathbf{B} \times \mathbf{C}$
  b. $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

• For each line, find $d\mathbf{l}$ if the z-component of $d\mathbf{l}$ is $dz$:
  a. $x=3$, $y=4$
  b. $x+y=0$, $y+z=1$
  c. the line passing from $(2,0,0)$ thru $(0,0,1)$
Surface Vector

Easy example surface: Flat surface in the yz plane

Pick a point on the surface

Find two vectors at that point that are tangent to the surface

A **surface vector** is always **normal** (perpendicular) to the surface

You get **normal** vectors by performing a **cross product**

\[ \vec{S} = \vec{l}_1 \times \vec{l}_2 \]
Surface Vectors

How do we come up with a surface vector when a surface is curved?

If we zoom into a small enough area, the surface will “look” flat.
Differential Surface Vectors

Find two differential length vectors at the point on the surface.

Think of $d\mathbf{S}$ as a tiny parallelogram with sides bounded by $d\mathbf{l}_1$ and $d\mathbf{l}_2$ and direction perp to the surface.

$$d\mathbf{S} = d\mathbf{l}_1 \times d\mathbf{l}_2$$
Cartesian Coordinates

\[(x, y, z)\]

\[
\vec{d\ell} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z
\]

\[
\vec{dS} = \pm dydz\hat{a}_x
\]

\[
\vec{dS} = \pm dzdx\hat{a}_y
\]

\[
\vec{dS} = \pm dx dy\hat{a}_z
\]
Differential area or surface vectors for unit vectors in cartesian coords

\[ d\vec{S} = dx dy \hat{\mathbf{a}}_z \]

\[ d\vec{S} = dy dz \hat{\mathbf{a}}_x \]

\[ d\vec{S} = dz dx \hat{\mathbf{a}}_y \]
Lecture 1 Summary

• Dot Product  \( \vec{A} \cdot \vec{B} = \text{________} = \text{________} \)
• Cross Product  
\[
\vec{A} \times \vec{B} = \begin{vmatrix}
\_ & \_ & \_ \\
\_ & \_ & \_ \\
\_ & \_ & \_ \\
\end{vmatrix} = \langle \text{______}, \text{______}, \text{______} \rangle
\]

• Next two lectures:
  – Scalar and Vector Fields (1.3)
  – The Lorentz Force (1.6)
  – Coulomb’s Law (1.4-1.6)
  – Surf. Integrals/Gauss’ Law (2.2,2.5)
Lectures 2-3
Sections 1.3-1.6, 2.2, 2.5

Scalar and Vector Fields
Lorentz Force Equation
Coulomb’s Law
Surface Integrals
Connecting Coulomb’s and Gauss’ Law
Scalar Field

Temperature

Tornadoes

Population Density

Elevation
Vector Field
Vector Field

Example

\[ \vec{E}(x, y) = y\hat{a}_x + \sin x \hat{a}_y \]
Vector Field

Example \( \vec{E}(x, y) = y \hat{a}_x + \sin x \hat{a}_y \)

“Direction Line” or “stream line” or “flux line”

\[ \rho \vec{E} \]

\[ \hat{\sin \hat{\theta}} \]

\[ + = \]

\[ \vec{E} \]

\[ \text{E vectors are tangent to the direction line (parallel to differential length vectors)} \]

Thus, slope of direction line is \( m = \frac{dy}{dx} = \frac{E_y}{E_x} \)
Electric force

\[ \vec{F}_E = q\vec{E} \]

Force vectors \( \vec{F}_E \) are parallel to \( \vec{E} \) for a positive charge \( q \).
Magnetic force

\[ \vec{F}_M = q \vec{v} \times \vec{B} \]

Direction: Perpendicular to velocity vector – does no work! Perpendicular to B vector
Lorentz Force Equation

So if a region of space contains BOTH an \( \mathbf{E} \) field and a \( \mathbf{B} \) field, a moving charge will experience force from both at the same time...

\[
\vec{F}_{TOTAL} = \vec{F}_E + \vec{F}_M
\]

\[
\vec{F}_{TOTAL} = q\vec{E} + q\vec{v} \times \vec{B}
\]
Application: Mass Spectrometers

• Part I: Velocity Selector
  – Particles with a specific velocity in crossed EM fields are undeflected

\[ \vec{E} = E_0 \hat{a}_z \]
\[ \vec{B} = -B_0 \hat{a}_y \]
\[ \vec{v} = v_0 \hat{a}_x \]
\[ \vec{F}_{TOTAL} = q(E_0 - v_0 B_0) \hat{a}_z = 0 \iff v_0 = \frac{E_0}{B_0} \]
Application: Mass Spectrometers

• Part II: Mass Selector
  – Mass sets the radius of B-field orbit since particle velocity is the same

\[ \vec{B} = B_0 \hat{a}_z \]
\[ \vec{v} = v_0 \hat{a}_x \]

Problem: Solve for \( R \) as a function of the particle’s mass, \( m \), and velocity, \( v_0 \)

\[ F_c = \frac{m v_0^2}{R} = q v_0 B_0 \]
Current = Moving Charge

What is CURRENT? CHARGES IN MOTION!!

\[ I \, d\vec{l} = q \, \vec{v} \]

\[ \frac{coul}{sec} \cdot m = \frac{coul}{sec} \cdot \frac{m}{sec} \]

So the current in a wire, \( I \), flowing across a magnetic field will feel a force…
Magnetic force

\[ F_M = (I \, d\vec{l}) \times \vec{B} = q \, \vec{v} \times \vec{B} \]

Direction:  
Perpendicular to velocity vector  
Perpendicular to B vector
Electrostatic Force

What is the force $F_2$ on a point charge $Q_2$ due to a single point charge $Q_1$ located a distance $R$ away?
Coulomb’s observations

• The magnitude of $F$ is
  – proportional to the product of the charges
  – inversely proportional to the square of the distance
  – depends on the medium

• $F$ points along the joining line

• Like charges repel; unlike charges attract
Coulomb’s Law

\[ \vec{F}_1 = \frac{Q_1 Q_2}{4 \pi \varepsilon_0 R^2} \hat{a}_{21} \]

\[ \vec{F}_2 = \frac{Q_1 Q_2}{4 \pi \varepsilon_0 R^2} \hat{a}_{12} \]
Electric Field

- The electric field $\mathbf{E}$ is the force per unit charge caused by the source charges.

\[ \mathbf{E} = \lim_{Q_2 \to 0} \frac{\mathbf{F}_2}{Q_2} = \frac{Q_1}{4\pi\varepsilon_0 R^2} \hat{a}_R \]
Electric Field Around a Point Charge

\[ \vec{E} = \frac{Q}{4 \pi \varepsilon_0 R^2} \hat{a}_R \]

Field strength is proportional to the length of vectors.
Surfaces of Constant E Magnitude are Spheres
Field Lines

- Another way to graphically represent vector fields
- The field strength is proportional to the density of field lines
- E-field lines begin on + charges, initially emanating uniformly in all directions, and end on – charges
  - Can’t stop in midair but can extend to ∞
- They never intersect
Electric Field Around a Point Charge

\[ \vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2} \hat{a}_R \]

Field strength is proportional to the **density** of field lines
Electric Field Around a Point Charge

\[ \vec{E} = \frac{Q}{4 \pi \varepsilon_0 R^2} \hat{a}_R \]

Gauss’ Law
Number of field lines passing thru any surface that encloses \( Q_1 \) is constant
Calculating the Electric Field

Point Charge at position $(x_1, y_1, z_1)$

Position where we want to calculate electric field at Position $(x_2, y_2, z_2)$

\[ \mathbf{E} = \frac{Q}{4 \pi \varepsilon_0 R^2} \hat{a}_R \]

\[ R = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

\[ \hat{a}_R = \frac{(x_2 - x_1)\hat{a}_x + (y_2 - y_1)\hat{a}_y + (z_2 - z_1)\hat{a}_z}{R} \]

Unit vector pointing along direction from Q to Point
Superposition of E Fields
Example: Calculate the E-field of a Dipole at a Point

Tip: We only need to work in 2D. Why?
Example: E of Dipole

Tip: Use symmetry to eliminate components that cancel. Here, we only need to calculate the y-component and only for one of the charges (let’s say for +Q). Why?
Example: E of Dipole

\[ \vec{E}_{TOT} = \vec{E}_1 + \vec{E}_2 = 2E_1y\hat{a}_y \]
**Example: E of Dipole**

\[ \vec{E}_{TOT} = 2E_1 y \hat{a}_y \]

\[ \vec{R}_{from +Q to P} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle D, -a/2 \rangle \]

\[ \hat{a}_R = \frac{\langle D, -a/2 \rangle}{\sqrt{D^2 + a^2/4}} \]

\[ \vec{E}_1 = \frac{Q}{4\pi\epsilon_0 R^2} \hat{a}_R \]
Example: E of Dipole

\[ E_{1y} = \left( \frac{Q}{4\pi\varepsilon_0 R^2} \hat{a}_R \right) \cdot \hat{a}_y \]

\[ \vec{E}_{TOT} = 2 \frac{Q}{4\pi\varepsilon_0 R^2} \left( -\frac{a/2}{R} R \right) \hat{a}_y \]

UNITS (Newtons per Coulomb)
Patented 5-Step Program for Problem Solving

1. MAKE A **LARGE CLEAR DRAWING**
   a. Also draw cross-sections if the problem is in 3D
   b. Pick a coordinate system that is appropriate for the symmetry of the problem

2. Divide charge distributions into tiny pieces

3. Find \( \text{dE} \) of one tiny piece

4. Use SYMMETRY to eliminate any components that cancel (i.e. add to ZERO)

5. INTEGRATE to add contribution of ALL the tiny pieces
Example Charge Distributions

- Several discrete points of charge
- Line of charge
- Ring of charge
- Nonuniform lines or rings of charge
- An infinite sheet of charge
- Infinite box of charge
- Spherical surface of charge
- Cylindrical surface of charge
- Spherical volume of charge
- Cylindrical volume of charge
Example 1: Find Total Charge of a Linear Charge Distribution

Linear Charge $\lambda_0$ (C/m) distribution in a circular loop of radius $= a$

One little piece has a charge (in coulombs)

$$dQ = (\lambda_0)(ad\phi)$$
Example 1: Find Total Charge of a Linear Charge Distribution

Linear Charge $\lambda_0$ (C/m) distribution in a circular loop of radius $= a$

One little piece has a charge (in coulombs)

$$dQ = (\lambda_0)(ad\phi)$$

Integrate to get the entire charge of the loop:

$$Q = \int_{\phi=0}^{2\pi} (\lambda_0)(ad\phi) = 2\pi\lambda_0 a$$

Units: Coulombs

(Review at home)
Example 2: F due to line of charge

Rod has a TOTAL charge = Q (coul)

So the rod has a charge DENSITY $\rho_l = \frac{Q}{L}$  (coul/m)

Find the FORCE exerted by the whole charged rod on the charge q
Example 2: F due to line of charge

What is the small amount of force, \( dF \), applied by a small sliver of the rod?

Differential force applied to \( q \)

\[
d\vec{F} = \frac{Q}{L} dx \frac{q\hat{a}_x}{4\pi\varepsilon_0 R^2}
\]

\[
R = (L + a) - x
\]
Example 2: F due to line of charge

INTEGRATE the force from each part of the rod to obtain the force due to the whole thing:

\[
\vec{F} = \int_0^L \frac{qQ}{4\pi \varepsilon_0 L} \frac{dx}{(L + a - x)^2} \hat{a}_x
\]

Fortunately, \(a_x\) is constant and can be taken outside of the integral. Not so simple in cylindrical/spherical coordinates!
Example 2: \( F \) due to line of charge

INTEGRATE the force from each part of the rod to obtain the force due to the whole thing:

\[
\vec{F} = \frac{qQ}{4\pi \varepsilon_0 L} \hat{a}_x \left[ \frac{1}{a} - \frac{1}{L+a} \right] = \frac{qQ}{4\pi \varepsilon_0 a(L+a)} \hat{a}_x
\]

Units: N

(Review at home)
Example 3: \( \mathbf{E} \) due to line of charge

Linear charge distribution \( \rho_1 \) (coul/m) from \(-a<z<a\)

Find \( \mathbf{E} \) at point on xy plane

Next, consider what happens when the line is infinitely long
Example 3: \( \mathbf{E} \) due to line of charge

Cylindrical symmetry so use cylindrical coordinates \((r, \phi, z)\)

\[ \mathbf{E}_{\text{total}} \] will point in \( a_r \) direction, why?
Example 3: \( \mathbf{E} \) due to line of charge

\[ \mathbf{dE} = \frac{dQ}{4\pi \varepsilon_0 R^2} \hat{a}_R \]

\[ dE_r = dE \cos(\alpha) = dE \frac{r}{R} \]

\[ R = \sqrt{r^2 + z^2} \]

\[ E_r = \frac{\rho_l}{4\pi \varepsilon_0} \int_{z=-a}^{a} \frac{r}{(r^2 + z^2)^{3/2}} dz(\hat{a}_r) \]

Slightly difficult to integrate directly
Example 3: \( \mathbf{E} \) due to line of charge

\[
E_r = \frac{\rho_l}{4\pi\varepsilon_0} \int_{z=-a}^{a} \frac{r}{(r^2 + z^2)^{3/2}} dz (\hat{a}_r)
\]

\[
R = \frac{r}{\cos(\alpha)}, \quad z = r \tan(\alpha), \quad dz = r \sec^2(\alpha) d\alpha
\]

\[
E_r = \frac{\rho_l}{4\pi\varepsilon_0} \int_{z=-a}^{a} r \frac{\cos^3(\alpha)}{r^3} (r \sec^2(\alpha) d\alpha) (\hat{a}_r)
\]

\[
= \frac{\rho_l}{4\pi\varepsilon_0 r} (\hat{a}_r) \int_{\tan(\alpha)=+a/r}^{\tan(\alpha)=-a/r} \cos(\alpha) d\alpha = \frac{\rho_l}{4\pi\varepsilon_0 r} (\hat{a}_r) \sin(\alpha)
\]

\[
= \frac{\rho_l}{4\pi\varepsilon_0 r} (\hat{a}_r) 2 \frac{a}{\sqrt{a^2 + r^2}}
\]
Example 4: $\mathbf{E}$ due to $\infty$ line of charge

Field strength now drops off as $1/r$, Not as $1/r^2$ like a point charge
Lecture 2 Summary

• The electric field $\mathbf{E}$ is the ______ per ______ caused by the source charges. It points along the ______ line. For a point charge,

\[ \mathbf{E} = \]  

• Next class:
  Surface Integrals (2.2)
  Connecting Coulomb’s and Gauss’ Law (2.5)
Lecture 3
Sections 2.2, 2.5

Surface Integrals
Connecting Coulomb’s and Gauss’ Law
Electric Field Around a Point Charge

\[ \vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2} \hat{a}_R \]

**Gauss’ Law**
- Number of field lines passing thru **any** surface that encloses \( Q_1 \) is constant
Surface Integrals

• Flux = # of arrows that pass thru a surface; it depends on:
  – The density of vectors
  – The angle of the surface
  – The area of the surface
Surface Integral describes the Flux of a Vector Field

\[ Flux = \vec{B} \cdot \Delta \tilde{S} \]

\[ = (\vec{B} \cdot \hat{a}_n) |\Delta S| \]

\[ = (B \cos \alpha) |\Delta S| \]

\[ = |B_n| |\Delta S| \]
Trick that works sometimes

If the flux is
  • Uniform (has equal magnitude across whole surface)
  • Perpendicular to the surface

\[
\psi = \int \int_B \mathbf{B} \cdot d\mathbf{S} = \mathbf{B} \cdot \mathbf{A}
\]
In the general case ...

\[
\text{Flux} = \sum_{j=1}^{n} \Delta \psi_j = \sum_{j=1}^{n} \mathbf{B}_j \cdot \Delta S_j
\]

In the limit \( n \to \infty \),

\[
\text{Flux, } \psi = \int_S \mathbf{B} \cdot dS = \text{Surface integral of B over } S.
\]
Differential surface vectors for unit vectors in cartesian coords

\[ d\vec{S} = \pm dydz \hat{a}_x \]

\[ d\vec{S} = \pm dzdx \hat{a}_y \]

\[ d\vec{S} = \pm dxdy \hat{a}_z \]
Electric Flux = Charge Enclosed

- Coulomb’s Law for the Electric field of a point charge:
  \[ \vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2} \hat{a}_R \]

Define \( \mathbf{D} = \varepsilon_0 \mathbf{E} \) to be the “displacement flux density”

\[
\psi_E = \iint_S \mathbf{D} \cdot d\mathbf{S} = \iint_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S}
\]

\[= \varepsilon_0 E (Surf \ Area)\]

\[= \varepsilon_0 \frac{Q}{4\pi\varepsilon_0 R^2} (4\pi R^2) = Q\]
Same Flux Out of Any Surface

- Same # of field lines pass thru any surface that encloses Q

\[ \psi_E = \int\int_S \vec{D} \cdot d\vec{S} = Q_{\text{enclosed}} \]

Our first Maxwell Equation!
Superposition

- The flux for an arbitrary distribution of charges is obtained using superposition.

\[ \psi_E = \iiint_S \vec{D} \cdot d\vec{S} = Q_{enclosed} \]

FLUX OUT = CHARGE ENCLOSED
Simple Example 1

6-sided cube with Q at the center:

- Flux out of entire box = Q
- Flux out of one side = Q/6

What if Q is not at the center?
Simple Example 2

Flux out of hemisphere with Q at the center = Q/2
Challenge Question 1

- What is the flux across the xy plane \( \mathbf{a}_n = \mathbf{a}_z \) for a dipole: \( +Q \) at \((0,0,a/2)\) and \(-Q\) at \((0,0,-a/2)\)?

(a) \( Q/2 \)
(b) \(-Q/2\)
(c) \(-Q\)
(d) \(-3Q/2\)
(e) \(-2Q\)
Challenge Question 2

- What is the flux across the xy plane \( a_n = a_z \) for the charge distribution: +2Q at \((0,0,a/2)\) and -Q at \((0,0,-a/2)\)? (Hint: Use superposition.)

(a) Q/2
(b) -Q/2
(c) -Q
(d) -3Q/2
(e) -2Q
Example 3: \( \mathbf{E} \) due to \( \infty \) line of charge

Linear charge distribution \( \rho \) (coul/m)

Find \( \mathbf{E} \) at point on xy plane
Example 3: $\mathbf{E}$ due to $\infty$ line of charge
Example 4: $\mathbf{E}$ due to a surface of charge

What surface shall we draw?
The (Fictional) Yadaraf Bug: Flux and Surface Integrals

- The Yadaraf Bug
  - They live in the ground
  - They only come out at night to search for food
  - Very hard to see - REALLY small
  - After gathering food, they always return to their hole in the ground
Yadaraf Bug Travel Paths
Yadaraf Bug Travel Paths

Bug Density Vector $\vec{B}$

(Bugs/m$^2$)
**Yadaraf Bug Counter**

- Register a +1 count for each bug going through in one direction
- Register a -1 count for each bug going through in the opposite direction
- Has a known area for the bugs to pass through
Yadaraf Bug Travel Paths

Bug Density
Vectors
(Bugs/m²)

$\vec{B}$

$\vec{S}$
Bug Counting Net

Bug Density Vectors
(Bugs/m²)

\( \vec{B} \)

\( \text{dx} \)
\( \text{dy} \)

\( d\vec{S} \) (normal to net section)
“Closed” Bug Counting Net

Bug Density Vectors
(Bugs/m²)

\[ \vec{B} \]

\[ \rho \, dx \, dy \]

\[ d\rho \, S \] (normal to net section)

(normal to net section)
So magnetic flux, $\psi_B$, through a CLOSED surface is always zero.

Magnetic field lines form closed paths (they do not begin or end).
Gauss’ Law for B Fields

Net flux of magnetic field lines through any closed surface MUST be zero.

\[ \oint_S \vec{B} \cdot d\vec{S} = 0 \]

Our Second Maxwell Equation!
Lecture 3 Summary

• Gauss’ Law

\[ \psi_E = \iiint_S \vec{D} \cdot d\vec{S} = Q_{\text{enclosed}} \]

FLUX OUT = CHARGE ENCLOSED

\[ \iiint_S \vec{B} \cdot d\vec{S} = 0 \]

MAGNETIC FLUX LINES DO NOT BEGIN/END
Lectures 4-5
Sections 3.1-3.3

Review of Vector Calculus
Curl and Divergence
Maxwell’s Equations in Differential Form
Fundamental Theorem of Single Variable Calculus

\[ \int_{a}^{b} f'(x) \, dx = f(b) - f(a) \]

df = f'(x) \, dx \text{ is the infinitesimal change of } f \text{ in going from } x \text{ to } x + dx

Thus, chopping up the interval \((a,b)\) into pieces \(dx\) and adding up \(df\) gives the total change
Fundamental Theorem of Multi-Variable Calculus

\[ \int_a^b \nabla f \cdot d\vec{l} = f(b) - f(a) \]

\[ \nabla f = \frac{\partial f}{\partial x} \hat{a}_x + \frac{\partial f}{\partial y} \hat{a}_y + \frac{\partial f}{\partial z} \hat{a}_z \]

df = \nabla f \cdot d\vec{l} \text{ is the infinitesimal change of } f \text{ in going from } (x,y,z) \text{ to } (x+dx,y+dy,z+dz)

Thus, chopping up the path \((a,b)\) into pieces \(d\vec{l}\) and adding up \(df\) gives the total change
\n\n∇f is a conservative field

\[ \int_{a}^{b} \nabla f \cdot d\vec{l} = f(b) - f(a) \]

\[ \nabla f = \frac{\partial f}{\partial x} \hat{a}_x + \frac{\partial f}{\partial y} \hat{a}_y + \frac{\partial f}{\partial z} \hat{a}_z \]

The right hand side doesn’t depend on path so

∇f is conservative

\[ \oint_{C} \nabla f \cdot d\vec{l} = 0 \]

∇f is curl-free
Curl

\[ \vec{v} = \langle -3y, 2x, 0 \rangle \]

\[ \nabla \times \vec{v} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = 5\hat{a}_z \]

\( \nabla \times \vec{v} \) is a measure of how much the vector field \( \vec{v} \) circulates at a given point.

A place of high curl is like a whirlpool.

– Everywhere here is a whirlpool of strength 5.
Physical Interpretation of Curl

Centre of river flows at maximum speed. Now put the “curl meter” into the river and see what happens!
Physical Interpretation of Curl

River

No rotation!

River

Anti-clockwise rotation.

River

Clockwise rotation.
Meaning of Curl

• The curl meter only spins if there is a non-uniformity in the vector field in a direction *perpendicular* to the field
  – Curl describes variation ACROSS the flow of the field
• Rotation rate is proportional to the degree of non-uniformity
• Rotation is described with a magnitude and a direction – so it’s a VECTOR and it’s given by the right hand rule
Curl

\[
\begin{align*}
\frac{dE_x}{dx} &\neq 0 \\
\text{NO CURL} \\
\text{Varies ALONG}
\end{align*}
\]

\[
\begin{align*}
\frac{dE_y}{dx} &\neq 0 \\
\text{YES CURL} \\
\text{Varies ACROSS}
\end{align*}
\]
Example: Curl

- Find the curl of $\mathbf{A} = (x^2 - 4) \mathbf{a}_y$
Stokes’ Theorem

\[ \oint_C \mathbf{v} \cdot d\mathbf{l} = \sum \oint_{C_i} \mathbf{v} \cdot d\mathbf{l} = \sum \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} \]

\( \nabla \times \mathbf{v} \) is a measure of how much the vector field \( \mathbf{v} \) circulates at a given point
A place of high curl is like a whirlpool
Thus, adding up the circulation from each whirlpool inside a region = the total circ. along the boundary
Optional: Derivation of Stokes’ Theorem

\[ \oint_C \vec{v} \cdot d\vec{l} = \left[ v_y(x + dx, y + dy/2) - v_y(x, y + dy/2) \right] \cdot dy \]

\[ - \left[ v_x(x + dx/2, y + dy) - v_x(x + dx/2, y) \right] \cdot dx \]

\[ = \left( \frac{dv_y}{dx} - \frac{dv_x}{dy} \right)_{x+dx/2, y+dy/2} \cdot dx \cdot dy \]

\[ \oint_C \vec{v} \cdot d\vec{l} = (\nabla \times \vec{v}) \cdot d\vec{S} \approx \iint_S (\nabla \times \vec{v}) \cdot d\vec{S} \]
\( \mathbf{v} \) is a conservative field
iff \( \nabla \times \mathbf{v} = 0 \)

\[
\oint_C \mathbf{v} \cdot d\mathbf{l} = 0 = \iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S}
\]

The following are therefore equivalent:

- \( \mathbf{v} \) is conservative
- \( \mathbf{v} \) is curl-free

\[
\oint_C \mathbf{v} \cdot d\mathbf{l} = 0 \quad \text{is path independent}
\]

\( \mathbf{v} = -\nabla f \) for some scalar potential \( f \)
Curl at a Point

“E has CURL” means \( EMF \neq 0 \) around a tiny closed path at a particular point – line integral is path dependent

If there is any DIFFERENCE in \( E_y \) in the x direction
\[
\frac{dE_y}{dx} \neq 0
\]
Path 123 \( \neq \) Path 143

Or if there is any DIFFERENCE in \( E_x \) in the y direction
\[
\frac{dE_x}{dy} \neq 0
\]
Path 123 \( \neq \) Path 143
Maxwell’s Equations in Differential Form

Faraday’s Law
\[ \oint_{C} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{S} \vec{B} \cdot d\vec{S} \]

Stokes’ Thm
\[ \oint_{C} \vec{E} \cdot d\vec{l} = \iint_{S} (\nabla \times \vec{E}) \cdot d\vec{S} = -\frac{d}{dt} \iint_{S} \vec{B} \cdot d\vec{S} \]

\[ \Rightarrow \nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \]
Faraday’s Law In Differential Form

\[ \nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \]
Challenge Question

- For \( \mathbf{E} = E_0 \cos(\omega t - \beta z) \mathbf{a}_x \), which direction will \( \frac{d\mathbf{B}}{dt} \) point at \( t=0, \ z=\frac{\pi}{4\beta} \)?

(a) \( \mathbf{a}_x \), (b) \( \mathbf{a}_y \), (c) \( -\mathbf{a}_y \), (d) \( \mathbf{a}_z \), (e) \( \frac{d\mathbf{B}}{dt} = 0 \)

- Hint: find \( \frac{d\mathbf{B}}{dt} \) directly or use the sketch

From D3.1 (p 141) of old book
Maxwell’s Equations in Differential Form

**Ampere’s Law**
\[
\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \iint_S \mathbf{D} \cdot d\mathbf{S}
\]

**Stokes’ Thm**
\[
\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \iint_S (\mathbf{J} + \frac{d\mathbf{D}}{dt}) \cdot d\mathbf{S}
\]

\[
\Rightarrow \nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}
\]
Ampere’s Law In Differential Form

$$MMF = \oint \vec{H} \cdot d\vec{l} \neq 0$$

If there is any current going through a particular point:

H has “CURL” at a point if there is current going through the point:

CONDUCTION current
DISPLACEMENT current
Divergence

\[ \vec{v} = \langle 3x, 2y, 0 \rangle \]

\[ \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 5 \]

\( \nabla \cdot \vec{v} \) is a measure of how many field lines for \( \vec{v} \) are created at a given point. A place of high divergence is like a water faucet. Everywhere here is a faucet of strength 5.
Meaning of Divergence

Flux in = flux out so no sources or sinks inside V.

Flux out > flux in Positive divergence. Must be a source inside V.

Flux out < flux in Negative divergence. Must be a sink or drain inside V.
Meaning of Divergence

• There is divergence if there is a non-uniformity in the vector field in a direction parallel to the field
  – Divergence describes variation ALONG the flow of the field

• Divergence is proportional to the degree of non-uniformity

• Divergence is described only by the magnitude – so it’s a SCALAR
Divergence

\[ \nabla \cdot \mathbf{E} = \frac{dE_x}{dx} \neq 0 \]

YES
DIVERGENCE
Varies ALONG

\[ \frac{dE_y}{dx} \neq 0 \]

NO
DIVERGENCE
Varies ACROSS
Example: Divergence

• Find the divergence of $\mathbf{A} = (x-2)^2 \mathbf{a}_x$
**Divergence Theorem**

\[ \iiint_{V} \nabla \cdot \vec{v} \ dV = \iint_{\partial V} \vec{v} \cdot dS \]

\( \nabla \cdot \vec{v} \) is a measure of how many field lines for \( \vec{v} \) are created at a given point.

Thus, adding up the net # lines created inside a volume = the flux of lines out its boundary.
Optional: Derivation of Divergence Theorem

\[ \iiint \mathbf{\nabla} \cdot \mathbf{\bar{v}} \, dV = (\mathbf{\nabla} \cdot \mathbf{\bar{v}}) \, dV \approx \iiint (\mathbf{\nabla} \cdot \mathbf{\bar{v}}) \, dV \]

\[ \iiint \mathbf{\bar{v}} \cdot d\mathbf{\bar{S}} = \left[ v_x (x + dx) - v_x (x) \right] \cdot dy \cdot dz \]

\[ + \left[ v_y (y + dy) - v_y (y) \right] \cdot dx \cdot dz \]

\[ + \left[ v_z (z + dz) - v_z (z) \right] \cdot dx \cdot dy \]
Maxwell’s Equations in Differential Form

Gauss’ Law
\[ \iiint_B \nabla \cdot \mathbf{B} \, dV = 0 \]

Divergence Thm
\[ \iiint_S \mathbf{B} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{B} \, dV = 0 \]
\[ \Rightarrow \nabla \cdot \mathbf{B} = 0 \]
Does \( \mathbf{B} \) satisfy \( \nabla \cdot \mathbf{B} = 0 \)

\[
\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{a}_\phi
\]
Maxwell’s Equations in Differential Form

Gauss’ Law
\[ \iiint_D \mathbf{D} \cdot d\mathbf{S} = \iiint_V \rho \, dV \]

Divergence Thm
\[ \iiint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{D} \, dV = \iiint_V \rho \, dV \]
\[ \Rightarrow \nabla \cdot \mathbf{D} = \rho \]
E-field of pn junction (ECE 340)

- Find the $\mathbf{E}$-field for an evenly doped: $N_d = N_a$

pn junction: $\rho = \begin{cases} -\rho_0 = -qN_a & \text{for } -a < x < 0, \\ \rho_0 = qN_d & \text{for } 0 < x < a, \\ 0 & \text{for } |x| > a \end{cases}$

From Example 3.5 (p 145) of old book
Blank space for work
Challenge Question

• Can $\mathbf{A} = y\mathbf{a}_x + x\mathbf{a}_y$ be an E or B field in a region of free space where $\mathbf{J} = 0$, $\rho = 0$, and electrostatics applies? Workspace:

(a) Yes, but $\mathbf{A}$ can only be an E-field
(b) Yes, but $\mathbf{A}$ can only be a B-field
(c) Yes, $\mathbf{A}$ can be either an E or B field
(d) No, $\mathbf{A}$ cannot be either an E or B field

From Problem 3.11a (p 198) of old book
Continuity Equation in Differential Form

Continuity Eqn
\[ \iiint \vec{J} \cdot d\vec{S} = -\frac{d}{dt} \iiint \rho dV \]

Divergence Thm
\[ \iiint \vec{J} \cdot d\vec{S} = \iiint \nabla \cdot \vec{J} \ dV = -\frac{d}{dt} \iiint \rho dV \]

\[ \nabla \cdot \vec{J} = -\frac{d\rho}{dt} \]
Useful Relationships

\( \nabla \cdot (\nabla \times \vec{A}) = 0 \)

\( \nabla \times (\nabla f) = 0 \)  We already knew \( \nabla f \) is conservative

\( \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \) Must use cartesian coordinates!

\[
\nabla^2 \vec{A} \equiv (\nabla^2 A_x) \hat{a}_x + (\nabla^2 A_y) \hat{a}_y + (\nabla^2 A_z) \hat{a}_z
\]

\[
0 = \nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot (\vec{J} + \frac{d\vec{D}}{dt})
\]

\[
= \nabla \cdot \vec{J} + \frac{d}{dt} (\nabla \cdot \vec{D}) = \nabla \cdot \vec{J} + \frac{d\rho}{dt}
\]

\[\therefore \ \nabla \cdot \vec{J} = -\frac{d\rho}{dt}\]

The continuity eqn. is contained in Maxwell's Eqns.
Example: Continuity Eqn.

- For \( \mathbf{J} = J_0(x^2 \mathbf{a}_x + y^2 \mathbf{a}_y + z^2 \mathbf{a}_z) \), find \( \frac{d\rho}{dt} \) at the point \((0.02, 0.01, 0.01)\)
Summary of Maxwell’s Equations

Faraday’s Law
\[ \oint_{c} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{S} \mathbf{B} \cdot d\mathbf{S} \] \[ \nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \]

Ampere’s Law
\[ \oint_{c} \mathbf{H} \cdot d\mathbf{l} = \int_{S} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_{S} \mathbf{D} \cdot d\mathbf{S} \] \[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt} \]

Gauss’ Law
\[ \iiint_{V} \mathbf{B} \cdot d\mathbf{S} = 0 \] \[ \nabla \cdot \mathbf{B} = 0 \]

Gauss’ Law
\[ \iiint_{V} \mathbf{D} \cdot d\mathbf{S} = \iiint_{V} \rho dV \] \[ \nabla \cdot \mathbf{D} = \rho \]

Continuity Eq.
\[ \iiint_{S} \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \iiint_{V} \rho dV \] \[ \nabla \cdot \mathbf{J} = -\frac{d\rho}{dt} \]
Lectures 6-7
Sections 6.1-6.2

Potential Functions for Static Fields
Poisson’s and Laplace’s Equation
PN Junction
Electric Potential

How much potential energy does a charge have in an electric field?

Answer: It depends on the work that you had to do to get it to that spot. You have to work against the E-field.

\[ \Delta PE = \vec{F}_{applied} \cdot d\vec{l} = -\vec{F}_e \cdot d\vec{l} \]

\[ \vec{F}_e = q\vec{E} \]

\[ \Delta PE = -q\vec{E} \cdot d\vec{l} \]

Answer depends on the amount of charge \( q \)
Can we define something determined by the field only

Analogy to gravity:

\[ \vec{F}_g = m\vec{g}, \ PE = mgh, \ \frac{PE}{m} = gh \]
Electric Potential - Definition

Work you do per unit charge is the ELECTRIC POTENTIAL DIFFERENCE between the two points

$$\Delta V = \frac{\Delta P E}{q} = -\vec{E} \cdot d\vec{l} = -E_x \Delta x \text{ if } d\vec{l} = \Delta x \hat{a}_x$$

Units: $$\frac{Nm}{C} = \frac{J}{C} = VOLTS$$

$$E_x = -\frac{\Delta V}{\Delta x}, \quad E_y = -\frac{\Delta V}{\Delta y}, \quad E_z = -\frac{\Delta V}{\Delta z} \quad \text{Units: } \frac{V}{m}$$

$$\vec{E} = -\nabla V$$
Potential: Mount Electron

What is the steepest slope on the mountain?
Potential

What direction is the steepest slope at a given point?

Ans: The steepest downslope is opposite the gradient! This is always perpendicular to surfaces of constant potential.
UNIFORM FIELD

(a)

LONGITUDINAL SECTIONS

(b)

NONUNIFORM (DIVERGING) FIELD

(c)

3-D

(d)

Field uniform

Equipotentials

10 V

Field stronger

Equipotentials

Field weaker

10 V

10 V
Gradient Definition

\[ \vec{E} = -\left( \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z \right) \]

Most important definition of today:

\[ \vec{E} = -\nabla V \]

\[ \nabla = \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z} \]

Del \textbf{Operator} (Cartesian Coordinates) used to do gradient, divergence, and curl
Gradient Operator

\[ \vec{E} = -\nabla V \]

\[ \nabla = \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z} \]

V is a SCALAR FIELD (potential as a function of position)

\nabla V \] is a VECTOR FIELD
Has magnitude and direction -E
Is the DIRECTION with the FASTEST INCREASE in V
Fastest direction for an electron (negative charge) to decrease its potential
Scalar Potentials

• For: \( \Phi_1(x, y, z) = x^2 + y^2 + z^2 \)
  \( \Phi_2(x, y, z) = x + 2y + 2z \)

Find the following quantities at (3, 4, 12):
(a) the maximum rate of increase of \( \Phi_1 \)
(b) the maximum rate of increase of \( \Phi_2 \)
(c) the rate of increase of \( \Phi_1 \) along the direction of the maximum rate of increase of \( \Phi_2 \)
Example: Potentials for a Point Charge

\[ \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r \]

In spherical coords

How much work is required to move a **unit** charge from infinity to a radial distance = \( r_1 \)
Example: Potentials for a Point Charge

\[ \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r \]  

In spherical coords

\[ \vec{E} = -\nabla V \]  

“Absolute” potential at \( r_1 \) using zero potential at \( r = \infty \)

\[ V(r) = -\int_{\infty}^{r_1} \vec{E} \cdot d\vec{l} \]

\[ d\vec{l} = -|dr| \hat{a}_r = dr\hat{a}_r \]  

Since \( dr < 0 \) going from \( r = \infty \)

\[ V(r) = -\int_{\infty}^{r_1} \frac{Q}{4\pi\varepsilon_0 r^2} dr \]
Example: Potentials for a Point Charge

\[ V(r) = -\int_{\infty}^{r_1} \frac{Q}{4\pi\varepsilon_0 r^2} \, dr \]

\[ = \frac{Q}{4\pi\varepsilon_0} \left( \frac{1}{r_1} - \frac{1}{\infty} \right) \]

\[ = \frac{Q}{4\pi\varepsilon_0 r_1} \]

Surfaces of constant potential are spheres in 3D - same amount of work
Challenge Question: Scalar Potentials

- If the scalar potential $V$ can be defined, which of the following is true?

  (a) The potential difference between two points is identical, independent of path
  
  (b) The electric field is conservative
  
  (c) The electric field lines will always be perpendicular to equipotential surfaces
  
  (d) The greatest decrease in potential per unit length for a positive charge is along the electric field direction
  
  (e) All of the above
Potential Superposition

Potentials of more than one point charge are superimposed by addition

\[ V_{\text{point}} = \frac{1}{4 \pi \varepsilon_0} \left( \frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3} + \cdots \right) \]
Superposition Example: Potential of a Line Charge

Find Potential at point “P”, 1 m away from the line

\[ V_P = \frac{1}{4\pi\varepsilon_0} \int_{-a}^{a} \frac{dQ}{r} \]

\[ V_P = \frac{1}{4\pi\varepsilon_0} \int_{-a}^{a} \frac{\rho_l dz'}{r} \]

\[ V_P = \frac{1}{4\pi\varepsilon_0} \int_{-a}^{a} \frac{\rho_l dz'}{\sqrt{z'^2 + 1^2}} \]

\[ V_P = \left. \frac{\rho_l}{4\pi\varepsilon_0} \ln\left(z' + \sqrt{z'^2 + 1}\right) \right|_{-a}^{a} \]
Scalar Potentials

- For: \( \vec{E} = yz\hat{a}_x + (y + zx)\hat{a}_y + xy\hat{a}_z \)

(a) Find the scalar potential if \( V(0,0,0)=0 \). (Hint: Use a direct line path. At home, try using 3 separate segments along \( x, y, \) and \( z \) directions.)
(b) Evaluate the potential difference \( V_A - V_B \) for:
   1. \( A=(2,1,1) \) and \( B=(1,4,0.5) \)
   2. \( A=(2,2,2) \) and \( B=(1,1,1) \)
   3. \( A=(5,1,0.2) \) and \( B=(1,2,3) \)

Adapted from D 5.4 (p299) of old book
Useful facts about Potential

- Potential is always the DIFFERENCE between two points.
- If one point is at infinity, then the potential is an absolute potential.
- The gradient of $V$ gives the vector $-\mathbf{E}$ at a particular point.
- Potential between two points is identical, regardless of whether a straight or curved path is taken.
  - $\mathbf{E}$ is a conservative field.
- Electric field lines are always perpendicular to equipotential surfaces (constant voltage).
  - E-field lines are along the direction of the greatest decrease in potential.
Gradient Function in Cylindrical and Spherical Coordinates

\( \Phi \) = scalar field (like voltage)

\( \nabla \Phi = \begin{cases} 
\left( \frac{\partial \Phi}{\partial x} \hat{a}_x + \frac{\partial \Phi}{\partial y} \hat{a}_y + \frac{\partial \Phi}{\partial z} \hat{a}_z \right) & \text{Cartesian} \\
\left( \frac{\partial \Phi}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \hat{a}_\phi + \frac{\partial \Phi}{\partial z} \hat{a}_z \right) & \text{Cylindrical} \\
\left( \frac{\partial \Phi}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} \hat{a}_\phi \right) & \text{Spherical} 
\end{cases} \)
Laplacian Operator

$$\nabla \cdot (\nabla \Phi) = \nabla^2 \Phi$$

The “Laplacian” of a scalar field. (also called “Del Squared”)

$$\Phi(x, y, z)$$

Scalar Field

$$\nabla \Phi = \frac{\partial \Phi}{\partial x} \hat{a}_x + \frac{\partial \Phi}{\partial y} \hat{a}_y + \frac{\partial \Phi}{\partial z} \hat{a}_z$$

Vector

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Scalar
Poisson Equation for Potentials
(static fields, constant $\varepsilon$)

$$\nabla \cdot (\nabla \Phi) = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon}$$

Known as “Poisson’s Equation”
It’s just Gauss’ Law in terms of Potentials for a Static Field
If $\rho=0$, it is known as “Laplace’s Equation”
Potential of pn junction (ECE 340)

- Find the scalar potential $V$ for a step pn junction in silicon:

$$
\rho = \begin{cases} 
- qN_A & \text{for } -d_p < x < 0 \\
qN_D & \text{for } 0 < x < d_n \\
0 & \text{otherwise}
\end{cases}
$$

From Example 5.5 (p 300) of old book
Maxwell’s Eqns - Integral form

\[ \oint_{C} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_{S} \vec{B} \cdot d\vec{S} \]

\[ \oint_{C} \vec{H} \cdot d\vec{l} = \iint_{S} \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iint_{S} \vec{D} \cdot d\vec{S} \]

\[ \iint_{S} \vec{B} \cdot d\vec{S} = 0 \]

\[ \iint_{S} \vec{D} \cdot d\vec{S} = \iiint_{V} \rho dV \]

They are valid for ALL closed paths and closed surfaces, EVEN WHEN THEY SPAN A BOUNDARY BETWEEN TWO MATERIALS
Closed Path Through a Boundary

Medium 1 (above) \( \sigma_1, \varepsilon_1, \mu_1 \)

Medium 2 (below) \( \sigma_2, \varepsilon_2, \mu_2 \)

Closed path: abcd

Apply Faraday’s Law and Ampere’s Law to the closed path abcd
Normal Vectors

\( \hat{a}_n \) Vector NORMAL to the boundary. Points INTO medium 1

\( \hat{a}_S \) Vector normal to the path, TANGENT to the interface.

Use right hand rule for path to define direction
Faraday’s Law at Boundary

\[ \oint_c \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S} = 0 \]

Consider remaining \( E_1 \) and \( E_2 \) TANGENT TO SURFACE

Take limit as \( ad \) and \( bc \) go to zero

\( E_1 = E_2 \) i.e. \( E_t \) is continuous
The closed volume can enclose surface charges

Example
1. Free charges on the surface of a conductor
Gauss’ Law for D at Boundary

$$\iiint_D \cdot d\vec{S} = \iiint_V \rho dV$$

Take limit as $ae$, $bf$, $cg$, and $dh$ go to zero

Consider $D_1$ and $D_2$ NORMAL TO SURFACE

$$D_1 - D_2 = \rho \text{ i.e. } D_n \text{ is discontinuous because of } \rho_s$$
Challenge Question: Boundary Conditions

• What is $D_1$?

$$
\varepsilon_1 = \frac{1}{\varepsilon_0}, \quad \rho = 3 \frac{C}{m^2} \\
\varepsilon_2 = 2 \varepsilon_0, \quad \vec{D}_2 = 3\hat{x} + 2\hat{y} \frac{C}{m^2}
$$

(a) $3\hat{x} + 2\hat{y} \frac{C}{m^2}$
(b) $3\hat{x} + 5\hat{y} \frac{C}{m^2}$
(c) $3\hat{x} - 1\hat{y} \frac{C}{m^2}$
(d) $\frac{3}{2}\hat{x} + 5\hat{y} \frac{C}{m^2}$
(e) $\frac{3}{2}\hat{x} - 1\hat{y} \frac{C}{m^2}$
Connection of Concepts for Electrostatics

\[
\vec{E} = -\nabla V
\]

\[
V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l}
\]

\[
\nabla \cdot \vec{D} = \rho_{vol}
\]

\[
D_{1n} - D_{2n} = \rho_{s0}
\]

\[
Q = \rho_{s0} A \quad \text{or} \quad Q = \rho_{vol} V
\]

\[
Q = \int \rho_{s0} dS \quad \text{or} \quad Q = \int \rho_{vol} dV
\]

\[
\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}
\]

\[
\vec{P} \quad \text{polarization (material response)}
\]

\[
\vec{E} \quad \text{electric field}
\]

\[
\vec{D} \quad \text{displacement flux density}
\]

\[
V \quad \text{electrostatic potential}
\]

\[
Q \quad \text{charge}
\]

\[
\rho \quad \text{charge density}
\]
Lecture 6-7 Summary

• Since \( \text{div } \mathbf{D} = \rho \) and \( \mathbf{D} = \varepsilon \mathbf{E} \),

\[
\mathbf{E} = -\nabla \Phi
\]

satisfy Gauss’ Laws and is valid for static fields, if \( \varepsilon \) is constant and \( \Phi \) satisfies Poisson’s equation:

\[
\nabla^2 \Phi = -\frac{\rho}{\varepsilon}
\]

• At a boundary, \( E_{1t} = E_{2t} \) but \( D_{1n} - D_{2n} = \rho \)
Lectures 8-9
Section 5.1

Conductors
Dielectrics
Atomic Model of Conductivity

- Tightly bound inner orbitals
- Loosely bound outer orbitals
  - Free to escape the nucleus and move around inside the material
Conduction of Free Electrons

$e^-$

$E_{\text{Applied}}$
Conduction of free electrons

- The electron cannot travel in a straight path for very long
- It keeps running into the nuclei, getting deflected MANY times, meeting RESISTANCE
- Net motion is still in direction opposite of the applied $E$ and drift velocity is:

$$\vec{v}_d = -\mu_e \vec{E}$$ for electrons

$$\mu_e = \frac{|e| \tau}{m_e^*}$$ is the electron mobility,

$$\tau \approx 10^{-14} \text{ sec}$$ is the average collision time
Holes are missing electrons

- Holes are missing electrons
  - Move in direction of $\mathbf{E}$ field

$$\bar{v}_d = +\mu_h \bar{E} \quad \text{for holes}$$
Resistors

\[ R = \frac{V}{I} \]

\[ \Omega (\text{Ohms}) = \frac{\text{volts}}{\text{amps}} \]

R tells us how much total resistance is encountered, but does not tell us anything fundamental about the resisting material.
Resistivity

\[ \rho = \text{Resistivity, units of } (\Omega \cdot \text{m}) \]

- Tells us how much resistance the *material* provides, factoring out the dimensions of the resistor

\[ R = \rho \frac{l}{A} \]
Conductivity

\[ \sigma = \frac{1}{\text{Resistivity}} = \frac{l}{R \cdot A} = \frac{1}{\Omega \cdot m} \]

- More common to describe materials in terms of conductivity rather than resistivity
- High conductivity = low resistivity
- Special units:

\[ \sigma = \frac{1}{\Omega \cdot m} = \frac{\text{Siemens}}{m} = \frac{S}{m} \]
<table>
<thead>
<tr>
<th>Material</th>
<th>Conductivity, $\Omega \text{m}^{-1}$</th>
<th>Material</th>
<th>Conductivity, $\Omega \text{m}^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quartz, fused</td>
<td>$\sim 10^{-17}$</td>
<td>Silicon</td>
<td>$10^3$</td>
</tr>
<tr>
<td>Ceresin wax</td>
<td>$\sim 10^{-17}$</td>
<td>Carbon</td>
<td>$\sim 3 \times 10^4$</td>
</tr>
<tr>
<td>Polystyrene</td>
<td>$\sim 10^{-16}$</td>
<td>Graphite</td>
<td>$\sim 10^5$</td>
</tr>
<tr>
<td>Sulfur</td>
<td>$\sim 10^{-15}$</td>
<td>Cast iron</td>
<td>$\sim 10^6$</td>
</tr>
<tr>
<td>Mica</td>
<td>$\sim 10^{-15}$</td>
<td>Mercury</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Paraffin</td>
<td>$\sim 10^{-15}$</td>
<td>Nichrome</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Rubber, hard</td>
<td>$\sim 10^{-15}$</td>
<td>Stainless steel</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Porcelain</td>
<td>$\sim 10^{-14}$</td>
<td>Constantan</td>
<td>$2 \times 10^6$</td>
</tr>
<tr>
<td>Glass</td>
<td>$\sim 10^{-12}$</td>
<td>Silicon steel</td>
<td>$2 \times 10^6$</td>
</tr>
<tr>
<td>Bakelite</td>
<td>$\sim 10^{-9}$</td>
<td>German silver</td>
<td>$3 \times 10^6$</td>
</tr>
<tr>
<td>Distilled water</td>
<td>$\sim 10^{-4}$</td>
<td>Lead</td>
<td>$5 \times 10^6$</td>
</tr>
<tr>
<td>Dry, sandy soil</td>
<td>$\sim 10^{-3}$</td>
<td>Tin</td>
<td>$9 \times 10^6$</td>
</tr>
<tr>
<td>Marshy soil</td>
<td>$\sim 10^{-2}$</td>
<td>Phosphor bronze</td>
<td>$10^7$</td>
</tr>
<tr>
<td>Fresh water</td>
<td>$\sim 10^{-2}$</td>
<td>Brass</td>
<td>$1 \times 10^7$</td>
</tr>
<tr>
<td>Animal fat‡</td>
<td>$4 \times 10^{-2}$</td>
<td>Zinc</td>
<td>$1.7 \times 10^7$</td>
</tr>
<tr>
<td>Animal muscle (⊥ to fiber)‡</td>
<td>0.08</td>
<td>Tungsten</td>
<td>$1.8 \times 10^7$</td>
</tr>
<tr>
<td>Animal, body (ave)‡</td>
<td>0.2</td>
<td>Duralumin</td>
<td>$3 \times 10^7$</td>
</tr>
<tr>
<td>Animal muscle (∥ to fiber)‡</td>
<td>0.35</td>
<td>Aluminum, hard-drawn</td>
<td>$3.5 \times 10^7$</td>
</tr>
<tr>
<td>Animal blood</td>
<td>0.7</td>
<td>Gold</td>
<td>$4.1 \times 10^7$</td>
</tr>
<tr>
<td>Germanium (semiconductor)</td>
<td>$\sim 2$</td>
<td>Copper</td>
<td>$5.7 \times 10^7$</td>
</tr>
<tr>
<td>Seawater</td>
<td>$\sim 4$</td>
<td>Silver</td>
<td>$6.1 \times 10^7$</td>
</tr>
<tr>
<td>Ferrite</td>
<td>$10^2$</td>
<td>Hg (at &lt;4.1 K)</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Tellurium</td>
<td>$\sim 5 \times 10^2$</td>
<td>Nb (at &lt;9.2 K)</td>
<td>$\infty$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Nb$_3$(Al-Ge) (at &lt;21 K)</td>
<td>$\infty$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>YBa$_2$Cu$_3$O$_7$ (at &lt;80 K)</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

† or low frequencies. At 20°C except where noted.
Conductivity

- Conductivity takes everything into account in one number
  - Number of free electrons
  - Frequency of collisions between electrons and the nuclei
  - Scattering properties of the nuclei
  - Motion for electrons and holes

\[ \sigma = \mu_e N_e |e| + \mu_h N_h |e| \]

\( N_e, N_h = \# \) of free electrons or holes per cm\(^3\)
Ohm’s Law

\[ |E| = \frac{V}{l} \left( \frac{V}{m} \right) \]

\[ |J| = \frac{I}{A} \left( \frac{A}{m^2} \right) \]

\[ R = \frac{l}{\sigma A} \ (\Omega) \]

\[ V = IR \]

\[ E \cdot l = (J \cdot A) \left( \frac{l}{\sigma A} \right) = \frac{1}{\sigma} J \cdot l \]

\[ J = \sigma E \]
Maxwell’s Equations in a Conductor

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

In free space

\[ \nabla \times \vec{H} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} \]

In a conducting material

Conduction current: Moving charge

Displacement Current: Time varying E-field
Perfect Conductor (PC) in an Applied Electric Field

Air

Perfect Conductor

$E_{\text{Applied}}$

What Happens?
Perfect Conductor (PC) in an Applied Electric Field

What Happens?
- Free e\(^{-}\) move in direction opposite of applied $E$
- When e\(^{-}\) moves away from its nucleus, it leaves a + charge behind
- The material as a whole is still NEUTRARLY CHARGED but the charge has now been redistributed
$E=0$ inside perfect conductor

What Happens Next?
- The separated + and - charges create their own SECONDARY INTERNAL $E$ field that **EXACTLY CANCELS** the applied $E$ field
- The TOTAL $E$ field inside the perfect conductor is **ALWAYS ZERO**
- If it weren’t zero, free charge would continue drifting till it is!
Infinite plane conducting slab

- An electrically neutral infinite plane conducting slab lies between two infinite plane sheets of uniform charge density \( \rho_A \) and \( \rho_B \). Find the surface charge densities on the two slab surfaces.

From D4.2 (p 217) of old book
If $\mathbf{E}=0$ inside a PC, how about $\mathbf{H}$?

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad 0 = -\frac{\partial \mathbf{B}}{\partial t} \]

B and H must be STATIC

Technically, a non-zero static H-field can exist inside a PC, but how did it get there? It must have existed there for all time. But, then what created it in the first place. This is an ill-posed problem so in 329, we will assume $\mathbf{H}=0$ inside a PC.
Challenge Question:
Point charge above a sheet

- A positive charge is located above a perfectly conducting infinite ground plane at \( z=0 \)

(a) \( \mathbf{E}=0 \) above the plane
(b) \( \mathbf{E}=0 \) below the plane
(c) A uniform charge density is induced on the plane
(d) The total induced charge on the plane is \(+Q\)
(e) The electric field lines have a radial component at the plane
Lecture 8 Summary

• Ohm’s Law:

• Conductivity: \( \sigma = \frac{1}{\text{Resistivity}} = \frac{l}{R \cdot A} \)

• Inside a perfect conductor, \( \mathbf{E} = \)

• Next class
  – Dielectrics
ECE 329
Lecture 9
Dielectrics
Atomic Model of Dielectric Polarization

- Tightly bound inner orbitals
- Not many loosely bound outer orbital electrons
Polarization of an atom

Externally applied electric field separates electron and nucleus

Bound charge

$E_a$
Polarization of an atom

External field distorts atom slightly
$E$ inside is reduced but is non-zero
Polarization or Electric Dipole Moment per Unit Volume

“mini” dipole moment for one atom

\[ d \] is the distance VECTOR going from “-” to “+” side

\[ p = Qd \] is the electric dipole moment
Apply \( \mathbf{E} \) to Dielectric Material

\[
\vec{P} = N \vec{p} = \varepsilon_0 \chi_e \vec{E}_{\text{tot}}
\]
Dielectric Susceptibility $\chi_e$

- Measures how easy it is to shift electrons from their centered orbit around the nuclei of a material to form internal dipoles

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}_{tot}$$
Apply $\mathbf{E}$ to Dielectric Material

Cancellation of internal charges

“Polarization Surface Charges” Remain
Apply $\mathbf{E}$ to Dielectric Material

Cancellation of internal charges

Secondary Electric Field Produced

* Does not cancel applied field inside dielectric
\( \mathbf{E}_{\text{tot}} \) is reduced but not zero

Unlike conductors where \( \mathbf{E} = 0 \), in the dielectric slab, the total field is **reduced (but NOT eliminated)** by the secondary field produced by the surface polarization charge

\[
\mathbf{E}_{\text{tot}} = \mathbf{E}_a + \mathbf{E}_s
\]

And the polarization vector, \( \mathbf{P} \), is the polarization of atoms in the dielectric due to the TOTAL field (after it has been reduced by \( \mathbf{E}_s \))

\[
\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}_{\text{tot}}
\]
Example

- Apply external electric field to a slab of dielectric material
- Apply field by placing the slab between two equal and opposite charge densities

\[ \vec{E}_{\text{Applied}} = \frac{\rho_{S0}}{\varepsilon_0} \hat{a}_z \]

Dielectric slab, \( \chi_e \)
Example

$$\vec{E}_{\text{Secondary}} = -\frac{\rho_p}{\varepsilon_0} \hat{a}_z$$

$$\vec{E}_{\text{Applied}} = \frac{\rho_{s0}}{\varepsilon_0} \hat{a}_z$$

Dielectric slab, $\chi_e$
Example

Dielectric slab, $\chi_e$

$$\vec{E}_{Secondary} = -\frac{\rho_P}{\varepsilon_0} \hat{a}_z$$

$$\vec{E}_{Applied} = \frac{\rho_{S0}}{\varepsilon_0} \hat{a}_z$$

$$\vec{E}_{Total} = \vec{E}_{Applied} + \vec{E}_{Secondary}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}_{total} = \chi_e (\rho_{S0} - \rho_P) \hat{a}_z$$

But how much charge $\rho_P$ is generated by $\rho_{S0}$??
Example

\[ \vec{P} = P_0 \hat{a}_z \]

where \( P_0 \) is the dipole moment per unit volume

\[ \vec{P} V = P_0 (d\Delta S) \hat{a}_z \]

the total dipole moment of the column

\[ Q \vec{d} = (\rho_p \Delta S) d \hat{a}_z \]

also the total dipole moment of the column

\[ \therefore P_0 = \rho_p \]

dipole moment per unit volume

= surface charge density

Goddard/Cunningham
ECE329 Lectures 8-9
Example

Dielectric slab, $\chi_e$

$E_{\text{Secondary}} = \frac{-P_0}{\varepsilon_0} \hat{a}_z$

$P = \varepsilon_0 \chi_e \vec{E}_{\text{total}} = \chi_e (\rho_{S0} - \rho_P) \hat{a}_z$

$P_0 \hat{a}_z = \chi_e (\rho_{S0} - \rho_P) \hat{a}_z = \chi_e (\rho_{S0} - P_0) \hat{a}_z$

$P_0 = \chi_e \rho_{S0} - \chi_e P_0$

$\therefore \rho_P = P_0 = \frac{\chi_e \rho_{S0}}{1 + \chi_e}$
Example

So finally, the “final answer” is:

\[ \vec{E}_{\text{Total}} = \frac{P_0}{\varepsilon_0 \varepsilon_r} = \frac{\rho_{s0}}{\varepsilon_0} / (1 + \varepsilon_r) \]

The total electric field strength inside the dielectric is reduced from its “free space” value by \((1 + \varepsilon_r)\)

Free space: \(\varepsilon_r = 0\)  Perfect Conductor: \(\varepsilon_r \to \infty\)
Moving charges means a current

Definition of the “Polarization Current”

\[ I_P = \frac{dQ}{dt} \]
Recall from the Example

Where $P_0$ is the dipole moment per unit volume

The total dipole moment of the column

Also the total dipole moment of the column

$P_0 \Delta S = Q$

$
\therefore \quad J_p = \frac{I_p}{\Delta S} = \frac{dQ/dt}{\Delta S} = \frac{dP_0}{dt}
$
Polarization Current

- Application of an alternating $\mathbf{E}$ field results in a polarization current due to motion of charge between the two surfaces of the dielectric material

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$
Ampere’s Law in a Dielectric

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

In a dielectric medium, we need to include \( J_P \) with the total current

\[ \nabla \times \vec{H} = \vec{J} + \vec{J}_P + \frac{\partial (\varepsilon_0 \vec{E}_{total})}{\partial t} \]

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{P}}{\partial t} + \frac{\partial (\varepsilon_0 \vec{E}_{total})}{\partial t} \]
Ampere’s Law in a Dielectric

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{P}}{\partial t} + \frac{\partial (\varepsilon_0 \vec{E}_{\text{total}})}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t} \left( \vec{P} + \varepsilon_0 \vec{E}_{\text{total}} \right)$$

So Ampere’s law is the same and we modify the displacement vector’s definition

$$\vec{D} = \vec{P} + \varepsilon_0 \vec{E}_{\text{total}}$$
Definition of Dielectric Constant

\[ \vec{D} = \vec{P} + \varepsilon_0 \vec{E}_{\text{total}} \]

\[ \vec{D} = \varepsilon_0 \chi_e \vec{E}_{\text{total}} + \varepsilon_0 \vec{E}_{\text{total}} \]

\[ \vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E}_{\text{total}} \]

\[ \vec{D} = \varepsilon_0 \varepsilon_r \vec{E}_{\text{total}} \]
Definition of Dielectric Constant

\[ \vec{D} = \varepsilon_0 \varepsilon_r \vec{E}_{total} \]

Relative Permittivity = “Dielectric Constant”

Dielectric Permittivity of the Material

\[ \varepsilon_r = \left(1 + \chi_e\right) \]

\[ \varepsilon = \varepsilon_0 \varepsilon_r \]

\[ \vec{D} = \varepsilon \vec{E}_{total} \]

Units for \( \varepsilon_r \): None
Ampere’s Law in a Dielectric

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

with

\[ \vec{D} = \varepsilon \vec{E}_{total} \]

No need to worry about polarization current - it is all incorporated into the “new” definition for \( \vec{D} \)

Generally we can use Maxwell’s equations to solve for \( \vec{D} \) and then calculate \( \vec{E}_{total} \)
Recall Example

Dielectric slab, $\chi_e$

$E_{Secondary} = \frac{-P_0}{\varepsilon_0} \hat{a}_z$

$E_{Applied} = \frac{\rho_{s0}}{\varepsilon_0} \hat{a}_z$

$E_{Total} = \frac{P_0}{\varepsilon_0 \chi_e} = \frac{\rho_{s0}/\varepsilon_0}{1 + \chi_e}$

$D = \varepsilon \overrightarrow{E}_{Total} = \varepsilon_0 (1 + \chi_e) \overrightarrow{E}_{Total} = \rho_{s0}$

If instead voltage were fixed, then $E$-field would be **same** as free space

---

E-field strength reduced by $(1 + \chi_e)$

D-field strength is **same** as free space because charge is fixed

Goddard/Cunningham
ECE329 Lectures 8-9
Infinite plane dielectric slab

- An infinite plane dielectric slab lies between two infinite plane sheets of uniform charge density of \( \rho = \pm 1 \mu \text{C/m}^2 \). Find \( \mathbf{D} \), \( \mathbf{E} \), and \( \mathbf{P} \) inside the slab.

\[
\mathbf{D} = \frac{\rho}{2} \hat{\mathbf{z}}
\]

From D4.3 (p 226) of old book
Challenge Question: Spherical shell dielectric

- If we have:
  \[
  \begin{cases}
  r < a & \text{perfect conductor} \\
  a < r < b & \text{perfect dielectric } \varepsilon = 3\varepsilon_0 \\
  r > b & \text{perfect conductor}
  \end{cases}
  \]

with equally and oppositely charged PCs and \( \mathbf{D} \) points radially inward, which is true:

(a) \( V(b) > V(a) \)
(b) \( V(b) = V(a) \)
(c) \( V(b) < V(a) \)
Lecture 9 Summary

- Electric dipole moment $\mathbf{p} = q \mathbf{d}$
- Polarization or electric dipole moment per unit volume

$$\mathbf{P} = N \mathbf{p} = \varepsilon_0 \chi_e \mathbf{E}_{\text{total}} = \varepsilon_0 \chi_e (\mathbf{E}_a + \mathbf{E}_s)$$
  - Simple linear isotropic dielectric
    - Reduces $\mathbf{E}$-field strength by $(1 + \chi_e)$
    - $\mathbf{D}$ has same value as free space
  - Polarization current $\mathbf{J}_p = \frac{d\mathbf{P}}{dt}$
  - New definition $\mathbf{D} = \mathbf{P} + \varepsilon_0 \mathbf{E}_{\text{total}} = \varepsilon_{ij} \mathbf{E}_{\text{total}}$

- Next class
  - Capacitance and Conductance
ECE 329
Lectures 10-11
Sections 6.3, 5.1

Capacitance and Conductance
Conductivity and Susceptibility
Parallel-plate Capacitor

Capacitance is: \[ C = \frac{Q}{V_0} \]
Steps to Find Capacitance

- Laplace Equation \( \nabla^2 V = 0 \)
- Find \( V \) using boundary conditions
- Find \( \mathbf{E} \) using \( \mathbf{E} = -\nabla V \)
- Find \( \mathbf{D} \) using \( \mathbf{D} = \varepsilon \mathbf{E} \)
- Get surface charge density on one conductor using BC
  \( \rho_s = \vec{a}_n \cdot (D_{n1} - D_{n2}) \)
- Charge
  \( Q = (\text{Area})(\rho_s) \)
- Capacitance
  \( C = \frac{Q}{V_0} \)

\[ V(x) = V_0 \frac{x}{d} \]

\[ \rho = \varepsilon V_0 / d \]

\[ C = \frac{\varepsilon A}{d^3} \]
Non-uniform permittivity

Find capacitance if material has $\varepsilon(x)$:

$$
\varepsilon(x) = \frac{\varepsilon_0}{1 - x/2d}
$$

---

Adapted from P5.20 (p352) of old book
Coaxial Cable

Perfect conductors

Cylindrical Coordinates

\[ E = E_r \]

\[ V = V_0 \]

\[ V = 0 \]
Coaxial Cable

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) = 0$$

$$\rho = \begin{cases} \varepsilon V_0 / (a \ln(b / a)), & r = a \\ -\varepsilon V_0 / (b \ln(b / a)), & r = b \end{cases}$$

$$C = \frac{2\pi \varepsilon L}{\ln(b / a)}$$
Challenge Question: Coaxial cable capacitance

- Consider a coax with \((\varepsilon_{\text{old}}=3\varepsilon_0)\) for \(a<r<b\). If we remove the dielectric and replace it with free space \((\varepsilon_{\text{new}}=\varepsilon_0)\) but keep the same amount of charge on each PC, which is false:

  (a) the voltage across the capacitor is reduced by 3x
  (b) the capacitance is reduced by 3x
  (c) the E-field for \(a<r<b\) is increased by 3x
  (d) the D-field for \(a<r<b\) is unchanged
Challenge Question: Coaxial cable capacitance

- Consider a coax with \((\varepsilon_{\text{old}} = 3\varepsilon_0)\) for \(a<r<b\). If we remove the dielectric and replace it with free space \((\varepsilon_{\text{new}} = \varepsilon_0)\) but keep the same voltage across the coax, which is false:

(a) the charge across the capacitor is reduced by 3x
(b) the capacitance is reduced by 3x
(c) the E-field for \(a<r<b\) is reduced by 3x
(d) the D-field for \(a<r<b\) is reduced by 3x
Energy Stored in a Capacitor

• The work by the battery to move a charge \( dq \) from the bottom to the top plate is:

\[
dU = (dq)V_C = dq \frac{q}{C}
\]

\( V_C \) is instantaneous voltage on the capacitor. It starts at zero and increases to \( V \)

• Thus the total stored energy while charging is:

\[
U = \int_{0}^{Q} dq \frac{q}{C} = \frac{Q^2}{2C} = \frac{1}{2} CV_C^2
\]

\( U_{\text{fully charged}} = \frac{1}{2} CV_C^2 \)

because \( V_C = V \) after charging
Current Flow for a Capacitor

- Charge moves onto a capacitor at a rate of:

\[
I = \frac{dQ}{dt} = \frac{d(CV_C)}{dt} = CV_C \frac{dV_C}{dt}
\]

- Thus the instantaneous rate of power absorption by the capacitor is:

\[
P = IV_C = CV_C \frac{dV_C}{dt} = \frac{d}{dt} \left( \frac{1}{2} CV_C^2 \right) = \frac{dU}{dt}
\]

Rate of power absorption = Rate of increase of stored energy
Conductance

Medium in capacitor now has conductivity

Now a current can flow $x=d$ to $x=0$

$$\mathbf{J}_c = \sigma \mathbf{E} = \sigma \left( \frac{V_0}{d} \right) (-\hat{a}_x)$$

Current density (A/m²)
Conductance

\[ G = \frac{|I_c|}{V_0} \]

Conductance
Units: Siemens

For the parallel plate capacitor

\[ G = \frac{\sigma A}{d} \]
What is radial current \((I_c)\) for a fixed length of the cable?

\[
\bar{E} = \frac{V_0}{\ln(b/a)} \left( \frac{1}{r} \right) \bar{a}_r
\]

so

\[
\bar{J}_c = \sigma \bar{E} = \frac{\sigma V_0}{\ln(b/a)} \left( \frac{1}{r} \right) \bar{a}_r
\]
Conductance/Length of Coaxial Cable

\[ I_c = \int \vec{J} \cdot d\vec{S} \]  

\[ I_c = \int_0^L \int_{\phi=0}^{2\pi} J_c (rd\phi dz) \]

\[ I_c = \frac{2\pi \sigma V_0 L}{\ln(b/a)} \]

\[ G = \frac{I_c}{V_0} = \frac{2\pi \sigma L}{\ln(b/a)} \quad [\text{Siemens}] \]

\[ G = \frac{G}{L} = \frac{2\pi \sigma}{\ln(b/a)} \quad [\text{S/m}] \]
Steps for Finding Conductance

- Find Electric Field

- Find Conduction Current Density (A/m²)  \( \vec{J}_c = \sigma \vec{E} \)

- Conduction Current (A)  \( I_c = \int \vec{J} \cdot d\vec{S} \)

- Conductance  \( G = \frac{I_c}{V_0} \)

- Conductance/Length
(Optional) The Laplace Transform

• Can be used for Transmission Line analysis
  – Method to solve Initial Value Differential Equations

\[ \mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} e^{-st} f(t)dt \]

• Most useful property - it converts differential equations in time to algebraic ones in s-space:

\[ \mathcal{L}\{f'(t)\} = sF(s) - f(0) \]

\[ \mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0) \]
(Optional) The Laplace Transform

- Key Transformations are the following:

<table>
<thead>
<tr>
<th>f(t) = $\mathcal{L}^{-1}{F(s)}$</th>
<th>F(s) = $\mathcal{L}{f(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/s</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$1/(s-a)$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$n!/s^{n+1}$</td>
</tr>
<tr>
<td>$u(t-c)$</td>
<td>$e^{-cs}/s$</td>
</tr>
<tr>
<td>$u(t-c)f(t-c)$</td>
<td>$e^{-cs}F(s)$</td>
</tr>
<tr>
<td>$e^{ct}f(t)$</td>
<td>$F(s-c)$</td>
</tr>
<tr>
<td>$f(ct)$</td>
<td>$1/c \ F(s/c)$</td>
</tr>
<tr>
<td>$\delta(t-c)$</td>
<td>$e^{-cs}$</td>
</tr>
<tr>
<td>$\int_0^t f(t-\tau)g(\tau),d\tau$</td>
<td>$F(s)G(s)$</td>
</tr>
</tbody>
</table>
(Optional) Application of the Laplace Transform

\[ IR + L \frac{dI}{dt} = 0 \]

\[ \mathcal{L}\left\{ IR + L \frac{dI}{dt} \right\} = 0 \]

\[ F(s)R + (sF(s) - I_0) \cdot L = 0 \]

\[ F(s) = \frac{I_0L}{R + sL} = \frac{I_0}{s + R / L} \]

\[ \mathcal{L}\{e^{at}\} = \frac{1}{s - a} \Rightarrow I(t) = I_0 e^{\frac{R}{L}t} \]
Lecture 10 Summary

• Capacitance \( C = \frac{Q}{V_0} \)

• Conductance \( G = \frac{|I_c|}{V_0} \)

• Next Up
  – Conductivity and Susceptibility
Lecture 11

Bound charge
Modeling $\chi$ and $\varepsilon$
The Lorentz-Drude model

- With applied $\mathbf{E}$, the mean position of free charge $q>0$ drifts in direction $\mathbf{E}$ with mean velocity $\mathbf{v}$: balance of acceleration due to $\mathbf{E}$ and friction from collisions with lattice at random intervals with mean time $\tau$

\[
m \frac{d\mathbf{v}}{dt} = q\mathbf{E} - m \frac{T}{\tau}
\]
The Lorentz-Drude model: DC conductivity

- If \( E = 0 \), charge eventually slows to \( \mathbf{v} = 0 \): \( \mathbf{v}(t) = v_0 e^{-t/\tau} \)

- But, with a constant \( E \), the steady state solution is:

\[
\mathbf{v}(t = \infty) = \frac{q\tau}{m} \mathbf{E} = \mu \mathbf{E} \quad \text{where} \quad \mu = \frac{q\tau}{m}
\]

charge mobility. For \( N \) charges per unit volume, the total current becomes:

\[
\vec{I} = \frac{dQ}{dt} = qN \Delta V = qNA\vec{v}\Delta t = qNA\vec{v}
\]

What does this look like?
Ohm’s Law!

\[ \vec{J} = \frac{\vec{I}}{A} = qN\vec{v} = \frac{Nq^2\tau}{m} \vec{E} \]

\[ \vec{J} = \sigma \vec{E} \text{ where } \sigma = \frac{Nq^2\tau}{m} = \frac{Nq^2}{m \bar{\omega}} \]

where \( \bar{\omega} \equiv \frac{1}{\tau} \) is the collision frequency
Review of PHASORS

Spock, set your TI-89 to “STUN”, not “KILL”!!!

Sorry Captain, I promise not to harm another ECE329 student. Next time, I will set the Phasor to “AWAKEN” instead!
Phasor Review - Complex #’s

\[ a + jb = \overline{A} = Ae^{j\theta} \]

\[ Ae^{j\theta} = A \cos \theta + jA \sin \theta \]

\[ A = \sqrt{a^2 + b^2} \]

\[ \theta = \tan^{-1} \frac{b}{a} \]
Phasor Review - Complex #'s

\[ Ae^{jx} = A \cos(x) + jA \sin(x) \]

\[ A \cos(x) = \text{Re}[Ae^{jx}] \]

\[ A \sin(x) = \text{Im}[Ae^{jx}] = \text{Re}[-jAe^{jx}] = \text{Re}[Ae^{j(x-\pi/2)}] \]

Write \( 5 \cos(\omega t) + 10 \sin(\omega t - 30^\circ) \) as a phasor
Phasor Review with vectors

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

\[ \cos(\theta) = \text{Re}[e^{j\theta}] \]

\[ \sin(\theta) = \text{Re}[-je^{j\theta}] = \text{Re}[e^{j(\theta-\pi/2)}] \]

\[ \text{Re}[z] = (z + z^*)/2 \]

\[ E_x(z, t) = E_x(z) \cos(\omega t + \beta z + \theta) \]

\[ = \text{Re}[E_x(z)e^{\jmath \beta z}e^{j\theta}e^{j\omega t}] \]

\[ = \text{Re}[\tilde{E}_x(z)e^{j\omega t}] \]
The Lorentz-Drude model: AC conductivity

- If $E$ is time varying, we use phasors to analyze the response:

$$E(t) = \text{Re}\{\tilde{E}e^{j\omega t}\}$$
$$\bar{v}(t) = \text{Re}\{\tilde{v}e^{j\omega t}\}$$
$$\bar{J}(t) = \text{Re}\{\tilde{J}e^{j\omega t}\}$$

- Note that: $\tilde{E}$ and $\tilde{J}$ are vectors and complex valued

(e.g. $\tilde{E} = 3.2e^{j\pi/4}\hat{x} - 1.3e^{j\pi/6}\hat{y}$)
The Lorentz-Drude model: AC conductivity

- We can now transform the force equation from a differential one to an algebraic one (d/dt = jω):

\[
m \frac{d\vec{v}}{dt} = q\vec{E} - m\frac{\vec{v}}{\tau} \implies mj\omega\vec{v} = q\vec{E} - m\omega\vec{v}
\]

and thus, we get:

\[
\vec{v} = \frac{q\vec{E}}{mj\omega + m\omega} \implies \vec{J} = qN\vec{v} = \frac{Nq^2}{m(j\omega + \omega)} \vec{E}
\]

\[
\sigma = \frac{Nq^2}{m(j\omega + \omega)}
\]
Challenge Question: AC conductivity

- We found: \[ \sigma = \frac{Nq^2}{m(j\omega + \omega)} \]

I. At all finite frequencies, energy is being lost to Joule heating
II. At low frequency (DC), J and E are in phase
III. At very high frequency (\( \omega >> \bar{\omega} \)), J and E are 90° out of phase

Which of the following is true:
(a) I only, (b) II only, (c) III only,
(d) I, II, and III, (e) None are true
(Preview) Time Averaged Poynting Vector

\[
\vec{E} = \text{Re}[\tilde{E} e^{j\omega t}] = \frac{\tilde{E} e^{j\omega t} + \tilde{E}^* e^{-j\omega t}}{2} \\
\vec{H} = \text{Re}[\tilde{H} e^{j\omega t}]
\]

\[
\langle \tilde{S} \rangle = \langle \vec{E} \times \vec{H} \rangle = \left\langle \frac{\tilde{E} e^{j\omega t} + \tilde{E}^* e^{-j\omega t}}{2} \times \frac{\tilde{H} e^{j\omega t} + \tilde{H}^* e^{-j\omega t}}{2} \right\rangle
\]

\[
= \frac{1}{4} \left\langle \vec{E} \times \vec{H} e^{2j\omega t} + \vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H} + \vec{E}^* \times \vec{H}^* e^{-2j\omega t} \right\rangle
\]

\[
\langle \tilde{S} \rangle = \frac{1}{2} \text{Re}[\tilde{E} \times \tilde{H}^*]
\]
Susceptibility

- A perfect dielectric is defined by $\sigma = 0$ so since $\sigma$, there can’t be any free charge inside ($N=0$)
- The dielectric can have bound charge and we know it can be polarized:

\[
p = q\vec{d} = -q\vec{r}
\]

if the nucleus is located at the origin
- The electron’s motion is described by:

\[
\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} = -q\vec{E} - 2m\alpha \frac{d\vec{r}}{dt} - m\omega_0^2 \vec{r}
\]

where $-m\omega_0^2 \vec{r}$ describes the spring like restoring force to the nucleus and $-2m\alpha \vec{v}$ is a friction like damping force

\[
\sigma = \frac{Nq^2}{m(j\omega + \sigma)}
\]
DC Susceptibility

• For DC, \( \frac{d}{dt} = 0 \) so we get

\[
\vec{r} = -\frac{q\vec{E}}{m\omega_0^2}
\]

and so:

\[
p = -q\vec{r} = \frac{q^2}{m\omega_0^2} \vec{E}
\]

and thus:

\[
P = N_d p = \frac{N_d q^2}{m\omega_0^2} \vec{E} = \varepsilon_0 \chi_e \vec{E}
\]

where:

\[
\chi_e = \frac{N_d q^2}{m\omega_0^2 \varepsilon_0}
\]

Note: \( N_d \) (# of dipoles per vol) ≠ \( N \) (# of free charge per vol)
Can use phasors to derive AC susceptibility: \( \chi_e(\omega) \)
Lecture 11 Summary

• AC Conductivity:

\[ \sigma = \frac{Nq^2}{m(j\omega + \varpi)} \]

• DC Susceptibility:

\[ \chi_e = \frac{N_d q^2}{m\omega_0^2 \varepsilon_0} \]

• Next Up
  – Magnetic Force, Ampere’s law, current sheets, Faraday’s law (1.6, 2.4, 2.3)
Lectures 12-14
Sections 1.6, 2.4, 2.1, 2.3
Magnetic Flux and Magnetic Fields
Biot-Savart Law
Ampere’s Law
Displacement Current
Faraday’s Law
Magnetism and Electricity
Andre Marie Ampere - 1820

Parallel Currents in Same Direction ATTRACT
Parallel Currents in Opposite Direction REPEL
Ampere’s observations

- The magnitude of $F$ is
  - proportional to the product of the currents AND to the product of their lengths
  - inversely proportional to the square of the distance
  - depends on the medium
- The direction of $F$ on current 1 is
  - perpendicular to $dI_1$
  - perpendicular to $dI_2 \times a_{21}$
- The forces $dF_1$ and $dF_2$ are not always equal and opposite
Magnetic Field of Bar Magnet

UNLIKE poles ATTRACTION
LIKE poles REPEL
(same as electric charges)
Magnetism and Electricity

Hans Oersted - 1821

Magnets line up in the presence of a nearby current
Magnetic Flux Lines

Magnetic flux lines form a VECTOR FIELD

Density of lines indicates MAGNITUDE

Direction = the way our compass would point

Unlike electric field lines which begin on positive charges and end on negative charges, magnetic flux lines **NEVER** begin or end
Magnetic Flux Lines

Units for $B$ are Tesla = Newt/(Amp-meter) or Webers/m$^2$ (Magnetic flux density)
Current = Moving Charge

What is CURRENT? CHARGES IN MOTION!!

\[ I \, d\vec{l} = q \, \vec{v} \]

So the current in a wire, \( I \), flowing across a magnetic field will feel a force...

\[ F_M = (I \, d\vec{l}) \times \vec{B} = q \, \vec{v} \times \vec{B} \]
Ampere’s Force Law

The force on current 1 due to current 2 depends on the magnetic flux density at 1 due to current 2.

\[ d\vec{F}_1 = I_1 d\vec{l}_1 \times d\vec{B}_1 \]
Biot-Savart Law for finding $\mathbf{B}$

$$d\mathbf{B}_{\text{at } 1} = \frac{\mu_o}{4\pi} \frac{I_2 d\mathbf{l}_2 \times \mathbf{\hat{a}}_R}{R^2}$$

Magnetic flux at point 1 due to current 2
Force between two wires

Combining these results,

\[ dF = (I_1 \, d\vec{l}_1) \times \left( \frac{\mu_0}{4\pi} \cdot \frac{I_2 \, d\vec{l}_2 \times \hat{a}_R}{R^2} \right) \]

Magnetic flux density caused by #2 at #1

\[ dF_{at \, 1} = (I_1 \, d\vec{l}_1) \times \vec{B} \]

To find the total force on wire 1, we add up (integrate) the contributions from segments \(d\vec{l}_2\) for every \(d\vec{l}_1\) (double integral)
Example: Find $\mathbf{B}$ for an Infinite Line of Current

Wire carrying current in $+z$ direction = $I$ (A)
Find $\mathbf{B}$ for any arbitrary point $P(r, \phi, z)$ in cylindrical coords
Patented 5-Step Program for Problem Solving

1. MAKE A **LARGE CLEAR DRAWING**
   a. Also draw cross-sections if the problem is in 3D
   b. Pick a coordinate system that is appropriate for the symmetry of the problem

2. Divide current distributions into tiny pieces

3. Find $d\mathbf{B}$ of one tiny piece

4. Use SYMMETRY to eliminate any components that cancel (i.e. add to ZERO)

5. INTEGRATE to add contribution of ALL the tiny pieces
Step 1 - draw a picture
Step 2 - divide the line into segments \( dl \)
Step 3 - find \( dB \) for one small segment

\[
Idl = Idz a_z = \]

\[
\vec{a}_z \quad \vec{a}_\phi \quad \vec{a}_R
\]

\[
P(r, \phi, 0)
\]

\[
dB = \frac{\mu_o}{4\pi} \frac{Idl \times \hat{a}_R}{R^2}
\]

\[
dB = \frac{\mu_o}{4\pi} \frac{Idz \sin(90 + \alpha)}{R^2} \hat{a}_\phi
\]

\[
dB = \frac{\mu_o}{4\pi} \frac{Idz r}{R^3} \hat{a}_\phi
\]
Step 4: Use symmetry to eliminate components
Step 5: Integrate over the whole object

\[ t \mathbf{B} = \frac{\mu_0 I}{4\pi r} \hat{a}_\phi \int_{-\pi/2}^{\pi/2} \cos(\alpha) \, d\alpha = \frac{\mu_0 I}{2\pi r} \hat{a}_\phi \]
Magnetic flux density ($\mathbf{B}$) around a line of current

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{a}_\phi$$

Right hand rule
Curl fingers around current

Practice sketching field lines:
“Static” Ampere’s Law

Let’s integrate $\mathbf{B}$ over any circular path centered on the wire:

$$\oint_{C} \mathbf{B} \cdot d\mathbf{l} = \oint_{\phi=0}^{2\pi} \left( \frac{\mu_0 I}{2\pi r} \hat{a}_\phi \right) \cdot (rd\phi \hat{a}_\phi)$$

$$= \oint_{\phi=0}^{2\pi} \left( \frac{\mu_0 I}{2\pi} \right)$$

$$= \mu_0 I$$

So...

$$\oint_{C} \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = I$$

for any radius circle
New Definition: Magnetic Field Intensity Vector

\[ \oint_C \vec{H} \cdot d\vec{l} = I_{\text{enclosed}} \]

\[ \vec{H} = \frac{\vec{B}}{\mu_0} \]

Units: (A/m)
Current Distributions

Line Current

\[ \vec{I} = \text{Amps}(A) \]

Surface Current

\[ \vec{J}_s = \frac{A}{m} \]

Volume Current

\[ \vec{J}_v = \frac{A}{m^2} \]
A coaxial cable has a solid center wire and an outer cylindrical shell wire. A uniform current density $J$ flows in the direction $-\mathbf{a}_x$ on the inner wire and the total current returns on the outer shell. In which region(s) will the H-field be constant?

(a) $r<a$
(b) $a<r<b$
(c) $r>b$
(d) both (b) and (c)
Static Ampere’s Law

Open Surface, where the Path forms a “Mouth”

\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}} = \iint_S \mathbf{J} \cdot d\mathbf{S} \]

In general, all the current entering the “mouth” will end up passing out of any surface that is formed around the mouth.
Example: Infinite Plane Sheet of Current

xz plane is an infinite sheet of current

Current density = $J_s = J_{s0}a_z (A/m)$

Solve for magnetic field $H$
Step 1 - draw a picture

\[ I_s = J_{s0} a_z dx \]
Step 2 - divide into segments

\[ x \]

\[ \text{dx} \]
Step 3 - find $d\mathbf{H}$ for one small segment

At every point $(x,y)$, only the $x$-component remains, why? For $y>0$, it is in -$a_x$ direction
Step 4 – use symmetry to eliminate components that are zero
Aha! Now that we know $H$ is along $a_x$, we can apply Ampere’s Law to a simple path.
And now for the math ...

\[ \mathbf{J}_s = \mathbf{J}_{s0} \mathbf{a}_z \]

\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = I \quad \Rightarrow \quad |\mathbf{H}| \cdot L + |\mathbf{H}| \cdot L = J_{s0} L \]

\[ |\mathbf{H}| = \frac{1}{2} J_{s0} \]

\[ \mathbf{H} = \frac{1}{2} J_{s0} (\pm \mathbf{a}_x) \]
General Formula for $\mathbf{H}$ due to infinite current sheet

The direction of current flow can be in ANY direction
The current sheet might not be on a coordinate plane

$$\mathbf{H} = \frac{1}{2} \mathbf{J}_s \times \hat{a}_n$$

Unit vector normal to the surface
Sample problem

- Infinite plane sheets of current lie in the $x=0$, $y=0$, and $z=0$ planes with uniform surface current densities $J_s a_z$, $2J_s a_x$, and $-J_s a_x$, respectively. Find $H$ at the points: (a) $(1,2,2)$, (b) $(2,-2,-1)$ and (c) $(-2,1,-2)$. 

Problem taken from D1.20 p58 in old book
Lecture 12 Summary

- Ampere’s Force Law
- Biot-Savart Law
- Infinite line of current
- Lorentz Force Equation
- Magnetic Field Intensity
- Next class
  - Ampere’s Law (Section 2.4)
Lecture 13
Section 2.4

Ampere’s Law
Displacement Current
“Static” Ampere’s Law

Using the Biot-Savart Law, we solved for the $\mathbf{B}$ field around a straight wire and expressed it as $\mathbf{H}$:

\[
\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{a}_\phi
\]

Magnetic flux density (Wb/m²)

\[
\mathbf{H} = \frac{I}{2\pi r} \hat{a}_\phi
\]

Magnetic field (A/m)
"Static" Ampere’s Law

We integrated $\mathbf{H}$ over any circular path centered on the wire:

\[
\oint_{C} \mathbf{H} \cdot d\mathbf{l} = \int_{\phi=0}^{2\pi} \left( \frac{I}{2\pi r} \mathbf{\hat{a}}_{\phi} \right) \cdot (r d\phi \mathbf{\hat{a}}_{\phi})
\]

\[
= \int_{\phi=0}^{2\pi} \frac{I}{2\pi} d\phi
\]

\[
= I
\]

So...

\[
\oint_{C} \mathbf{H} \cdot d\mathbf{l} = I
\]

for any radius circle
Ampere’s Law Physical Meaning

\[ \oint_C \vec{H} \cdot d\vec{l} = I_{\text{enclosed}} \]

Taking the integral of \( \vec{H} \) around a closed path equals the enclosed current.

\[ \oint_C \vec{H} \cdot d\vec{l} = \text{Magneto Motive Force (MMF)} \]

similar to:

\[ \oint_C \vec{E} \cdot d\vec{l} = \text{Electro Motive Force (EMF)} \]

Caution: MMF does no work!

\[ \vec{F}_M = q\vec{v} \times \vec{B} \] is \( \perp \) to \( d\vec{l} = \vec{v}dt \)
Static Ampere’s Law

According to this static law, all the current entering the “mouth” will end up passing out of any surface that is formed around the mouth. Is this true?
Problem with Static Ampere’s Law

Consider a simple capacitor
- No DC current goes through
- But, AC voltage results in current flow

How does AC current get through a capacitor if it is not conducted through by a wire?
Problem with Static Ampere’s Law

There is a second “way” to get current to flow

Somehow, a TIME-VARYING E field results in current flow
Problem with Static Ampere’s Law

Flux of E field lines crossing a surface, S

\[ \psi_E = \iint_S \varepsilon_0 \vec{E} \cdot d\vec{S} \]

Recall the Definition:
Electric Flux \( \psi_E \) Units: (C)
There are TWO sources of MMF:
1. Flow of charges due to current
2. Time-varying electric field

Called (by Maxwell) “Displacement Current”

\[
\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \iint_S \mathbf{D} \cdot d\mathbf{S}
\]

\[
\text{MMF (Amps)} = \text{“Regular” Current (Amps)} + \text{Displacement Current (Amps)}
\]
“New” Definition: Displacement Flux Density Vector

\[ \vec{D} = \varepsilon_0 \vec{E} \quad \text{“Displacement Flux Density Vector”} \]

Units:
\[ \vec{D} = \frac{\text{Charge}}{\text{Area}} = \frac{C}{m^2} \]

\[ \psi_E = \iint_S \varepsilon_0 \vec{E} \cdot d\vec{S} = \iint_S \vec{D} \cdot d\vec{S} \quad \text{Coul} \]

\[ \frac{d\psi_E}{dt} = \frac{d}{dt} \iint_S \vec{D} \cdot d\vec{S} \quad \text{Coul/sec (Amps)} \]
Displacement current from a time varying E-field

- Find the displacement current crossing an area $A=0.1\,\text{m}^2$ in the $xy$ plane from the $-z$ to $+z$ side for:

$$\mathbf{E} = E_0 \, t \, e^{-t^2} \mathbf{a}_z$$

From Discussion 2.8 (p 106) of old book
Ampere’s Law (Non static)

The Third Maxwell Equation

\[ \oint_C \vec{H} \cdot d\vec{l} = \iint_S \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iint_S \vec{D} \cdot d\vec{S} \]

MMF (Amps) = “Regular” Current (Amps) + Displacement Current (Amps)

\[ \nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt} \]

After using Stokes’ theorem to convert to differential form
Ampere’s Law Rules

1. Right Hand Rule:
   Choose direction of C so \( dS \) points OUT of the surface
2. Must use same surface when evaluating surface integrals for conduction current and displacement current
Displacement Current

- The displacement current bridges the gap in the capacitor plates

  - Regular current flowing into the CLOSED surface = displacement current flowing out

The MMFs cancel if we apply Ampere’s Law to two loops going in opposite directions:

\[- \oint_{S} \mathbf{J} \cdot d\mathbf{S} = \frac{d}{dt} \oint_{S} \mathbf{D} \cdot d\mathbf{S} \]
Displacement Current

- Current flow changes amount of charge \( I_{in} = \frac{dQ}{dt} \)
  - Since the charge changes, the electric flux out of the surface changes, i.e. a displacement current

\[
\bar{E}(t) = \frac{Q(t)}{4\pi \varepsilon_0 R^2} \hat{a}_R
\]

\[
\psi_E = \iint_S \varepsilon_0 \bar{E} \cdot d\bar{S} = \varepsilon_0 E(Surf \ Area)
\]

\[
= \varepsilon_0 \frac{Q(t)}{4\pi \varepsilon_0 R^2} (4\pi R^2) = Q(t)
\]

\[
I_d = \frac{d\psi_E}{dt} = \frac{dQ}{dt}
\]

\[
\therefore I_d = \frac{dQ}{dt} = I_{in} \quad \text{so displacement current out} = \text{regular current in}
\]
Displacement Current from time varying charge

- 3 point charges, $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ are at the corners of an equilateral triangle and connected by wires. Currents of I and 3I flow from $Q_1$ to $Q_2$ and $Q_3$ respectively. The displacement current emanating from a small surface surrounding $Q_2$ is $-2I$. Find: (a) the current flowing from $Q_2$ to $Q_3$ and (b) the displacement current from a small surface surrounding $Q_1$ and surrounding $Q_3$.
Challenge Question 1

- Current moves along a wire connecting two point charges. For which closed surface can the displacement current be non-zero?
Current moves along a wire connecting two point charges. For which infinite plane (open surface) can the displacement current be non-zero?

(a)  (b)  (c)
Lecture 13 Summary

- Ampere’s Circuit Law

- Next class
  - Faraday’s Law (Sections 2.1 and 2.3)
Lecture 14
Sections 2.1 and 2.3

Review of Line Integrals
Faraday’s Law
Work is a Line Integral

\[ W_{AB} = \sum_{j=1}^{n} \vec{F}_j \cdot \Delta \vec{l}_j = \int_{A}^{B} \vec{F} \cdot d\vec{l} \]
Conservative Forces

• Conservative means the work done by the force is independent of path
  – E.g. Gravity, Static Electric/Magnetic

• No work done along any closed loop

• Described by a potential energy
  – Energy conservation
    • Work done increases KE & decreases PE
    • Friction, Drag & Time Dependent Electric or Magnetic Forces are non-conservative

• Curl-free (non-rotational)
  – Field strength does not vary perpendicular to the field direction
The Force From a **Static EM Field** is Conservative

- **Definition of Voltage**

\[ V_B - V_A = \frac{W_{AB}}{q} = -\sum_{j=1}^{n} \mathbf{E}_j \cdot d\mathbf{l}_j \]

Voltage drop from \( B \) to \( A \) is equal to the work you need to do to move a unit charge from \( A \) to \( B \) against the electric field \( \mathbf{E} \).

- **In the limit** \( n \to \infty \),

\[ V_B - V_A = -\int_{A}^{B} \mathbf{E} \cdot d\mathbf{l} \]

= Line integral of \( \mathbf{E} \) from \( A \) to \( B \). Well defined since integral is independent of path.
Non-conservative Fields: Electromotive Force (emf)

*Electromotive Force (emf)*

\[ \text{emf} = \oint_C E \cdot dl \]

= Line integral of \( E \) around the closed path \( C \) (counter-clockwise).

EMF can be non-zero if EM field varies in time. EMF is a difference in potential that can give rise to an electric current. Think of it as a battery between \( A \) and \( B \).
Summary of Conservative Fields

• Conservative fields
  – Line integral around a closed path is ZERO
  – Gravity, static EM Field

• Non-conservative fields
  – Line integral around a closed path is NONZERO
  – Friction, time-varying EM Field
Faraday’s Law

When the magnetic flux enclosed by a loop of wire CHANGES WITH TIME, a current is produced in the loop.

The EMF in the loop is the NEGATIVE of the rate of change of the magnetic flux enclosed in the loop:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$$

Our third Maxwell Equation!!!

$$emf = -\frac{d\psi_B}{dt}$$
Lenz’s Law

• The direction of the induced EMF always OPPOSES the CHANGE in magnetic flux that produces it.

• It opposes the CHANGE in flux, not the flux itself!

– Explains why it is “-d/dt”
The curve C and surface S

• Outward magnetic flux thru the closed surface: \( S_1 \cup S_2 \) is zero
  - The flux out any open surface \( S_2 \) = minus the flux out \( S_1 \) = plus the flux into \( S_1 \) so all surfaces \( S \) bounded by \( C \) have the same flux
Faraday’s Law Rules

- Right Hand Rule
  - Right hand curls around C so thumb points in direction of dS

\[ \oint_{C} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_{S} \vec{B} \cdot d\vec{S} \]
Experiments for Faraday’s Law

[Diagram showing the concept of induced magnetic fields and Faraday’s Law]
Induced emf around rectangular loop in a time-varying $\mathbf{B}$ field

Rectangular wire loop
In the xz-plane

$$\mathbf{B} = B_0 \cos \omega t \hat{\mathbf{y}}$$

Steps:
1. Write down Faraday’s law
2. Write down expression for $dS$. DIRECTION!!
3. Perform dot product $\mathbf{B} \cdot dS$
4. Solve double integral over limits of the loop
5. Take time derivative of result. Put in “-” sign!
Faraday’s Law Rules

- EMF increases in proportion to the number of turns of wire
  - If the loop, C, contains N turns of wire, the EMF is multiplied by N

\[ EMF = -N \frac{d\psi_B}{dt} \]
Example Problem: Induced emf around closed path

\[ \vec{B} = B_0 (\sin \omega t \hat{a}_x - \cos \omega t \hat{a}_y) \text{ Wb/m}^2 \]

We can make the problem easier by solving for the flux going into the three surfaces on the coordinate planes.

Setting up the limits of integration here would be painful!

We can make the problem easier by solving for the flux going into the three surfaces on the coordinate planes.

From Discussion 2.5b p100 in old book.
Steps

1. Identify surfaces on the coordinate axes that are bounded by the same closed path C
2. Solve for $\int\int_{S} \vec{B} \cdot d\vec{S}$ separately for each surface
3. Add up contribution of each surface for the final result

$\vec{B} = B_0 (\sin \omega t \hat{a}_x - \cos \omega t \hat{a}_y)$
Solving for $\int\int_{S} \vec{B} \cdot d\vec{S}$ of each component surface

1. Write down expression for $dS$. PAY ATTENTION TO DIRECTION!
2. Perform dot product with $B$. Is the dot product zero for a particular surface?
3. Shortcut if $B$ is uniform over the surface. Then $\int\int_{S} \vec{B} \cdot d\vec{S} = $(Surface Area)$(B)$
4. Otherwise, perform double integral over the dimension limits of the surface.
Example Problem: Motional EMF

- A non-uniform static magnetic field given by $B = B_0/x \hat{a}_z$ exists in the region $x > 0$. A square loop with side length, $s$, and situated in the $xy$ plane ($x > 0$) moves in the $+a_x$ direction with speed $v$. Find the induced emf in the loop.

Adapted from Problem 2.11 (p 123) of old book
Challenge Question

A loop, radius $R$, centered at the origin, sits in the $xy$ plane in the presence of a uniform field $\mathbf{B} = B_0 \mathbf{a}_z$ ($B_0 > 0$). If the loop radius begins to decrease, which direction is the induced emf?

(a) $\mathbf{a}_\phi$
(b) $-\mathbf{a}_\phi$
(c) $\mathbf{a}_z$
(d) cannot be determined
Comparing Faraday and Ampere’s Law

• Faraday’s Law
  – Time varying magnetic fields generate emf (voltage)

• Ampere’s Law
  – Time varying electric fields generate magnetic fields
  – Electric currents generate magnetic fields (Ampere’s static law)
**EMF and MMF**

\[ \oint_C \mathbf{E} \cdot d\mathbf{l} = EMF \]

**EMF** = Volts

\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = MMF \]

**MMF** = Amps/\(m\)

\[ B(t) \]

\[ E(t) \]

\[ E = Volts/m \]

\[ H = Amps/m \]
Lecture 14 Summary

- Faraday’s Law: _________________
  - The generated emf opposes ____________________ and for a loop with N turns is ___ times larger
  - The direction for C and dS determined by right hand rule

- Next class
  - Magnetic Vector Potential (Section 10.6 – just pp. 406-407)
  - Inductance (Section 6.3)
ECE 329
Lectures 15-17
Sections 10.6, 6.3, 2.5, 5.5, 5.2

Magnetic Vector Potential
Inductance
Conservation of Charge (Continuity)
Boundary Conditions
Magnetic Materials
Magnetic Potentials

\[ \nabla \cdot \vec{B} = 0 \quad \text{Gauss’ Law} \]

If the divergence is zero, then \( \vec{B} \) can be written as the curl of a vector (not obvious \( \vec{A} \) should exist, but it does)

**New Definition: Magnetic Potential Vector**

\[ \vec{B} = \nabla \times \vec{A} \]

Oddly familiar:

\[ \nabla \cdot (\nabla \times \vec{A}) = 0 \]
Faraday’s Law

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} (\nabla \times \vec{A}) \]

\[ \nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{A}}{\partial t} \]

\[ \nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0 \]

If curl is zero, then can be written as the gradient of a scalar

Oddly familiar:

\[ \nabla \times \left( \nabla \Phi \right) = 0 \]
Electric and Magnetic Potentials

\[ \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \]

Electric Potential  Magnetic Potential

\[ \vec{B} = \nabla \times \vec{A} \]

With these definitions, we automatically satisfy Faraday’s Law & Gauss’ Magnetic Law

Instead of 6 unknowns: \((E_x, E_y, E_z) \) & \((B_x, B_y, B_z) \)

we have 4: \((A_x, A_y, A_z) \) and \(V \)
The magnetic potential $\mathbf{A}$ and its relations to $\mathbf{E}$ and $\mathbf{B}$

$$\text{EMF} = -\frac{d\psi_B}{dt} = -\frac{d}{dt} \int S \mathbf{B} \cdot d\mathbf{S}$$

$$= -\frac{d}{dt} \int S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = -\frac{d}{dt} \oint_c \mathbf{A} \cdot d\mathbf{l} = -\oint_c \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l}$$

assuming the loop geometry is constant

$$\text{EMF} = \oint_c \mathbf{E} \cdot d\mathbf{l} = \oint_c (-\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}) \cdot d\mathbf{l} = -\oint_c \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l}$$

why can we drop gradient of $\Phi$?
Gauss’ Electric Law for Potentials

\[ \nabla \mathbf{D} = \rho \]

\[ \nabla \mathbf{E} = \rho \]

Assuming \( \epsilon \) is constant

\[ \nabla \cdot \left( -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\rho}{\epsilon} \]

1 equation
Ampere’s Law for Potentials

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

\[ \nabla \times \frac{\vec{B}}{\mu} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \]

\[ \nabla \times \vec{B} = \mu \vec{J} + \mu \varepsilon \frac{\partial \vec{E}}{\partial t} \]

\[ \nabla \times \vec{B} - \mu \varepsilon \frac{\partial \vec{E}}{\partial t} = \mu \vec{J} \]

\[ \nabla \times \left( \nabla \times \vec{A} \right) - \mu \varepsilon \frac{\partial}{\partial t} \left( -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \right) = \mu \vec{J} \]

3 equations
Special Case: Static Fields

Gauss

\[ \nabla \cdot \left( -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\varepsilon} \]

\[ \nabla \cdot (\nabla \Phi) = -\frac{\rho}{\varepsilon} \]

Ampere

\[ \nabla \times \left( \nabla \times \vec{A} \right) - \mu \varepsilon \frac{\partial}{\partial t} \left( -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \right) = \mu \vec{J} \]

\[ \nabla \times (\nabla \times \vec{A}) = \mu \vec{J} \]

We studied this one already (Poisson):

Relationship between a charge distribution and the potential field
Suppose \( \mathbf{E} \) and \( \mathbf{B} \) can be represented by the scalar and vector potentials: \( \Phi \) and \( \mathbf{A} \).

\[
\begin{align*}
\vec{E} &= -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \\
\vec{B} &= \nabla \times \vec{A}
\end{align*}
\]

Which of the following is true:

(a) \( \Phi \) and \( \mathbf{A} \) are uniquely defined
(b) \( \vec{A}' = \vec{A} + \nabla \lambda \), \( \Phi' = \Phi - \frac{\partial \lambda}{\partial t} \) also represents \( \mathbf{E} \) and \( \mathbf{B} \)
(c) The divergence of \( \mathbf{A} \) must be 0
(d) The laplacian of \( \Phi \) must be \(-\rho/\varepsilon\)
Lecture 15a Summary

- Since \( \text{div} \, \text{curl} \, \mathbf{A} = 0 \) and \( \text{curl} \, \text{grad} \, f = 0 \), satisfy Faraday’s and Gauss’ Mag. Laws
- For static fields, Poisson’s equation is

\[
\vec{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}
\]

\[
\vec{B} = \nabla \times \mathbf{A}
\]

satisfy Faraday’s and Gauss’ Mag. Laws
- For static fields, Poisson’s equation is

\[
\nabla^2 \Phi = -\frac{\rho}{\varepsilon}
\]
Inductance
Example: Find $B$ for an infinitely long solenoid with $n$ turns per unit length.
Inductance of a Coax Cable

Now, instead of applying a voltage across the inner and outer conductor, a current, $I$, flows down the length of the outer conductor and returns in the opposite direction through the inner conductor.

Results in magnetic field

$$H_\phi = \frac{I}{2\pi r}$$

in between the coax.
Inductance of Coaxial Cable

\[ \vec{B} = \mu \vec{H} = \frac{\mu I}{2\pi r} \hat{a}_\phi \]

\[ \psi = \int B \cdot dS = \int_0^b \int_0^z \left( \frac{\mu I}{2\pi r} \right) (drdz) \]

\[ \psi = \frac{\mu I z}{2\pi} \ln(b/a) \]
Inductance

\[ L = \frac{\psi}{I} \quad \text{Units: Henry (H)} \]

\[ L = \frac{\mu z}{2\pi} \ln\left(\frac{b}{a}\right) \]

\[ L = \frac{L}{z} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{Inductance/Length (H/m)} \]
Induced emf

Faraday’s Law

\[ \text{emf} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\psi_B}{dt} \]

\[ \text{emf} = -\frac{d(LI)}{dt} = -L \frac{dI}{dt} \]

The inductance of the wire creates an emf that opposes rapid changes in the current

Assuming \( \frac{dI}{dt} > 0 \)
Steps for Finding Inductance

- Find $H(r)$
- Find $B \quad \vec{B} = \mu \vec{H}$
- Find Magnetic flux by integrating $\psi = \int B \cdot dS$
- Inductance $L = \frac{\psi}{I}$
- Inductance/Length
Relationships between Capacitance, Conductance & Inductance

Notice in the above examples,

\[ C = \varepsilon \cdot \text{GeometricalFactor} \]
\[ \mathcal{L} = \mu / \text{GeometricalFactor} \]
\[ \mathcal{G} = \sigma \cdot \text{GeometricalFactor} \]

This is true in general for any pair of infinitely long, parallel perfect conductors and so we have the following:

\[ \mathcal{L}C = \mu \varepsilon \quad \mathcal{G} / C = \sigma / \varepsilon \]

If you know one (L, C, or G), you can find the other two from the material parameters.
Challenge Question: Parallel plate capacitor

- For the parallel plate capacitor, we found

\[ C = \frac{\varepsilon A}{d} \Rightarrow \frac{1}{C} = \frac{d}{\varepsilon A} \]

If the plate separation \( d \) increases, which is true:

(a) \( G \) and \( L \) will both increase

(b) \( C \) and \( G \) will both increase

(c) \( L \) will increase, but \( C \) will decrease

(d) \( G \) will increase, but \( L \) will decrease
There now is also $H$ field inside the inner wire that will also contribute to inductance.

**Last Example**

**New Example**

\[
L = \frac{\psi}{I} \quad \text{and} \quad L = \frac{\Lambda}{I} \quad \text{“Flux Linkage”}
\]
(Optional) Flux Linkage

This flux line is “linked” to a small amount of current

This flux line is “linked” to a large amount of current

AND, since flux $\psi_B$ is dependent on $r$

Flux here is LOW

Flux here is HIGH
(Optional) Definition of Flux Linkage

\[ \Lambda = (\text{Flux Magnitude}) \cdot (\text{Fraction of Current Linked to the Flux Line}) \]

Accounts for two factors:
1. The magnitude of the flux line
2. The amount of current linked to a flux line

The self inductance is then defined as:

\[ L_{\text{int}} = \frac{\Lambda}{I_{\text{total}}} = \int_S \frac{d\Lambda}{I_{\text{total}}} = \int_S \frac{N \cdot d\psi}{I_{\text{total}}} = \frac{1}{I_{\text{total}}} \int_S I_{\text{linked}} d\psi \]

Fraction of linked current:

\[ N = \frac{I_{\text{linked}}}{I_{\text{total}}} \]
(Optional) Example: Self-Inductance of a Wire

Step 1: Ampere’s Law inside the Wire

\[ \oint \vec{H} \cdot dl = I_{\text{enclosed}} \]

\[ \oint \frac{\vec{B}}{\mu} \cdot dl = I_{\text{enclosed}} \]

\[ \frac{1}{\mu} \int_{\phi=0}^{2\pi} B_\phi(rd\phi) = I_{\text{enclosed}} \]

\[ \frac{1}{\mu} B_\phi(2\pi r) = I_{\text{enclosed}} \]

\[ \frac{1}{\mu} B_\phi(2\pi r) = J_s(Area_{\text{Enclosed}}) \]

\[ \frac{1}{\mu} B_\phi(2\pi r) = \left( \frac{I}{\pi r_0^2} \right) (\pi r^2) \quad (r < r_0) \]
(Optional) Example, continued

\[ \frac{1}{\mu} B_\phi(2\pi r) = \left( \frac{I}{\pi r_0^2} \right) (\pi r^2) \quad (r < r_0) \]

\[ B_\phi = \frac{\mu I}{2\pi r_0^2} \quad \text{(Wb/m}^2\text{)} \]

**Step 2: Determine One Differential Piece of Flux**

\[ d\psi_B = B(r) \cdot dr \cdot l \quad \text{(Wb)} \]

\[ d\psi_B = \frac{\mu I}{2\pi r_0^2} \cdot dr \cdot l \]
(Optional) Example, continued

Step 3: Differential piece of flux linkage

\[ d\Lambda = d\psi_B \cdot N \]

\[ d\Lambda = d\psi_B \left( \frac{\pi r^2}{\pi r_0^2} \right) \]

\[ d\Lambda = \frac{\mu I}{2\pi} \frac{r^3}{r_0^4} \cdot dr \cdot l \]

N = Fraction of current inside radius \( r \)

Step 4: Total flux linkage

\[ \Lambda = \frac{\mu I \cdot l}{2\pi r_0^4} \int_0^{r_0} r^3 \, dr = \frac{\mu I \cdot l}{8\pi} \]
(Optional) Example, continued

Step 5: Internal inductance

\[ L = \frac{\Lambda}{I} = \frac{\mu l}{8\pi} \]  

(units: Henry (H))

Internal inductance per unit length of the wire:

\[ \frac{L}{l} = \frac{\mu}{8\pi} \]  

(units: H/m)
Lecture 15b Summary

• Capacitance \( C = \frac{Q}{V_0} \)
• Conductance \( G = \frac{|I_c|}{V_0} \)
• Inductance \( L = \frac{\psi}{I} \)
• Relationships \( \mathcal{L}C = \mu \varepsilon \) \( \mathcal{G}/\mathcal{C} = \sigma / \varepsilon \)
• Self-inductance \( L_{\text{int}} = \int_{S} \frac{Nd\psi}{I_{\text{total}}} = \int_{S} \frac{I_{\text{linked}}^2}{I_{\text{total}}^2} d\psi \)
• Next Up
  – Conservation of Charge
  – Boundary Conditions
Lecture 16
Sections 2.5 and 5.5

Conservation of Charge

Review of Maxwell’s Equations in Integral Form

Boundary Conditions
Conservation of Charge

Say we have a container that can accumulate charge.

Start pouring charges into the container.

Flow of charges is a current, \( I \), in Amps.

If the charges don’t leave the container, the charge inside the container increases.

Current flow IN = Charge INCREASE
Current flow OUT = Charge DECREASE
Conservation of Charge

In general, we can pour charges in from more than one direction, or take some out from other parts of the container.

\[
\text{Net Rate of Current flow OUT} = \text{Net Rate of Charge DECREASE}
\]

\[
\iint \vec{J} \cdot d\vec{S} = -\frac{dQ_{\text{enc}}}{dt} = -\frac{d}{dt} \iiint \rho dV
\]

or,

\[
\nabla \cdot \vec{J} = -\frac{d\rho}{dt}
\]

using the divergence theorem
Conservation of Charge

Can be derived by combining two of Maxwell’s equations

\[ 0 = \oint C_1 \vec{H} \cdot d\vec{l} + \oint C_2 \vec{H} \cdot d\vec{l} \]

\[ = \iint_{S_1} \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iiint_{S_1} \vec{D} \cdot d\vec{S} + \iint_{S_2} \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iiint_{S_2} \vec{D} \cdot d\vec{S} \]

\[ = \iint_{S} \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iiint_{S} \vec{D} \cdot d\vec{S} \]

\[ = \iiint_{V} \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iiint_{V} \rho dV \]

\[ \therefore \iiint_{S} \vec{J} \cdot d\vec{S} = -\frac{d}{dt} \iiint_{V} \rho dV = -\frac{dQ_{enc}}{dt} \]
Conservation of Charge

Differential equation derivation is much faster!

\[ 0 = \nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \left( J + \frac{\partial \vec{D}}{\partial t} \right) = \nabla \cdot J + \frac{\partial (\nabla \cdot \vec{D})}{\partial t} = \nabla \cdot J + \frac{\partial \rho}{\partial t} \]

\[ \therefore \nabla \cdot J = -\frac{\partial \rho}{\partial t} \]
Application of multiple Maxwell’s Equations

- Current I flows from a point charge Q(t) at the origin along the z-axis off to infinity. Find the counterclockwise MMF for a circular path of radius a in the xy plane centered at the origin. Hint: consider a sphere and a hemisphere.

From Example 2.5 (p 111) of old book.
Application of charge conservation

• For $\mathbf{J} = \langle x, y, z \rangle$, find the rate of decrease of charge contained in the unit cube: corner vertices $(0,0,0)$ and $(1,1,1)$. From Problem 2.24a (p 126) of old book
$\vec{E} = \frac{Q}{4\pi \varepsilon_0 R^2} \hat{a}_R$

$\vec{E} = \frac{\rho_l}{2\pi \varepsilon_0 r} \hat{a}_r$

$\vec{E} = \frac{\rho_{so}}{2\varepsilon_0} (\pm \hat{a}_z)$

$\vec{B}_\phi = \frac{\mu_0 I}{2\pi R}$

$B = \frac{\mu_0}{2} \vec{J}_s \times \hat{a}_n^{35}$
Lecture 16a Summary

- Maxwell’s Equations
  - Gauss Magnetic: _______________
  - Gauss Electric: _______________
  - Faraday: _______________
  - Ampere + Maxwell: ____________________
Lecture 16b
Sections 5.5

Boundary Conditions
Maxwell’s Eqns - Integral form

\[ \oint_{C} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_{S} \vec{B} \cdot d\vec{S} \]

\[ \oint_{C} \vec{H} \cdot d\vec{l} = \iint_{S} \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iint_{S} \vec{D} \cdot d\vec{S} \]

\[ \oiint_{S} \vec{B} \cdot d\vec{S} = 0 \]

\[ \oiint_{S} \vec{D} \cdot d\vec{S} = \iiint_{V} \rho dV \]

They are valid for ALL closed paths and closed surfaces, EVEN WHEN THEY SPAN A BOUNDARY BETWEEN TWO MATERIALS
Closed Path Through a Boundary

Medium 1 (above)
\( \sigma_1, \varepsilon_1, \mu_1 \)

Medium 2 (below)
\( \sigma_2, \varepsilon_2, \mu_2 \)

Closed path: abcd
Apply Faraday’s Law and Ampere’s Law to the closed path
Normal Vectors

\( \hat{a}_n \) Vector NORMAL to the boundary. Points INTO medium 1

\( \hat{a}_S \) Vector normal to the path, TANGENT to the interface. Use right hand rule for path to define direction
Faraday’s Law at Boundary

\[ \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iiint_S \vec{B} \cdot d\vec{S} = 0 \]

Medium 1 (above) \( \sigma_1, \varepsilon_1, \mu_1 \)

Medium 2 (below) \( \sigma_2, \varepsilon_2, \mu_2 \)

Take limit as \( ad \) and \( bc \) go to zero
Consider remaining \( E_1 \) and \( E_2 \) TANGENT TO SURFACE

\[ E_1 = E_2 \text{ i.e. } E_t \text{ is continuous} \]
The closed path can enclose surface current

Example:
1. Current on surface of a conductor
Ampere’s Law at Boundary

\[ \oint_C \vec{H} \cdot d\vec{l} = \iint_S \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iint_S \vec{D} \cdot d\vec{S} = I_{\text{enclosed}} \]

Medium 1 (above) \( \sigma_1, \varepsilon_1, \mu_1 \)

Medium 2 (below) \( \sigma_2, \varepsilon_2, \mu_2 \)

Take limit as \( ad \) and \( bc \) go to zero

Consider remaining \( H_1 \) and \( H_2 \) TANGENT TO SURFACE

\[ H_1 - H_2 = J_s \quad \text{i.e.} \quad H_t \text{ is discontinuous because of } J_s \]

\[ \left( \vec{H}_1 - \vec{H}_2 \right)_t = \vec{J}_s \times \hat{a}_n \]

Right Hand Triangle

\[ \hat{a}_n \times \left( \vec{H}_1 - \vec{H}_2 \right)_t = \vec{J}_s \]

Note: we know nothing about \( H_n \)
Closed surface through the volume

Medium 1 (above) \( \sigma_1, \varepsilon_1, \mu_1 \)

Medium 2 (below) \( \sigma_2, \varepsilon_2, \mu_2 \)
The closed volume can enclose surface charges

Example
1. Free charges on the surface of a conductor
Gauss’ Law for D at Boundary

\[ \int_{S} \vec{D} \cdot d\vec{S} = \int_{V} \rho dV \]

Take limit as \( a, b, c, \) and \( d \) go to zero

Consider \( D_1 \) and \( D_2 \) NORMAL TO SURFACE

\[ D_1 - D_2 = \rho \] i.e. \( D_n \) is discontinuous because of \( \rho_s \)
Finding surface charge

- Given \( \mathbf{D} \) on the surface of a perfect conductor, find the surface charge, \( \rho \), at that point (Hint: what is the vector \( \mathbf{a}_n \) for each surface):
  
  (a) \( \mathbf{D} = D_0 <1, -2, 2> \) pointing away from surface  
  (b) \( \mathbf{D} = D_0 <1, 0, \sqrt{3}> \) pointing towards surface  
  (c) \( \mathbf{D} = D_0 <0.8, 0, 0.6> \) pointing away

From D4.13 (p262) of old book
Gauss’ Law for B at Boundary

\[ \iiint \vec{B} \cdot d\vec{S} = 0 \]

Take limit as \( ae, bf, cg, \) and \( dh \) go to zero

Consider \( B_1 \) and \( B_2 \) NORMAL TO SURFACE

\[ B_1 = B_2 \quad \text{i.e.} \quad B_n \text{ is continuous} \]
Challenge Question: Boundary conditions

• Which of the following are realizable as the field outside a perfect conductor

(a) 1, 2, and 3 can be either $E$ or $H$
(b) 1 can be either $E$ or $H$, but 2 is $E$, 3 is $H$
(c) 1 can be either $E$ or $H$, but 2 is $H$, 3 is $E$
(d) 1 can be neither $E$ nor $H$, but 2 is $E$, 3 is $H$
(e) 1 can be neither $E$ nor $H$, but 2 is $H$, 3 is $E$
Remember this drawing!!

\[ \vec{J}_s = \hat{a}_n \times (\vec{H}_1 - \vec{H}_2) \]

- \( \vec{E}_{t1} = \vec{E}_{t2} \)
- \( (\vec{H}_1 - \vec{H}_2)_t = \vec{J}_s \times \hat{a}_n \)
- \( D_{n1} - D_{n2} = \rho_s \)
- \( B_{n1} = B_{n2} \)
Boundary between Two Perfect Dielectrics

\[ \sigma = 0 \quad J_s = \sigma E = 0 \quad \rho_s = 0 \]
### Surface of Perfect Conductor

<table>
<thead>
<tr>
<th>$E_{\text{Cond}} = 0$</th>
<th>$H_{\text{Cond}} = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{t1} = 0$</td>
<td>$H_{t1}$</td>
</tr>
<tr>
<td>$J_s \times H_{t1} = J_s$</td>
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<tr>
<td>$\hat{a}<em>n \times \vec{H}</em>{1t} = \vec{J}_s$</td>
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<tr>
<td>$\vec{H}_{1t} = \vec{J}_s \times \hat{a}_n$</td>
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</tr>
<tr>
<td>$D_{n1}$</td>
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</tr>
<tr>
<td>$\rho_s$</td>
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</tr>
<tr>
<td>$B_{n1} = 0$</td>
<td></td>
</tr>
<tr>
<td>$D_{n2} = 0$</td>
<td></td>
</tr>
<tr>
<td>$B_{n2} = 0$</td>
<td></td>
</tr>
<tr>
<td>Perf Cond</td>
<td></td>
</tr>
<tr>
<td>Medium 1</td>
<td></td>
</tr>
</tbody>
</table>

$\hat{a}_n$ is the normal vector to the surface.
(Time permitting) BCs for a rectangular cavity resonator

- The region $0<x<a$, $0<y<b$, $0<z<d$ is a perfect dielectric $\varepsilon=4\varepsilon_0$ and the boundary is a perfect conductor on all 6 sides. Inside the resonator, the fields are:

$$\vec{E} = E_0 \sin \frac{\pi x}{a} \sin \frac{\pi z}{d} \cos \omega t \hat{a}_y$$

$$\vec{H} = H_{01} \sin \frac{\pi x}{a} \cos \frac{\pi z}{d} \sin \omega t \hat{a}_x - H_{02} \cos \frac{\pi x}{a} \sin \frac{\pi z}{d} \sin \omega t \hat{a}_z$$

Find $\rho_s$ and $\mathbf{J}_s$ on all 6 walls.
Lecture 16b Summary

- **Never** use the differential form of Maxwell’s equations at a boundary – only use integral form.

\[ \hat{J}_s = \hat{a}_n \times (\vec{H}_1 - \vec{H}_2) \]

\[ (\vec{H}_1 - \vec{H}_2) \times \hat{a}_n = \vec{J}_s \times \hat{a}_n \]
ECE 329
Lecture 17
Section 5.2

Magnetic Materials
Magnetic Moments at the atomic scale

Internal magnetic fields are produced by electrons orbiting the nucleus, $\mathbf{L}$, or by the internal spin of electrons, $\mathbf{S}$. The atom has a magnetic dipole moment, $\mathbf{m}$.
Net magnetic moment

A volume of material contains many magnetic moments. They might be randomly oriented.
External $\mathbf{B}$-field can rotate Magnetic Moments or change Orbital Velocity

\[ \mathbf{F} = I \mathbf{d} \times \mathbf{B}_0 \]

Torque
\[ T = \mathbf{m} \times \mathbf{B} \]

Moment $\mathbf{m}$

Area $A$

Nucleus of atom

Electron

Current $I$
So after momentarily applying an external field, they can get magnetized and are mostly aligned in one direction. The internal field can keep them aligned to each other.
Magnetization Vector

\[ \vec{M} = N \vec{m} \]

Magnetic dipole moment per unit volume

\( N = \# \) atoms per unit volume

\( \vec{m} = \) average dipole moment per molecule

Units for \( \vec{M} \): \((1/m^3) \cdot (A*m^2) = A/m\)

\[ \vec{B}_{\text{int}} = \mu_0 \vec{M} \]

is the magnetic flux per unit area from the dipoles
Magnetic Susceptibility

B INSIDE the material is a function of how strong and how well-aligned all the magnetic moments are

The INTERNAL magnetic flux is INDUCED by the application of an EXTERNAL magnetic field
Total flux = Applied + Secondary

$$\vec{B}_{total} = \mu_0 (\vec{H}_{ext} + \vec{M})$$

Some materials are more easily “magnetized” than others

$$\vec{M} = \chi_m \vec{H}_{ext}$$

Definition of magnetic susceptibility (has no units)
Relative Magnetic Permeability

\[ \vec{B}_{total} = \mu_0 (\vec{H}_{ext} + \vec{M}) \]
\[ \vec{M} = \chi_m \vec{H}_{ext} \]

\[ \vec{B}_{total} = \mu_0 (1 + \chi_m) \vec{H}_{ext} \]
\[ \vec{B} = \mu \vec{H} \]
\[ \mu = \mu_0 (1 + \chi_m) = \mu_0 \mu_r \]
\[ \mu_r = 1 + \chi_m \]

\[ \vec{D} = \varepsilon_0 \vec{E}_{tot} + \vec{P} \]
\[ \vec{P} = \varepsilon_0 \chi_e \vec{E}_{tot} \]

\[ \vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E}_{tot} \]
\[ \vec{D} = \varepsilon \vec{E}_{tot} \]
\[ \varepsilon = \varepsilon_0 (1 + \chi_e) = \varepsilon_0 \varepsilon_r \]
\[ \varepsilon_r = 1 + \chi_e \]
Diamagnetic Materials ($\chi_m<0$)

- With $H_{\text{ext}}$, the electron orbital speed changes depending on the relative orientation of $v$ and $H_{\text{ext}}$
  - Equivalent to a weak magnetic dipole that OPPOSES $H_{\text{ext}}$
- Magnetic susceptibility is a NEGATIVE number

Examples:
- Copper $\chi_m = -0.94 \times 10^{-5}$
- Lead $\chi_m = -1.70 \times 10^{-5}$
- Water $\chi_m = -0.88 \times 10^{-5}$
**Paramagnetic Materials \((\chi_m > 0)\)**

**Positive susceptibility**

- With no \(H_{\text{ext}}\), domains of orbital electrons and spinning electrons exist
- However, the domains are physically oriented in random directions (as a function of time), so overall \(B_{\text{int}} = 0\)
- With \(H_{\text{ext}}\), domains reorient themselves to generate \(B_{\text{int}}\) that ALIGNS WITH \(H_{\text{ext}}\)
- Magnetic susceptibility is a POSITIVE number

**Examples:**

- Platinum \(\chi_m = +2.90 \times 10^{-5}\)
- Aluminum \(\chi_m = +2.10 \times 10^{-5}\)
- Liquid Oxygen \(\chi_m = +3.50 \times 10^{-5}\)
Ferromagnetic Materials ($\chi_m >> 1$)

- Very high susceptibility
- Microscopic “domains” that have strongly oriented magnetic dipoles
- Direction of magnetic dipole differs from one domain to another
- Under an externally applied H field, the domains can orient coherently (i.e. in the same direction)
- Using a strong enough applied field, the domain orientation can become permanent
<table>
<thead>
<tr>
<th>Material</th>
<th>Magnetic Type</th>
<th>$\mu_r = 1 + \chi_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bismuth</td>
<td>Diamagnetic</td>
<td>0.99983</td>
</tr>
<tr>
<td>Silver</td>
<td>Diamagnetic</td>
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</tr>
<tr>
<td>Lead</td>
<td>Diamagnetic</td>
<td>0.99993</td>
</tr>
<tr>
<td>Copper</td>
<td>Diamagnetic</td>
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<td>Water</td>
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<td>0.999991</td>
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<tr>
<td>Vacuum</td>
<td>Nonmagnetic</td>
<td>1†</td>
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<tr>
<td>Air</td>
<td>Paramagnetic</td>
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</tr>
<tr>
<td>Aluminum</td>
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<td>1.00002</td>
</tr>
<tr>
<td>Palladium</td>
<td>Paramagnetic</td>
<td>1.0008</td>
</tr>
<tr>
<td>2-81 Permalloy powder (2 Mo, 81 Ni)‡</td>
<td>Ferromagnetic</td>
<td>130</td>
</tr>
<tr>
<td>Cobalt</td>
<td>Ferromagnetic</td>
<td>250</td>
</tr>
<tr>
<td>Nickel</td>
<td>Ferromagnetic</td>
<td>600</td>
</tr>
<tr>
<td>Ferroxcube 3 (Mn-An-ferrite powder)</td>
<td>Ferromagnetic</td>
<td>1,500</td>
</tr>
<tr>
<td>Mild steel (0.2 C)</td>
<td>Ferromagnetic</td>
<td>2,000</td>
</tr>
<tr>
<td>Iron (0.2 impurity)</td>
<td>Ferromagnetic</td>
<td>5,000</td>
</tr>
<tr>
<td>Silicon Iron (4 Si)</td>
<td>Ferromagnetic</td>
<td>7,000</td>
</tr>
<tr>
<td>78 Permalloy (78.5 Ni)</td>
<td>Ferromagnetic</td>
<td>100,000</td>
</tr>
<tr>
<td>Mumetal (75 Ni, 5 Cu, 2 Cr)</td>
<td>Ferromagnetic</td>
<td>100,000</td>
</tr>
<tr>
<td>Purified iron (0.05 impurity)</td>
<td>Ferromagnetic</td>
<td>200,000</td>
</tr>
<tr>
<td>Superalloy (5 Mo, 79 Ni)§</td>
<td>Ferromagnetic</td>
<td>1,000,000</td>
</tr>
</tbody>
</table>

†By definition.
‡Percentage composition. Remainder is iron and impurities.
§Used in transformer applications with continuous tape-wound (gapless) cores.
Magnetization

- An infinite plane ferromagnetic slab ($\mu_r=100$) lies between two infinite plane sheets of uniform current density of $\mathbf{J}=\pm 0.1 \mathbf{a}_y \text{A/m}$. Find $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{M}$ inside the slab and compare to if the slab were non-magnetic ($\mu_r=1$).

Hint:

$\mathbf{H} = \frac{\mathbf{J}}{2} \mathbf{\hat{a}}_n = -\frac{\mathbf{J}}{2} \mathbf{\hat{a}}_x$

$\mathbf{H} = \frac{\mathbf{J}}{2} \mathbf{\hat{a}}_n = \frac{\mathbf{J}}{2} \mathbf{\hat{a}}_x$

From D4.6 (p238) of old book
Magnetization Current

Due to the spatial variation of magnetic dipole moments

\[ M_y(z) = \frac{m_y}{Volume} = \frac{IA}{Volume} = \frac{I_{1x} (dx dz)}{dx dy dz} = \frac{I_{1x}}{dy} \]

\[ M_y(z + dz) = -\frac{I_{2x}}{dy} \]

\[ I_{tot} = I_{1x} + I_{2x} = (M_y(z) - M_y(z + dz)) dy \]

\[ J_M = \frac{I_{tot}}{dy dz} = -\frac{\partial M_y}{\partial z} \hat{a}_x \]

\[ \vec{J}_M = \nabla \times \vec{M} \]
Ampere’s Law in Mag. Material

\[ \nabla \times \left( \frac{\vec{B}}{\mu_0} \right) = \vec{J} + \vec{J}_M + \frac{\partial \vec{D}}{\partial t} \]

In magnetic medium, we need to include \( J_M \) with the total current

\[ \nabla \times \left( \frac{\vec{B}}{\mu_0} \right) = \vec{J} + \nabla \times \vec{M} + \frac{\partial \vec{D}}{\partial t} \]

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

\[ H \equiv \frac{\vec{B}}{\mu_0} - \vec{M} \]

\[ \vec{B} = \mu_0 (\vec{H} + \vec{M}) \]
Connection of Concepts for Electrodynamics

\[ \Phi \text{ and } \vec{A} \]

Scalar and vector potential

- Electric field: \( \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \)
- Magnetic flux density: \( \vec{B} = \mu \vec{H} = \mu_0 (\vec{H} + \vec{M}) \)
- Magnetic field: \( \vec{H} \)
- Electric displacement: \( \vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P} \)
- Magnetic induction: \( \vec{B} = \nabla \times \vec{A} \)

Polarization (material response):

- \( \vec{P} = \varepsilon_0 \chi_e \vec{E} \)

Magnetization (material response):

- \( \vec{M} = \vec{B} \left( \frac{\mu - \mu_0}{\mu \mu_0} \right) \)

Integral

- \( \nabla \cdot \vec{D} = \rho_{\text{vol}} \)
- \( \vec{D}_{1n} - \vec{D}_{2n} = \rho_s \)

Derivative

- Current density:
  \[
  \vec{J} = \vec{A}_n \times (\vec{H}_1 - \vec{H}_2)_t
  \]
- Charge density:
  \[
  \rho = \int \rho_s dS \quad \text{or} \quad \rho = \int \rho_{\text{vol}} dV
  \]
- Current:
  \[
  I = \int J_s (x) dx \quad \text{or} \quad I = \int \int \vec{J} \cdot d\vec{S}
  \]
- Charge:
  \[
  Q = \rho_s A \quad \text{or} \quad Q = \rho_{\text{vol}} V
  \]
3 types of materials

Conductors

Free electrons

\[ E = 0 \text{ inside} \]
\[ \rho = 0 \text{ inside} \]
\[ \rho = \rho_s \text{ only surface charge} \]
\[ V \text{ is same throughout} \]
\[ E_{\text{outside}} \text{ is } \downarrow \text{ to surface} \]

Dielectrics

Polarized atoms/molecules

Bound electrons

\[ E \neq 0 \text{ inside but it is reduced} \]
\[ E_{\text{tot}} = E_a + E_s \]
\[ D = \varepsilon E_{\text{tot}} = P + \varepsilon_0 E_{\text{tot}} \]

Magnetic

Magnetic moments

Bound electrons

\[ B_{\text{tot}} = B_a + B_s \]
\[ B_{\text{tot}} = \mu H = \mu_0 (H + M) \]
Lecture 17 Summary

• Magnetic dipole moment $m = I A$

• Magnetization or magnetic dipole moment per unit volume

$$M = N m = \chi_m H_{\text{external}}$$

  – Simple linear isotropic magnetic material
    • Strengthens $B$-field strength by $(1 + \chi_m)$
    • $H$ has same value as free space
  – Magnetization current $J_M = \nabla \times M$
  – New definition $H = B / \mu_0 - M$ and $B = \mu H$

• Upcoming schedule: Plane Waves
  – Sections 4.1, 4.2, 4.4, 4.5, 5.3, and 5.4
Uniform Plane Waves in Free Space
Poynting’s Theorem
Summary of Maxwell’s Equations

Faraday’s Law
\[ \oint_c \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \iint_s \mathbf{B} \cdot d\mathbf{S} \]
\[ \nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \]

Ampere’s Law
\[ \oint_c \mathbf{H} \cdot d\mathbf{l} = \iint_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \iint_s \mathbf{D} \cdot d\mathbf{S} \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt} \]

Gauss’ Law
\[ \iint_s \mathbf{B} \cdot d\mathbf{S} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]

Gauss’ Law
\[ \iiint_v \mathbf{D} \cdot d\mathbf{S} = \iiint_v \rho dV \]
\[ \nabla \cdot \mathbf{D} = \rho \]

Continuity Eq.
\[ \iiint_s \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \iiint_v \rho dV \]
\[ \nabla \cdot \mathbf{J} = -\frac{d\rho}{dt} \]
Motivation for Waves

• Maxwell’s equations say...
  – Time variation in \( J(t) \) leads to spatial variation of \( H(t) \)
    \[
    \nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}
    \n    \]
  – Time variation in \( H(t) \) leads to spatial variation in \( E(t) \)
    \[
    \nabla \times \vec{E} = -\frac{d\vec{B}}{dt}
    \n    \]
  – Time variation in \( E(t) \) leads to spatial variation in \( H(t) \)
    \[
    \nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}
    \n    \]

• This recursion relationship between \( E(t) \) and \( H(t) \) leads to electromagnetic wave propagation
Field Source

- Infinite plane sheet of current at \( z=0 \)

\[ \mathbf{J}_s(t) = -J_s(t) \hat{a}_x \]

At \( z=0 \)

\( J_s \) = surface current in Amps per meter

Free space \( z<0 \)

Free space \( z>0 \)
Which direction do we expect the generated fields to point?
Hmm, we did the static case

\[ \vec{H} = \frac{\vec{B}}{\mu_0} = \frac{1}{2} \vec{J} \times \hat{a}_n \]

\( \vec{H} \) direction:

\[ \perp \vec{J}_s \text{ and } \perp d\vec{S} \]
So we expect $\mathbf{H}$ to be along the $\pm y$-direction

What variables can $H_y$ depend on??

$$\vec{H} = H_y \hat{a}_y$$
$H_y$ only depends on $z$ and $t$

Infinite plane looks the same everywhere so $H_y$ can’t depend on $(x,y)$ coordinates

\[ \vec{H} = H_y(z,t) \hat{a}_y \]
\( \mathbf{E} \) depends only on \( z \) and \( t \), but in what direction is \( \mathbf{E} \)?

\[
\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} = -\mu_0 \frac{d\vec{H}}{dt} \\
= -\mu_0 \frac{dH_y}{dt} \hat{a}_y
\]

\[
(\nabla \times \vec{E})_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}
\]

\[
\frac{\partial E_x}{\partial z} = -\mu_0 \frac{dH_y}{dt}
\]

Infinite plane looks the same everywhere so \( \mathbf{E} \) can’t depend on \((x,y)\) coordinates.
$E_x \neq 0$, what about $E_y$ and $E_z$?

$$0 = -\frac{dB_x}{dt} = (\nabla \times \vec{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}$$

So $\frac{\partial E_y}{\partial z} = 0 \Rightarrow E_y = \text{const}$,

but const = 0 since we can turn off our source, i.e. set $J(t) = 0$.

$$0 = -\frac{dB_z}{dt} = (\nabla \times \vec{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$$

No new information.

$$0 = \frac{\rho}{\varepsilon_0} = \nabla \bullet \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

So $\frac{\partial E_z}{\partial z} = 0 \Rightarrow E_z = \text{const} = 0$
\[ \frac{\partial E_x}{\partial z} = -\mu_0 \frac{dH_y}{dt} \]

\[ \vec{E} = E_x(z, t) \hat{a}_x \]
Summary so far ...

- We have \( \mathbf{E} \) and \( \mathbf{H} \) that are
  - Perpendicular to each other
  - Perpendicular to the direction of propagation
  - Magnitude is constant ("uniform") in any plane perpendicular to the propagation direction

\[
\vec{J}_s = -J_x(t)\hat{a}_x \\
\vec{E} = E_x(z,t)\hat{a}_x \\
\vec{H} = H_y(z,t)\hat{a}_y
\]
Following Maxwell’s Footsteps

• We will **SIMULTANEOUSLY** solve Maxwell’s equations to find $\mathbf{E}$ and $\mathbf{H}$ caused $\mathbf{J}$

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}
\]

\[
\vec{E} = E_x(z,t)\hat{a}_x \\
\vec{H} = H_y(z,t)\hat{a}_y \\
\vec{J}_s = -J_x(t)\hat{a}_x
\]
Apply the two Maxwell’s Equations

- Performing the cross product, only two equations contain $E_x$ or $H_y$

\[
\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}
\]

\[
\frac{\partial H_y}{\partial z} = -J_x - \frac{\partial D_x}{\partial t}
\]
Simplify to $\mathbf{E}$ and $\mathbf{H}$

- Use constitutive relationships $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ for free space to express $\mathbf{D}$ and $\mathbf{B}$ in terms of $\mathbf{E}$ and $\mathbf{H}$
- We write the equations for everywhere EXCEPT at $z=0$ (where the current source is) so $J_x$ goes away - for now

\[
\frac{\partial E_x}{\partial z} = - \frac{\partial B_y}{\partial t} = -\mu_0 \frac{\partial H_y}{\partial t}
\]

\[
\frac{\partial H_y}{\partial z} = - \frac{\partial D_x}{\partial t} = -\varepsilon_0 \frac{\partial E_x}{\partial t}
\]
Eliminate $H$ in favor of $E$ only

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_y}{\partial z} = -\varepsilon_0 \frac{\partial E_x}{\partial t}$$

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0 \left( \frac{\partial}{\partial z} \left( \frac{\partial H_y}{\partial t} \right) \right) = -\mu_0 \frac{\partial}{\partial t} \left( \frac{\partial H_y}{\partial z} \right) = -\mu_0 \frac{\partial}{\partial t} \left( -\varepsilon_0 \frac{\partial E_x}{\partial t} \right)$$

- Coupled 1st order partial differential equation (PDE) $\rightarrow$ single 2nd order PDE
This is the “wave equation”

\[
\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}
\]

• Why is it called a wave equation?
  – The solution to this differential equation will be a function whose shape moves like a wave in the z-direction
  – How do we solve it?
    • Two techniques: Separation of variables (see book) or factorable operators (my notes)
  – We will do the solution for \( E \), and then we can come back and plug the solution into Maxwell’s equation to get \( H \)
Change of variables

• Makes the math a little cleaner

\[ \tau = z \sqrt{\mu_0 \varepsilon_0} \]

Has units of time

\[ \frac{\partial^2 E_x}{\partial \tau^2} = \frac{\partial^2 E_x}{\partial \xi^2} \]

Still the wave eqn
Two possible solutions

\[ \frac{\partial^2 E_x}{\partial \tau^2} = \frac{\partial^2 E_x}{\partial \alpha^2} \]

\[ \frac{\partial^2 E_x}{\partial \tau^2} - \frac{\partial^2 E_x}{\partial \alpha^2} = 0 \]

"\( x^2 - y^2 = (x + y)(x - y) \)"

\[ \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \alpha} \right) \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \alpha} \right) E_x = 0 \]

\[ \frac{\partial E_x}{\partial \tau} = -\frac{\partial E_x}{\partial \alpha} \]

\[ \frac{\partial E_x}{\partial \tau} = +\frac{\partial E_x}{\partial \alpha} \]

Now we have two first order diff eqns that each can be a solution to the wave equation
Two possible solutions

\[ \frac{\partial E_x}{\partial \tau} = - \frac{\partial E_x}{\partial t} \]

If \( E_x \) is a function of \((t-\tau)\)
The equation is satisfied

\[ E_x(\tau, t) = Af(t - \tau) \]

\( A = \text{a constant} \)

\[ \frac{\partial E_x}{\partial \tau} = + \frac{\partial E_x}{\partial t} \]

If \( E_x \) is a function of \((t+\tau)\)
The equation is satisfied

\[ E_x(\tau, t) = Bg(t + \tau) \]

\( B = \text{a constant} \)

Combining the two possible solutions...

\[ E_x(\tau, t) = Af(t - \tau) + Bg(t + \tau) \]
Changing variables *back* to $z$

\[ E_x(\tau, t) = Af(t - \tau) +Bg(t + \tau) \]

\[ E_x(z,t) = Af(t - z\sqrt{\mu_0\varepsilon_0}) +Bg(t + z\sqrt{\mu_0\varepsilon_0}) \]

- Traveling wave propagating in the +z direction
- Traveling wave propagating in the -z direction
Solving for $\mathbf{H}$

$$E_x(z,t) = Af(t - z\sqrt{\mu_0\varepsilon_0}) + Bg(t + z\sqrt{\mu_0\varepsilon_0})$$

Recall…

$$\frac{\partial H_y}{\partial z} = -\varepsilon_0 \frac{\partial E_x}{\partial t}$$

$$H_y(z,t) = \frac{1}{\sqrt{\mu_0/\varepsilon_0}} \left[ Af(t - z\sqrt{\mu_0\varepsilon_0}) - Bg(t + z\sqrt{\mu_0\varepsilon_0}) \right]$$
Two definitions

\[ v_p = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \text{speed of light} = 3 \times 10^8 \text{ (m/s)} \]

\[ \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \text{ (ohms)} \quad \text{Intrinsic impedance of free space} \]
Rewriting the solution

\[ E_x(z,t) = Af(t - \frac{z}{v_p}) + Bg(t + \frac{z}{v_p}) \]

\[ H_y(z,t) = \frac{1}{\eta_0} \left[ Af(t - \frac{z}{v_p}) - Bg(t + \frac{z}{v_p}) \right] \]

Solution is a superposition of traveling waves
- one going in +z direction
- one going in -z direction
Challenge Problem: Velocity of propagation

The velocities of propagation for the following waves: \( f = (0.05y-t)^2 \), \( g = u(t+0.02x) \), \( h = \cos(2\pi 10^8 t - 2\pi z) \) are:

(a) \( 0.05a_y, -0.02a_x, 10^8 a_z \)
(b) \( 0.05a_y, -0.02a_x, -10^8 a_z \)
(c) \( 20a_y, -50a_x, 10^8 a_z \)
(d) \( 20a_y, 50a_x, 10^8 a_z \)
(e) \( 20a_y, -50a_x, -10^8 a_z \)

From D3.11 (p 171) of old book
Two useful identities

Forward wave \( a_z \)

\[
E_x(z,t) \equiv f(t - \frac{z}{v_p}) \Rightarrow
E_x(z,t) = E_x(0,t - \frac{z}{v_p}) = E_x(z - v_p t,0)
\]

Backwards wave \(-a_z\)

\[
E_x(z,t) \equiv g(t + \frac{z}{v_p}) \Rightarrow
E_x(z,t) = E_x(0,t + \frac{z}{v_p}) = E_x(z + v_p t,0)
\]

These identities simply say that the forward wave moves like \( z = v_p t \) and the backwards wave like \( z = -v_p t \)

They allow you to express the wave solutions in terms of the wave at a fixed position or at a fixed time
Moving waveform

- A wave traveling in the \(-a_z\) direction with speed 100 m/s is measured at \(z=0\):

Find the wave amplitude at:
- (a) \(z=200\text{m}, t=0.2\text{s}\)
- (b) \(z=-300\text{m}, t=3.4\text{s}\)
- (c) \(z=100\text{m}, t=0.6\text{s}\)

From D3.13 (p 171) of old book
Lecture 18 Summary

• Differentiate Maxwell’s Equations

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

to derive wave equation:

\[ \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \]

which has solution:

\[ E_x(z,t) = Af(t - \frac{z}{V_p}) + Bg(t + \frac{z}{V_p}) \]

Traveling wave +z direction
Traveling wave -z direction
Lecture 19
Sections 4.4, 4.5

Uniform Plane Waves in Free Space
Summary so far ...

- We have $\mathbf{E}$ and $\mathbf{H}$ that are
  - Perpendicular to each other
  - Perpendicular to the direction of propagation
  - Magnitude is constant ("uniform") in any plane perpendicular to the propagation direction

\[
\vec{J}_S = -J_x(t)\hat{a}_x
\]

\[
\vec{E} = E_x(z,t)\hat{a}_x
\]

\[
\vec{H} = H_y(z,t)\hat{a}_y
\]
Our solution so far ...

\[ E_x(z, t) = Af(t - \frac{z}{v_p}) + Bg(t + \frac{z}{v_p}) \]

\[ H_y(z, t) = \frac{1}{\eta_0} \left[ Af(t - \frac{z}{v_p}) - Bg(t + \frac{z}{v_p}) \right] \]

Solution is a superposition of traveling waves

- one going in +z direction

- one going in -z direction
For a valid solution, the waves move away from source

- Generated waves travel in the +z direction for $z>0$ and –z direction for $z<0$

$\mathbf{z}<0$ then $A=0$

\[
E_x(z,t) = Bg(t + \frac{z}{v_p})
\]

\[
H_y(z,t) = -\frac{B}{\eta_0}g(t + \frac{z}{v_p})
\]

$\mathbf{z}>0$ then $B=0$

\[
E_x(z,t) = Af(t - \frac{z}{v_p})
\]

\[
H_y(z,t) = \frac{A}{\eta_0}f(t - \frac{z}{v_p})
\]
Final Step: Boundary conditions from the current source

We have to replace unknown functions $f$ & $g$ and constants $A$ & $B$ with something that relates them to the current source $J_s(t)$.

\[ \mathbf{J}_s = -J_s(t) \hat{a}_x \]

At $z=0$
Near the boundary $z \to 0$
from the $z<0$ and $z>0$ sides

\[
\begin{align*}
\text{z=0}^- & \quad E_x(z,t) = Bg(t) \\
& \quad H_y(z,t) = -\frac{B}{\eta_0} g(t)
\end{align*}
\]

\[
\begin{align*}
\text{z=0}^+ & \quad E_x(z,t) = Af(t) \\
& \quad H_y(z,t) = \frac{A}{\eta_0} f(t)
\end{align*}
\]
Apply Faraday’s Law to closed path cutting through sheet that is parallel to current flow

\[ \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S} \]

\[ \oint_C \vec{E} \cdot d\vec{l} = 0 \]

\[ \vec{E}_t \text{ is continuous} \]

\[ E_{1t} = E_{2t} \]
Use $E_t$ continuous to match the solutions for $z<0$ and $z>0$

$\tilde{E}_t$ is continuous

$E_x(z = 0^+)\Delta x - E_x(z = 0^-)\Delta x = 0$

$Af(t) - Bg(t) = 0$

$Af(t) = Bg(t)$

$E_{1t} = E_{2t}$

$\vec{J}_s = -J_s(t)\hat{a}_x$

At $z=0$
Eliminate Bg in favor of Af

\[ E_x(z,t) = Af(t) \]
\[ H_y(z,t) = -\frac{A}{\eta_0} f(t) \]

\[ E_x(z,t) = Af(t) \]
\[ H_y(z,t) = \frac{A}{\eta_0} f(t) \]
Apply Ampere’s Law to closed path cutting through sheet that is perpendicular to current

\[ \oint_C \vec{H} \cdot d\vec{l} = I_{\text{enclosed}} + \frac{d}{dt} \iint_S \varepsilon_0 \vec{E} \cdot d\vec{S} \]

\[ \oint_C \vec{H} \cdot d\vec{l} = I_{\text{enclosed}} \]

\( \vec{H}_t \) is discontinuous at current sheet

\[ (\vec{H}_1 - \vec{H}_2)_t = \vec{J}_s \times \hat{a}_n \]

At \( z=0 \)

\( \vec{J}_s = -J_s(t)\hat{a}_x \)

\( J_s \) is a surface current in Amps per meter
Use $H_{||}$ discontinuity to relate the solutions to the current sheet

$\vec{H}_t$ is discontinuous at sheet

$H_y(z = 0^+)\Delta y - H_y(z = 0^-)\Delta y = J_s\Delta y$

$$\frac{A}{\eta_0} f(t) + \frac{A}{\eta_0} f(t) = J_s(t)$$

$$(\vec{H}_1 - \vec{H}_2)_y = (\vec{J}_s \times \hat{a}_n)_y$$

$$\frac{2A}{\eta_0} f(t) = J_s(t)$$

$$Af(t) = \frac{\eta_0}{2} J_s(t)$$

At $z=0$

$\vec{J}_s = -J_s(t)\hat{a}_x$

$J_s$ is a surface current in Amps per meter
The final answer at last

\[
\begin{align*}
\text{\emph{z<0}} \\
E_x(z,t) &= \frac{\eta_0}{2} J_s(t + \frac{z}{v_p}) \\
H_y(z,t) &= -\frac{1}{2} J_s(t + \frac{z}{v_p}) \\
\text{\emph{z>0}} \\
E_x(z,t) &= \frac{\eta_0}{2} J_s(t - \frac{z}{v_p}) \\
H_y(z,t) &= \frac{1}{2} J_s(t - \frac{z}{v_p})
\end{align*}
\]
Combining the expressions...

\[ \vec{E}(z, t) = \frac{\eta_0}{2} J_s(t \mp \frac{z}{v_p}) \hat{a}_x \]

\[ \vec{H}(z, t) = \pm \frac{1}{2} J_s(t \mp \frac{z}{v_p}) \hat{a}_y \]

z \geq 0
Transforming time and space

- A wave travelling in the $a_z$ direction with speed 100m/s is measured at $t=0$:

Sketch:
(a) $E(z, 1s)$
(b) $E(0, t)$
(c) $E(200m, t)$

From Problem 3.22 (p 200) of old book
Lecture 19a Summary

• Wave emanates from z=0 so must travel +a_z for z>0 and −a_z for z<0

• Use integral form of Maxwell’s Eqns to get BOUNDARY CONDITIONS

\[ \vec{E}_t \text{ is continuous} \]
\[ E_x(0^+) = E_x(0^-) \]

\[ \vec{H}_t \text{ is discontinuous} \]
\[ H_y(0^+) - H_y(0^-) = J_s \]
Sinusoidal Plane Waves

- Infinite plane sheet of current at $z=0$

\[ \vec{J}_s(t) = -J_{S0} \cos(\omega t) \hat{a}_x \]

At $z=0$
Solution ...

\[ \vec{J}_s = -J_{s0} \cos(\omega t) \hat{a}_x \]
\[ \vec{E}(z,t) = \frac{\eta_0}{2} J_s(t - \frac{z}{v_p}) \hat{a}_x \]
\[ \vec{H}(z,t) = \pm \frac{1}{2} J_s(t - \frac{z}{v_p}) \hat{a}_y \]

\[ \vec{E}(z,t) = \frac{\eta_0 J_{s0}}{2} \cos(\omega t \mp \beta z) \hat{a}_x \quad \text{for } z \geq 0 \]
\[ \vec{H}(z,t) = \pm \frac{J_{s0}}{2} \cos(\omega t \mp \beta z) \hat{a}_y \]
\[ \beta = \frac{\omega}{v_p} \]
Web Demo

http://www.phy.ntnu.edu.tw/java/emWave/emWave.html
Wave Parameters

Electric Field
\[
\vec{E}(z,t) = \frac{\eta_0 J_{s0}}{2} \cos(\omega t + \beta z) \hat{a}_x
\]
(V/m)

Phase
\[
\phi = \omega t + \beta z
\]
(radians)

Angular Frequency
\[
\omega = \frac{\partial \phi}{\partial t}
\]
(radians/sec)

Linear Frequency
\[
f = \frac{\omega}{2\pi}
\]
(1/sec)

Phase Constant
\[
\beta = \left| \frac{\partial \phi}{\partial z} \right|
\]
(radians/m)

Wavelength
\[
\lambda = \frac{2\pi}{\beta}
\]
(m)

Phase Velocity
\[
v_p = \frac{\omega}{\beta} = \lambda f = c
\]
(m/sec)

Impedance
\[
\eta_0 = \left| \frac{\vec{E}}{\vec{H}} \right|
\]
(\Omega)
Sinusoidal wave parameters

For a wave in free space, find the following:

a. \( f \) if the phase of the field at a point changes \( 3\pi \) in \( 0.1\mu s \)

b. \( \lambda \) if the phase changes \( 0.04\pi \) in \( 1\) m

c. \( f \) if \( \lambda = 25\) m

d. \( \lambda \) if \( f = 5\) MHz

From D3.14 (p 178) of old book
A few additional important properties

\[ v_p = \lambda f \]

Speed of light in free space

\[ \frac{|\vec{E}|}{|\vec{H}|} = \eta_0 \]

Intrinsic impedance of free space
Poynting Vector

- “Points” in the direction of energy flow
- Energy flux density

\[ \vec{S} = \vec{E} \times \vec{H} = \pm \frac{\eta_0 J_s^2}{4} \cos^2 (\omega t \mp \beta z) \hat{a}_z \] 

\( z \geq 0 \)
Notice the Triad (permutations)

\[ \hat{S} = \hat{E} \times \hat{H} \]
\[ \hat{E} = \hat{H} \times \hat{S} \]
\[ \hat{H} = \hat{S} \times \hat{E} \]

Very useful for finding directions

Similar to coordinate axes

\[ \hat{a}_z = \hat{a}_x \times \hat{a}_y \]
\[ \hat{a}_x = \hat{a}_y \times \hat{a}_z \]
\[ \hat{a}_y = \hat{a}_z \times \hat{a}_x \]
Challenge problem: Finding field directions

- If $\mathbf{H} = H_0 \cos (6\pi \times 10^8 t + 2\pi y) \mathbf{a}_x \text{ A/m}$, then the directions of: (1) $\mathbf{H}$ at $t=0$, $y=0$, (2) propagation, and (3) $\mathbf{E}$ at $t=0$, $y=0$ are:

(a) $\mathbf{a}_x$, $-\mathbf{a}_y$, $-\mathbf{a}_x$
(b) $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$
(c) $\mathbf{a}_x$, $\mathbf{a}_y$, $-\mathbf{a}_x$
(d) $\mathbf{a}_x$, $-\mathbf{a}_y$, $-\mathbf{a}_z$
(e) $\mathbf{a}_x$, $\mathbf{a}_y$, $-\mathbf{a}_z$

From D3.15 (p 178) of old book
Example: Antenna Array

- An antenna array consists of two or more antenna elements spaced appropriately and excited with currents of appropriate amplitude and phase. Find $\mathbf{E}$ everywhere if:
  
  $J_{s1} = -J_0 \cos \omega t \ a_x$ at $z=0$
  
  $J_{s2} = -J_0 \sin \omega t \ a_x$ at $z=\lambda/4$

From Example 3.12 (p 177) of old book
Lecture 19 Summary

• Sinusoidal waves

\[ \phi = \omega t + \beta z \]

\[ \omega = \frac{\partial \phi}{\partial t} \]

\[ f = \frac{\omega}{2\pi} \]

\[ \beta = \frac{\partial \phi}{\partial z} \]

\[ \lambda = \frac{2\pi}{\beta} \]

\[ v_p = \lambda f \]

\[ \left| \frac{E}{H} \right| = \eta_0 \]

• Poynting vector (power per unit area) is in direction of energy flow

\[ \vec{S} = \vec{E} \times \vec{H} \]

• \( \vec{E}, \vec{H}, \vec{S} \) form a triad
Lecture 20
Section 4.6

Power Flow/Energy Storage
Poynting’s Theorem
Field Source

- Infinite plane sheet of current at z=0

Free space z<0

\[ \mathbf{J}_s(t) \]

Free space z>0

\[ \mathbf{J}_s = -J_x(t) \hat{a}_x \]

At z=0

\[ \mathbf{E} = E_x(z,t) \hat{a}_x \]

\[ \mathbf{H} = H_y(z,t) \hat{a}_y \]

\( \mathbf{J}_s = \) surface current in Amps per meter
Combining the expressions...

\[ \vec{E}(z,t) = \frac{\eta_0}{2} J_s \left( t \pm \frac{z}{v_p} \right) \hat{a}_x \]

\[ \vec{H}(z,t) = \pm \frac{1}{2} J_s \left( t \pm \frac{z}{v_p} \right) \hat{a}_y \]

\( z \geq 0 \)
Definition: Poynting Vector

The POWER provided by the current source is used to generate the propagating \( \mathbf{E} \) and \( \mathbf{H} \) fields.

The \( \mathbf{E} \) and \( \mathbf{H} \) fields are carrying power with them as they propagate

\[
\mathbf{S} = \mathbf{E} \times \mathbf{H}
\]

Definition for the Power Flow Density of an EM Field

Units for \( \mathbf{S} \): Watts/m\(^2\)
Integral & Differential Forms

\[ \vec{S} = \vec{E} \times \vec{H} \]

“Instantaneous” Poynting vector

We can calculate power density magnitude and direction for any single place and time if we know \( \mathbf{E} \) and \( \mathbf{H} \) at that place and time.

\[ \oint \oint \vec{S} \cdot d\vec{S} = \oint \oint (\vec{E} \times \vec{H}) \cdot d\vec{S} \]

Power flow **out** of a CLOSED surface (units = Watts)
There’s ENERGY in the field!

If the EM field is carrying POWER, how much ENERGY is contained in any given VOLUME of space?

Free space \( z < 0 \)

Free space \( z > 0 \)

\[
\bar{J}_s = -J_x(t) \hat{a}_x
\]

At \( z = 0 \)
How much ENERGY?

\[ \vec{S} = \vec{E} \times \vec{H} = \left| E_x \right| H_y \hat{a}_z \]

Use integral form to get a special case of Poynting’s Theorem and calculate the power flow OUT of our little closed volume

\[ \iiint_S \vec{S} \cdot d\vec{S} = \]

How much energy is “stored” in \( \vec{E} \) and \( \vec{H} \) for this volume \( dV \)?

\[ \iiint_S \vec{S} \cdot d\vec{S} = \frac{\partial [E_x H_y]}{\partial z} \Delta V = \]

Hint: Use slide 14 result:

\[ \frac{\partial E_x}{\partial z} = - \frac{\partial B_y}{\partial t} = -\mu_0 \frac{\partial H_y}{\partial t} \]

\[ \frac{\partial H_y}{\partial z} = -J_x - \frac{\partial D_x}{\partial t} = -J_x - \varepsilon_0 \frac{\partial E_x}{\partial t} \]
A question we may ask on an exam

The power is not a constant value as a function of time. Remember - it has that $\cos^2(t)$ dependence.

What is the TIME-AVERAGED power being carried by the EM field?

Do derivation - it contains a trig identity
Power calculations

- For $H = H_0 \cos(6\pi \times 10^7 t - 0.2\pi z) \text{ a}_y \text{ A/m}$, find:
  - (a) the instantaneous power flow across a 1 m$^2$ area, $A$, in the $z=0$ plane at $t=0$
  - (b) the instantaneous power across $A$ at $t=1/8\mu$s
  - (c) the time averaged power flow across $A$
Challenge Question: Power flow

- For $\mathbf{H}=H_0\cos(6\pi \times 10^7 t - 0.2\pi z) \mathbf{a}_y \ A/m$, and $A_{@z0}$ is a $1m^2$ area at the plane $z=z_0$ which statement is true:

(a) the instantaneous power flow crossing $A_{@z0}$ in the $a_z$ direction can be negative
(b) the time averaged power flow across $A_{@z0}$ in the $a_z$ direction can be negative
(c) the instantaneous power flow across $A_{@z0}$ at $t=0$ depends on the position $z_0$
(d) the time averaged power flow across $A_{@z0}$ depends on the position $z_0$
Poynting’s Theorem

Begin with: \( \nabla \cdot \vec{S} = \nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) \)
Poynting’s Theorem

Thus,
\[
\nabla \cdot \vec{S} = -\frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon_0 E^2 \right) - \frac{\partial}{\partial t} \left( \frac{1}{2} \mu_0 H^2 \right) - \vec{E} \cdot \vec{J}
\]

\[
\nabla \cdot \vec{S} + \vec{E} \cdot \vec{J} = -\frac{\partial u_e}{\partial t} - \frac{\partial u_m}{\partial t}
\]

\[
-\frac{\partial}{\partial t} (u_e + u_m) = \nabla \cdot \vec{S} + \vec{E} \cdot \vec{J}
\]

\[
u_e = \frac{1}{2} \varepsilon_0 E^2, \quad u_m = \frac{1}{2} \mu_0 H^2
\]

\[
u_e = \text{electric field energy density}, \quad \nu_m = \text{magnetic field energy density}
\]
Poynting’s Theorem

\[- \frac{\partial}{\partial t} (u_m + u_e) = \nabla \cdot \vec{S} + \vec{E} \cdot \vec{J} \]

Integrate over the volume and Apply Divergence Theorem:

\[\iiint_V \vec{S} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{S} \ dV\]

Rate the fields **LOSE** energy = **Power flow** OUT of surface + **Rate of work done** BY the fields

Can be + or - Can be + or -

\[\mathbf{E} \cdot \mathbf{J}\] is non-negative when the fields move charge (resistive load):

\[\mathbf{J} = \sigma \mathbf{E} \quad \Rightarrow \quad \mathbf{E} \cdot \mathbf{J} = \sigma |\mathbf{E}|^2 \geq 0\]

Zero only if \(\sigma = 0\) (perfect dielectric)

\[\mathbf{E} \cdot \mathbf{J}\] is negative when the applied current injects energy into the fields (the current sheet at \(z=0\) is a power source)
Rate of work done by the fields

\[ \vec{E} \quad \Theta \quad \vec{E} \quad \Phi \]

\[ dq = \rho dV \]

\[ d\vec{l} = \vec{v} dt \]

Work done moving \( dq \) a distance \( d\vec{l} \) is \( dW = \vec{dF} \cdot d\vec{l} \)

\[ d\vec{F} = dq \vec{E} \]

Rate of work done **BY** the fields moving a small charge \( dq \):

\[ \frac{dW}{dt} = d\vec{F} \cdot \frac{d\vec{l}}{dt} = d\vec{F} \cdot \vec{v} = dq \vec{E} \cdot \vec{v} = \vec{E} \cdot (\rho \vec{v}) dV = (\vec{E} \cdot \vec{J}) dV \]

\[ \therefore \iiint_V \vec{E} \cdot \vec{J} \ dV = \text{Rate of work done by the fields (Joule heating)} \]
Poynting’s Theorem for Perfect Dielectric

\[ \vec{J} = \sigma \vec{E} = 0 \quad \Rightarrow \quad \oint_S \vec{S} \cdot d\vec{S} = -\frac{\partial}{\partial t} \iiint_V (u_m + u_e) dV \]

If net power flows out, the energy stored inside must decrease

\[ P_{\text{out}}^{\text{net}} > 0 \]

Energy stored = \[\iiint_V (u_m + u_e) dV\] is decreasing
Challenge Question: Poynting’s Theorem

- If \( \mathbf{H} = H_0 \cos(\omega t + x) \hat{a}_y \) A/m and \( \mathbf{E} = \eta_0 H_0 \cos(\omega t + x) \hat{a}_z \) V/m in the free space region \( x > 0 \), which statement is true for \( V \), the volume bounded by the \( x = 0 \) and \( x = 1 \) planes:
  
  (a) the net outward power flow is zero at all times
  (b) the fields do work and thus lose energy
  (c) the total electric field energy inside is constant in time
  (d) the time averaged electric field energy density at each position \( x \) is the same
  (e) the total electric and magnetic field energy density at fixed position \( x \) is constant in time
Time Averaged Poynting Vector

\[
\vec{E} = \text{Re}\left[\vec{E} e^{j\omega t}\right] = \frac{\vec{E} e^{j\omega t} + \vec{E}^* e^{-j\omega t}}{2}
\]

\[
\langle \hat{S} \rangle = \langle \vec{E} \times \vec{H} \rangle = \left\langle \frac{\vec{E} e^{j\omega t} + \vec{E}^* e^{-j\omega t}}{2} \times \frac{\vec{H} e^{j\omega t} + \vec{H}^* e^{-j\omega t}}{2} \right\rangle
\]

\[
= \frac{1}{4} \left\langle \vec{E} \times \vec{H} e^{2j\omega t} + \vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H} + \vec{E}^* \times \vec{H}^* e^{-2j\omega t} \right\rangle
\]

\[
\langle \hat{S} \rangle = \frac{1}{2} \text{Re}\left[\vec{E} \times \vec{H}^*\right] = \frac{1}{2} \text{Re}\left[(\eta \vec{H})\vec{H}^*\right] \hat{S} = \frac{1}{2} |\vec{H}|^2 \text{ Re}[\eta] \hat{S}
\]

"\(\vec{E} = \eta \vec{H}\)" but \(\vec{E}, \vec{H}\) point in different directions
Lecture 20 Summary

• Poynting’s Theorem:
  – Rate that stored field energy is lost = rate that energy flows out boundary surface plus rate that the fields do work (Joule heating) or minus the rate that a current source injects energy
    \[- \frac{\partial}{\partial t} \iiint_V (w_e + w_m) dV = \oiint_S \vec{S} \cdot d\vec{S} + \iiint_V \vec{E} \cdot \vec{J} dV\]
  – Time averaged Poynting vector
    \[\left\langle \vec{S} \right\rangle = \frac{1}{2} \text{Re}\left[\vec{E} \times \vec{H}^*\right]\]

• Next up: Waves in materials
  – Sections 5.3-5.4
Lectures 21-24
Sections 5.3-5.4

Plane Waves in Materials

Also Section 1.4: Polarization
3 types of materials

Conductors

- Free electrons
  - $E=0$ inside
  - $\rho=0$ inside
  - $\rho=\rho_s$ only surface charge
  - $V$ is same throughout
  - $E_{outside}$ is $\perp$ to surface

Dielectrics

- Polarized atoms/molecules (Bound electrons)
  - $E\neq 0$ inside but it is reduced
  - $E_{tot}=E_a+E_s$
  - $D=\varepsilon E_{tot}=P+\varepsilon_0 E_{tot}$

Magnetic

- Magnetic moments (Bound electrons)
  - $B_{tot}=B_a+B_s$
  - $B_{tot}=\mu H=\mu_0 (H+M)$
New Relations

\[ \vec{J} = \sigma \vec{E} \]

\[ \vec{D} = \varepsilon \vec{E} \quad \varepsilon = \varepsilon_0 \varepsilon_r \]

\[ \vec{B} = \mu \vec{H} \quad \mu = \mu_0 \mu_r \]
Inside a material, Maxwell’s Equations become:

**Free Space**

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

\[ \nabla \cdot \vec{D} = \rho \]

\[ \nabla \cdot \vec{B} = 0 \]

**Inside Material**

\[ \nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \]

\[ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \]

\[ \nabla \cdot (\varepsilon \vec{E}) = \rho \]

\[ \nabla \cdot \vec{H} = 0 \]
A single material can have conductive, dielectric, and magnetic properties AT THE SAME TIME

\[ \sigma \neq 0 \]
\[ \varepsilon_r \neq 1 \]
\[ \mu_r \neq 1 \]

How does an EM wave propagate through this?
Field Source

- Infinite plane sheet of current at z=0

$\mathbf{J}_s(t)$

\[
\mathbf{J}_s = -J_{s0} \cos(\omega t) \hat{a}_x
\]

At z=0

$J_s = \text{surface current in Amps per meter}$

Free space $z<0$

Free space $z>0$
Solution

\[ \vec{E}(z,t) = \frac{\eta_0 J_s^0}{2} \cos(\omega t \mp \beta z) \hat{a}_x \]

\[ \vec{H}(z,t) = \pm \frac{J^s_0}{2} \cos(\omega t \mp \beta z) \hat{a}_y \quad z \geq 0 \]

\[ \beta = \frac{\omega}{v_p} \]
Web Demo

http://www.phy.ntnu.edu.tw/java/emWave/emWave.html
Key characteristics in FREE SPACE

- No attenuation
- $E_x$ and $H_y$ are IN PHASE for linear polarization
- Impedance: $|\vec{E}| = \eta_0 |\vec{H}|$
- Travel at Speed of Light: $v_p = c = \frac{\omega}{\beta}$
- Perpendicular: $\mathbf{E} \perp \mathbf{H} \perp \mathbf{S}$

$$\vec{S} = \vec{E} \times \vec{H}$$
Field Source in a MATERIAL

- Infinite plane sheet of current at $z=0$

Material $z<0$

$\sigma, \mu, \varepsilon$

Material $z>0$

$\sigma, \mu, \varepsilon$

$J_s(t)$

$J_s = \text{surface current in Amps per meter}$

$\mathbf{J}_s = -J_{S0} \cos(\omega t) \hat{a}_x$

At $z=0$
Similar Procedure for Maxwell’s Equations

- We will SIMULTANEOUSLY solve Maxwell's equations to find \( \mathbf{E} \) and \( \mathbf{H} \) caused by \( \mathbf{J} \)
  - Note: \( \mathbf{J} = \sigma \mathbf{E} \neq 0 \) inside the material

\[
\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}
\]
\[
\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}
\]
\[
\vec{J}_s = -J_x(t) \hat{a}_x
\]
\[
\vec{E} = E_x(z,t) \hat{a}_x
\]
\[
\vec{H} = H_y(z,t) \hat{a}_y
\]
Apply the two Maxwell’s Equations

• Performing the cross product, only two equations contain $E_x$ or $H_y$

\[
\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}
\]

\[
\frac{\partial H_y}{\partial z} = -\sigma E_x - \varepsilon \frac{\partial E_x}{\partial t}
\]

The key difference
Phasor Review

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

\[ \cos(\theta) = \text{Re}[e^{j\theta}] \]

\[ \sin(\theta) = \text{Re}[-je^{j\theta}] = \text{Re}[e^{j(\theta-\pi/2)}] \]

\[ \text{Re}[z] = (z + z^*)/2 \]

\[
E_x(z, t) = E_x(z) \cos(\omega t + \beta z + \theta) \\
= \text{Re}[E_x(z)e^{j\beta z}e^{j\theta}e^{j\omega t}] \\
= \text{Re}[\tilde{E}_x(z)e^{j\omega t}] 
\]
Solve PDEs with Phasors

• Technique simplifies the algebra

\[ E_x(z, t) = \text{Re}[\tilde{E}_x(z)e^{j\omega t}] \]
\[ H_y(z, t) = \text{Re}[\tilde{H}_y(z)e^{j\omega t}] \]
\[ \frac{\partial E_x}{\partial t} = \text{Re}[j\omega \tilde{E}_x(z)e^{j\omega t}] \]
\[ \frac{\partial}{\partial t} \equiv j\omega \]

\[ \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \]
\[ \frac{\partial H_y}{\partial z} = -\sigma E_x - \varepsilon \frac{\partial E_x}{\partial t} \]

\[ \frac{\partial \tilde{E}_x}{\partial z} = -\mu(j\omega)\tilde{H}_y \]
\[ \frac{\partial \tilde{H}_y}{\partial z} = -\sigma\tilde{E}_x - \varepsilon(j\omega)\tilde{E}_x \]
Differentiate again in $z$

\[
\frac{\partial \tilde{E}_x}{\partial z} = -\mu(j\omega)\tilde{H}_y
\]

\[
\frac{\partial \tilde{H}_y}{\partial z} = -\sigma\tilde{E}_x - \varepsilon(j\omega)\tilde{E}_x
\]

\[
\frac{\partial^2 \tilde{E}_x}{\partial z^2} = -\mu(j\omega)\frac{\partial \tilde{H}_y}{\partial z} = -\mu(j\omega)[-\sigma\tilde{E}_x - \varepsilon(j\omega)\tilde{E}_x] = j\omega\mu(\sigma + j\omega\varepsilon)\tilde{E}_x
\]

\[
\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \bar{\gamma}^2 \tilde{E}_x
\]

\[
\bar{\gamma} \equiv \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta
\]
Solve simple PDE and re-write Phasors as Cosines

\[
\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \bar{\gamma}^2 \tilde{E}_x
\]

\[
\tilde{E}_x(z) = \bar{A} e^{-\bar{\gamma}z} + \bar{B} e^{+\bar{\gamma}z}
\]

\[
\bar{\gamma} = \alpha + j \beta
\]

\[
\tilde{E}_x(z) = \bar{A} e^{-\alpha z} e^{-j\beta z} + \bar{B} e^{\alpha z} e^{j\beta z}
\]

\[
= Ae^{j\theta} e^{-\alpha z} e^{-j\beta z} + Be^{j\phi} e^{\alpha z} e^{j\beta z}
\]

\[
\tilde{E}_x(z, t) = \text{Re}[\tilde{E}_x(z) e^{j\omega t}]
\]

\[
\tilde{E}_x(z, t) = Ae^{-\alpha z} \cos(\omega t - \beta z + \theta) + Be^{+\alpha z} \cos(\omega t + \beta z + \phi)
\]

\[z > 0\]

\[z < 0\]
Also Solve for $H_y$

\[ \vec{E}_x(z,t) = A e^{-\alpha z} \cos(\omega t - \beta z + \theta) + B e^{+\alpha z} \cos(\omega t + \beta z + \phi) \]

\[ \frac{\partial \vec{E}_x}{\partial z} = -\mu(j \omega) \vec{H}_y \]

\[ \bar{\eta} = \sqrt{\frac{j \omega \mu}{\sigma + j \omega \varepsilon}} = |\eta| e^{j \tau} \]

\[ \vec{H}_y(z,t) = \frac{A}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z + \theta - \tau) - \frac{B}{|\eta|} e^{+\alpha z} \cos(\omega t + \beta z + \phi - \tau) \]
Apply Faraday’s Law to closed path cutting through sheet that is parallel to current flow

\[ \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S} \]
\[ \oint_C \vec{E} \cdot d\vec{l} = 0 \]

\[ \vec{E}_{||} \text{ is continuous} \]

\[ E_x(z = 0^+) \Delta x - E_x(z = 0^-) \Delta x = 0 \]
\[ A \cos(\omega t + \theta) = B \cos(\omega t + \phi) \]
\[ \therefore A = B \text{ and } \theta = \phi \]

\[ \vec{J}_s = -J_s(t) \hat{a}_x \]

At z=0
Apply Ampere’s Law to closed path cutting through sheet that is perpendicular to current

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enclosed} + \frac{d}{dt} \iint_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S}$$

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enclosed}$$

\(\mathbf{H}\) is discontinuous at sheet

At \(z=0\)

\(\mathbf{J}_s = -J_s(t)\hat{a}_x\)

\(J_s\) is a surface current in Amps per meter

\(H_y(z = 0^+ )\Delta y - H_y(z = 0^- )\Delta y = J_s \Delta y\)

\(\frac{2A}{|\mathbf{n}|} \cos(\omega t + \theta - \tau) = J_{s0} \cos(\omega t)\)

\(\therefore A = \frac{|\mathbf{n}| J_{s0}}{2}\) and \(\theta = \tau\)
Final Solution for $E_x$ and $H_y$

$$
\vec{E}(z, t) = \frac{|\eta| J_s 0}{2} e^{+\alpha z} \cos(\omega t \mp \beta z + \tau) \hat{a}_x
$$

$$
\vec{H}(z, t) = \pm \frac{J_s 0}{2} e^{+\alpha z} \cos(\omega t \mp \beta z) \hat{a}_y
$$

Strength of fields drops exponentially according to the attenuation constant

Magnitudes $|E|$ and $|H|$ related through magnitude of the complex impedance, $|\eta|$

$E$ and $H$ are out of phase by the phase of the complex impedance, $\tau = \arg(\eta)$
Phasor Solution for $E_x$ and $H_y$

\[
J_x(z = 0, t) = \text{Re}[-\tilde{J}_s e^{j\omega t}]
\]
\[
E_x(z, t) = \text{Re}[\tilde{E}_x(z) e^{j\omega t}]
\]
\[
H_y(z, t) = \text{Re}[\tilde{H}_y(z) e^{j\omega t}]
\]
\[
\tilde{E}_x(z) = \frac{\eta \tilde{J}_s}{2} e^{\mp jz} \hat{a}_x
\]
\[
\tilde{H}_y(z) = \frac{\pm \tilde{J}_s}{2} e^{\mp jz} \hat{a}_y
\]

\[\tilde{H}\] is in phase with the current source

"\[\tilde{E}_x = \eta \tilde{H}_y\]"

holds for amplitude and for phase but they point in different directions
In this drawing, the current sheet is not at $z=0$. Need to shift functions right by $\theta = \tau$ for our case.

\[
\vec{E}(z, t) = \frac{\eta J_{S0}}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z + \tau) \hat{a}_x
\]

\[
\Delta z = \frac{\tau}{\beta}
\]

\[
\vec{H}(z, t) = \frac{\pm J_{S0}}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z) \hat{a}_y
\]
What is the same?

• Frequency of the wave = $\omega$
• Perpendicular: $\textbf{E} \perp \textbf{H} \perp \textbf{S}$

$$\vec{S} = \vec{E} \times \vec{H}$$
What is different?

1. The magnitude of the wave is ATTENUATED

   Gets weaker by $e^{-\alpha z}$ for a wave travelling in $+z$

   \[\alpha = \text{“Attenuation Constant”}\]
   \[\text{Units} = 1/m \text{ or Np/m or dB/m}\]
What is different?

2. Magnitude relationship between $E$ and $H$

$$|\vec{E}| = |\eta| |\vec{H}|$$

$|\eta| \neq \eta_0$

$\eta$ = “Complex Impedance” of the material (because it is a complex number)

$|\eta|$ = Magnitude of the complex impedance (Units = Ohms)
What is different?

3. **E** and **H** are OUT OF PHASE by \( \tau \)

\[ \tau = \text{“Phase Offset”} \]

Units = radians
What is different?

4. Speed of propagation is no longer $c$ in free space

$$v_p = \frac{\omega}{\beta}$$

Is still good

But $\beta$ will be related to properties of the material and actually, $\beta$ can depend on $\omega$

If relation $\beta(\omega)$ is nonlinear, then the material has dispersion (different frequencies travel at different speeds, e.g. colors separate in a prism due to this dispersion) and thus a pulse (Fourier series - sum of waves) can change its shape as the pulse propagates.
Challenge Question: Propagation in $\sigma=0$ medium

- Given the phasors: 
  \[ \tilde{E}_x(z) = \frac{\eta J_{s0}}{2} e^{\mp \bar{\gamma} z} \hat{a}_x \quad \tilde{H}_y(z) = \frac{\pm J_{s0}}{2} e^{\mp \bar{\gamma} z} \hat{a}_y \]

  and definitions: 
  \[ \eta = \sqrt{\frac{j \omega \mu}{\sigma + j \omega \varepsilon}} = |\eta| e^{j \tau} \quad \bar{\gamma} = \sqrt{j \omega \mu (\sigma + j \omega \varepsilon)} = \alpha + j \beta \]

If $\sigma=0$ and $\varepsilon=2\varepsilon_0$,
consider the veracity of the statements:

I. The fields will attenuate as they propagate
II. $E$ and $H$ will be out of phase
III. The velocity of propagation will be $c$

(a) I only, (b) II only, (c) III only, (d) I, II, and III, (e) none are true
Lecture 21-22a Summary

• Differentiate Maxwell’s Equations

\[
\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}
\]

to get complex wave equation:

\[
\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \bar{\gamma}^2 \tilde{E}_x
\]

which has decaying solutions:

\[
\tilde{E}(z, t) = \frac{\eta J_0}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z + \tau) \hat{a}_x \quad \tilde{E}_x(z) = \frac{\eta \tilde{J}_0}{2} e^{\mp \bar{\gamma} z} \hat{a}_x
\]

\[
\tilde{H}(z, t) = \frac{\pm J_0}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z) \hat{a}_y \quad \tilde{H}_y(z) = \frac{\pm \tilde{J}_0}{2} e^{\mp \bar{\gamma} z} \hat{a}_y
\]
ECE 329
Lectures 22b-23

Plane Waves in Materials
Hmm, Almost the entire Greek Alphabet!!!

These constants are material properties and affect wave propagation: $\alpha$, $\beta$, $\gamma$, $\tau$, $\eta$
Wave Equation for $E_x$

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = j \omega \mu (\sigma + j \omega \varepsilon) \tilde{E}_x$$

(P Propagation Constant)$^2$
Most important definition from last class

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \gamma^2 \tilde{E}_x$$

For wave in free space, wave eqn was:

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$
Propogation Constant

\[ \frac{\partial^2 \tilde{E}_x}{\partial z^2} = \tilde{\gamma}^2 \tilde{E}_x \]

\[ \tilde{\gamma} = \sqrt{j \omega \mu (\sigma + j \omega \varepsilon)} = \alpha + j \beta = |\gamma| e^{j\psi} \]

It's a COMPLEX NUMBER

REAL PART + IMAGINARY PART

\[ \text{Re}[\tilde{\gamma}^2] < 0, \text{Im}[\tilde{\gamma}^2] > 0 \]

\[ \therefore \tilde{\gamma}^2 \text{ is in Quadrant II} \]

\[ \Rightarrow 45^\circ \leq \psi \leq 90^\circ \]

\[ \therefore \beta \geq \alpha \geq 0 \]
Propagation Constant

\[ \vec{\gamma} = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \]

Calculating \( \vec{\gamma}^2 \) gives 2 equations and 2 unknowns \( \alpha, \beta \)

\[ \vec{\gamma}^2 = \alpha^2 - \beta^2 + 2j\alpha\beta = -\omega^2 \mu\varepsilon + j\omega\mu\sigma \]

\[ : \alpha^2 - \beta^2 = -\omega^2 \mu\varepsilon \text{ and } 2\alpha\beta = \omega\mu\sigma \]

Solving for \( \alpha \) and \( \beta \) using Mathematica:

\[ \alpha = \omega \sqrt{\frac{\mu\varepsilon}{2} \left( \sqrt{1 + \left( \frac{\sigma}{\omega\varepsilon} \right)^2} - 1 \right)^{1/2}} \]

\[ \beta = \omega \sqrt{\frac{\mu\varepsilon}{2} \left( \sqrt{1 + \left( \frac{\sigma}{\omega\varepsilon} \right)^2} + 1 \right)^{1/2}} \]
The key relationships

\[ \alpha = \omega \sqrt{\frac{\mu \varepsilon}{2}} \left( \sqrt{1 + \left( \frac{\sigma}{\omega \varepsilon} \right)^2} - 1 \right)^{1/2} \]

\[ \beta = \omega \sqrt{\frac{\mu \varepsilon}{2}} \left( \sqrt{1 + \left( \frac{\sigma}{\omega \varepsilon} \right)^2} + 1 \right)^{1/2} \]

Attenuation and velocity are functions of \( \omega \) (dispersion)

\[ e^{-\alpha z} \]

\[ v_p = \frac{\omega}{\beta} \]
Complex Material Impedance

\[ \bar{\eta} = \sqrt{\frac{j \omega \mu}{\sigma + j \omega \varepsilon}} \]

\[ \bar{\eta} = |\bar{\eta}| e^{j\tau} \]

Magnitude of Complex Impedance

Phase difference between \( E_x \) and \( H_y \)
Solving for $|\eta|$ and $\tau$

\[
\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \gamma^2 \tilde{E}_x
\]

\[
\tilde{E}_x(z) = \begin{cases} 
A e^{-\gamma z} & z > 0 \\
B e^{+\gamma z} & z < 0
\end{cases}
\]

\[
\frac{\partial \tilde{E}_x}{\partial z} = -\mu(j\omega)\tilde{H}_y
\]

\[
\tilde{H}_y(z) = \begin{cases} 
\frac{-\gamma A e^{-\gamma z}}{-j\omega\mu} = \frac{\gamma \tilde{E}}{j\omega \mu} = \frac{\tilde{E}}{\eta} & z > 0 \\
\frac{+\gamma B e^{+\gamma z}}{-j\omega\mu} = -\frac{\tilde{E}}{\eta} & z < 0
\end{cases}
\]

\[
\therefore \tilde{E} = \pm \bar{\eta} \tilde{H} \text{ where } \bar{\eta} = \frac{j\omega\mu}{\gamma}
\]

\[
\gamma \equiv \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}
\]

\[
\therefore \bar{\eta} \equiv \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}
\]

\[
\bar{\eta}^2 = |\bar{\eta}|^2 e^{i\tau} = \frac{j\omega\mu(\sigma - j\omega\epsilon)}{\sigma^2 + (\omega\epsilon)^2} = \frac{\omega\mu(\omega\epsilon + j\sigma)}{\sigma^2 + (\omega\epsilon)^2}
\]

\[
|\eta| = \sqrt{\frac{\omega\mu}{\sqrt{\sigma^2 + (\omega\epsilon)^2}}}
\]

\[
\therefore \tau = \frac{1}{2} \tan^{-1}(\frac{\sigma}{\omega\epsilon})
\]
Propagation in Dielectric

- For a uniform plane wave with \( f = 10^6 \) Hz in a nonmagnetic medium \( (\mu = \mu_0) \), the propagation constant is: \( \gamma = 0.05 + 0.1j \) m\(^{-1}\). Find:

(a) Distance for field to attenuate by \( e^{-1} \)

(b) Distance for field to change phase by 1 rad

(c) Distance that constant phase moves in 1\(\mu\)s

(d) The ratio of \(|E|\) to \(|H|\)

(e) The phase difference between \(E\) and \(H\)

From D4.8 (p249) of old book
Power flow in Dielectric

- Given $\mu = \mu_0$ and $\vec{H} = H_0 e^{-z} \cos(6\pi 10^7 t - \sqrt{3}z) \hat{a}_y$, find:
  
  (a) The instantaneous power flow across $A = 1m^2$ in the $z=0$ plane at $t=0$.
  
  (b) The time averaged power flow across $A_{@z=0}$
  
  (c) The time averaged power flow across $A_{@z=1}$

From D4.9 (p249) of old book
Perfect Dielectric Material

Definition of a PERFECT dielectric: \( \sigma = 0 \)

Propagation Constant:

\[
\bar{\gamma} = \sqrt{j \omega \mu (\sigma + j \omega \varepsilon)}
\]

\[
\bar{\gamma} = j \omega \sqrt{\mu \varepsilon}
\]
Perfect Dielectric Material

\[ \bar{\gamma} = j\omega \sqrt{\mu \varepsilon} \]

\[ \bar{\gamma} = \alpha + j\beta \]

\[ \alpha = 0 \]

NO ATTENUATION

\[ \beta = \omega \sqrt{\mu \varepsilon} \]

\[ v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \varepsilon}} \]

\[ v_p = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \]

Speed of wave is less than in free space:
Perfect Dielectric Material

\[ \eta = \sqrt{\frac{j \omega \mu}{\sigma + j \omega \varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \]

Is a REAL number

\[ \eta = |\eta|e^{j\tau} \]

\[ \tau = 0 \]

\( \mathbf{E} \) and \( \mathbf{H} \) are IN PHASE

\[ |\eta| = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}} \]

Impedance is different than free space - could be greater or less, depending on material
Finding parameters

- Given $\mu = \mu_0$ and $\vec{E} = 10\cos(3\pi 10^7 t - 0.2\pi x)\hat{a}_z$, find:
  (a) The frequency
  (b) The wavelength
  (c) The phase velocity
  (d) The relative permittivity
  (e) The associated $\mathbf{H}$ field
Imperfect Dielectric

Definition: \( \sigma \neq 0 \)

But material does not conduct enough to really be considered a conductor

So, how much conductivity is “enough”?  

Look at the loss tangent: 

\[ \sigma \quad \omega \varepsilon \]

conductivity  

dielectric
Imperfect Dielectric

Official math definition for what constitutes an imperfect dielectric (small loss tangent)

\[
\frac{\sigma}{\omega \varepsilon} \ll 1
\]

Very important definition

\[
\frac{\partial^2 \widetilde{E}_x}{\partial z^2} = j \omega \mu (\sigma + j \omega \varepsilon) \widetilde{E}_x
\]

Tells which term is dominant

Also valid definitions:

\[
\psi \approx 90^\circ
\]

\[
\alpha \ll \beta
\]
Behaves like a Dielectric or Conductor depending on $\omega$
Imperfect Dielectric:
How is it different from perfect dielectric?

Only important difference: There IS attenuation in an imperfect dielectric

\[ \alpha \neq 0 \]

\[ \alpha \approx \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}} \]

Everything else is the same as in a perfect dielectric:

\[ \beta \approx \omega \sqrt{\mu \varepsilon} \]

\[ \nu_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \varepsilon}} \]

\[ \lambda = \frac{2\pi}{\beta} \]

Pretty good approximation as long as \[ \frac{\sigma}{\omega \varepsilon} < 0.1 \]
Good Conductors

Definition: \( \sigma \neq \infty \)

But conductivity is still large

\[
\frac{\sigma}{\omega \varepsilon} \gg 1
\]

Official math definition of a good conductor

Large loss tangent

Note that any material with \( \sigma > 0 \) behaves like a good conductor at low enough frequency
Good Conductor

Propagation Constant

\[ \bar{\gamma} = \sqrt{j \omega \mu (\sigma + j \omega \varepsilon)} \approx \sqrt{j \omega \mu \sigma} \]

\[ = \sqrt{j} \sqrt{\omega \mu \sigma} \]

Warning: math trick

\[ \sqrt{j} = \frac{1 + j}{\sqrt{2}} \]

\[ \bar{\gamma} = \sqrt{\frac{\omega \mu \sigma}{2}} (1 + j) = \alpha + j \beta \]

\[ \alpha = \beta = \sqrt{\frac{\omega \mu \sigma}{2}} \]
**Good Conductor**

**Complex Impedance**

\[
\eta = \sqrt{\frac{j \omega \mu}{\sigma + j \omega \varepsilon}} \approx \sqrt{\frac{j \omega \mu}{\sigma}}
\]

\[
= \sqrt{j} \sqrt{\frac{\omega \mu}{\sigma}}
\]

\[
\eta = \frac{1 + j}{\sqrt{2}} \sqrt{\frac{\omega \mu}{\sigma}} = (1 + j) \sqrt{\frac{\omega \mu}{2 \sigma}}
\]

\[
|\eta| = \sqrt{\frac{\omega \mu}{\sigma}}
\]

\[
\tau = \frac{\pi}{4}
\]

\[\text{E and H are 45 deg out of phase}\]
Good Conductor: Skin Depth

$\alpha$ tells us how far the E and H fields can propagate into a good conductor

In a conductor, $\alpha = \text{LARGE}$, so fields drop off in a short distance

**SKIN DEPTH** = Distance that the field strength is attenuated by $e^{-1}$

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega \mu \sigma}}$$

Or

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{1}{\pi f \mu \sigma}}$$

Using $\omega = 2\pi f$
Skin Depth

FIGURE 4-10
Relative magnitude of electric field $E$ or current density $J$ ($= \sigma E$) as a function of depth of penetration $\delta$ for a plane wave traveling in $x$ direction into conducting medium. The abscissa gives the penetration distance $x$ and is expressed in $1/e$ depths ($\delta$). The wavelength in the conductor equals $2\pi\delta$. 

$\delta = \frac{1}{e}$ depth of penetration
Perfect Conductor

$$\sigma = \infty$$

$$\alpha = \omega \sqrt{\frac{\mu \varepsilon}{2}} \left( \sqrt{1 + \left( \frac{\sigma}{\omega \varepsilon} \right)^2} - 1 \right)^{1/2}$$

$$\alpha = \infty$$  \hspace{1cm} $$\delta = 0$$

Infinite attenuation

Means that neither $\mathbf{E}$ or $\mathbf{H}$ can propagate into a perfect conductor

NO TIME-VARYING FIELDS ($\mathbf{E}$ or $\mathbf{H}$) CAN EXIST IN A PERFECT CONDUCTOR!!!
Challenge Question: Material characteristics

- In what type of material would you expect dispersion (velocity depends on frequency)?

(a) free space
(b) perfect dielectric
(c) imperfect dielectric
(d) good conductor
(e) perfect conductor

LG’s question
Finding parameters

- For a uniform plane wave with $f=10^5$ Hz in a good conductor, the field is attenuated by $e^{-\pi}$ in 2.5m. Find the following:

  (a) Distance for a $2\pi$ phase change if $f=10^5$ Hz

  (b) Distance a constant phase plane travels in $1\mu$s for $f=10^5$ Hz

  (c) Distance a constant phase plane travels in $1\mu$s for $f=10^4$ Hz assuming the material properties are the same as at $f=10^5$ Hz

From D4.11 (p254) of old book
## Lecture 21-23 Summary

<table>
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<tr>
<th>Perfect Dielectric</th>
<th>Imperfect Dielectric</th>
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<tr>
<td><strong>Definition:</strong></td>
<td><strong>Definition:</strong></td>
</tr>
<tr>
<td>$\sigma = 0$</td>
<td>$\sigma / \omega \varepsilon &lt;&lt; 1$</td>
</tr>
<tr>
<td><strong>Attenuation:</strong></td>
<td><strong>Attenuation:</strong></td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>$\alpha \approx \sigma / 2 \sqrt{\mu / \varepsilon}$</td>
</tr>
<tr>
<td><strong>Speed:</strong></td>
<td><strong>Speed:</strong></td>
</tr>
<tr>
<td>$v_p = c / \sqrt{\mu_r \varepsilon_r} \leq c$</td>
<td>$v_p \approx c / \sqrt{\mu_r \varepsilon_r} \leq c$</td>
</tr>
<tr>
<td><strong>E, H In Phase:</strong></td>
<td><strong>E, H In Phase:</strong></td>
</tr>
<tr>
<td>$\tau = 0$</td>
<td>$\tau \approx 0$</td>
</tr>
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<td><strong>Impedance:</strong></td>
<td><strong>Impedance:</strong></td>
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<td>\vec{\eta}</td>
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<table>
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<th>Perfect Conductor</th>
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<td><strong>Attenuation:</strong></td>
</tr>
<tr>
<td>$\alpha \approx \sqrt{\omega \mu \sigma} / 2$</td>
<td>$\alpha \to \infty$</td>
</tr>
<tr>
<td><strong>Speed:</strong></td>
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<td><strong>E, H 45° Phase:</strong></td>
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</tr>
<tr>
<td>$\tau \approx \pi / 4$</td>
<td>$\tau \to \pi / 4$</td>
</tr>
<tr>
<td><strong>Impedance:</strong></td>
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</tr>
<tr>
<td>$</td>
<td>\vec{\eta}</td>
</tr>
</tbody>
</table>

$\rho \rightarrow 0$
Lecture 24
Section 1.4

Polarization
Definition of Polarization

• Describes the “tip” of the time-varying $\mathbf{E}$ field vector for a particular point in space as it varies in time

• $\mathbf{J} = -J_{s0}a_x$ example
  – Field with only one component is always linearly polarized

\[ E_x(z, t) = \frac{\eta_0 J_{s0}}{2} \cos(\omega t + \beta z) \]
Web Demo

http://www.enzim.hu/~szia/cddemo/edemo0.htm
How do we treat this mathematically?

Say we have two superimposed fields

- Propagating in same direction
- Possibly out of phase
- Possibly oriented out of plane with each other

How can we tell if the combined wave will be:

- Linear
- Circular
- Elliptical
Linear Polarization

Plane in space perpendicular to propagation direction

The tip of the E-field vector traces out a straight line
DIRECTION: Constant
MAGNITUDE: Changes w/ time

How do we get this mathematically?
Linear Polarization

Given two linearly polarized vector fields

\[ \vec{E}_1 = E_1 \cos(\omega t + \phi) \hat{a}_x \]
\[ \vec{E}_2 = \pm E_2 \cos(\omega t + \phi) \hat{a}_y \]

If the vectors are IN PHASE or 180° apart, \( \vec{E}_1 + \vec{E}_2 \) will also be linear polarization

\[ \vec{E}_1 + \vec{E}_2 = \cos(\omega t + \phi)(E_1 \hat{a}_x \pm E_2 \hat{a}_y) \]

Magnitude changes \quad Direction is constant

Polarization ANGLE depends on relative magnitudes of \( E_1 \) and \( E_2 \)

\[ \alpha = \pm \tan^{-1}\left(\frac{E_2}{E_1}\right) \]
Circular Polarization

Plane in space perpendicular to propagation direction

The tip of the E-field vector traces out a circle
  DIRECTION: Changes with time
  MAGNITUDE: Constant

How do we get this mathematically?
Circular Polarization

Given two linearly polarized vector fields

$$\vec{E}_1 = E_0 \cos(\omega t + \phi) \hat{a}_x$$  \hspace{1cm} E_1 \text{ is horizontal linear polarization (green)}

$$\vec{E}_2 = E_0 \sin(\omega t + \phi) \hat{a}_y$$  \hspace{1cm} E_2 \text{ is vertical linear polarization (red)}

The vectors have EQUAL MAGNITUDES
The vectors are OUT OF PHASE by $\pi/2$
The vectors are PERPENDICULAR

ALL THREE CONDITIONS MUST BE MET
for $E_1 + E_2$ to be circular polarization
How to tell: Left or Right Handed Circular Polarization

Method 1: Left/right thumb points in propagation direction

“Right-Hand” Polarized (cw as seen by source)
LHCP if moving in -a_z

“Left-Hand” Polarized (ccw as seen by source)
RHCP if moving in +a_z
How to tell: Left or Right Handed Circular Polarization

Method 2: It’s right handed if Ahead $\hat{E}$ x Behind $\hat{E}$ is in propagation direction

$$\vec{E}_1 = E_0 \cos(\omega t \pm \beta z) \hat{x} \quad \text{Ahead}$$

$$\vec{E}_2 = E_0 \sin(\omega t \pm \beta z) \hat{y}$$

$$= E_0 \cos(\omega t \pm \beta z - \frac{\pi}{2}) \hat{y} \quad \text{Behind}$$

It is behind by $\pi/2$ because $\omega t$ needs to be $\pi/2$ larger to be at the same part of the wave

$\hat{E}_1 \times \hat{E}_2 = \hat{z}$

so RHCP if moving in $+a_z$

else LHCP if moving in $-a_z$
How to tell: Left or Right Handed Circular Polarization

Method 3: It’s right handed if
\( \text{Re}[\hat{E}] \times \text{Im}[\hat{E}^\ast] \) is in propagation direction

\[
\hat{E}_1 = E_0 \cos(\omega t \pm \beta z) \hat{x} \quad \text{Ahead}
\]

\[
\hat{E}_2 = E_0 \cos(\omega t \pm \beta z - \frac{\pi}{2}) \hat{y} \quad \text{Behind}
\]

\[
\hat{E} = \hat{E}_1 + \hat{E}_2 = E_0 e^{\pm j\beta z} \hat{x} + E_0 e^{\pm j\beta z} e^{-j\frac{\pi}{2}} \hat{y} = E_0 e^{\pm j\beta z} (\hat{x} - j\hat{y})
\]

\[
\hat{E} \equiv \hat{x} - j\hat{y}
\]

I made up this notation. Note it is not a unit vector

\( \text{Re}[\hat{E}] = \hat{x} \) gives the Ahead field

\( \text{Im}[\hat{E}] = \hat{y} \) gives the Behind field
Elliptical Polarization

Most general: two combined linearly polarized waves will result in an elliptically polarized wave - if the conditions for linear or circular are not satisfied.

Plane in space perpendicular to propagation direction

The tip of the E-field vector traces out an ellipse

DIRECTION: Changes with time
MAGNITUDE: Changes
Sum of two fields

- For \( \mathbf{E}_1 = E_0 \cos(2\pi \times 10^8 t - 2\pi z) \mathbf{a}_x \), and \( \mathbf{E}_2 = E_0 \cos(2\pi \times 10^8 t - 3\pi z) \mathbf{a}_y \), find the polarization of \( \mathbf{E}_1 + \mathbf{E}_2 \) at the following points:

  (a) (3, 4, 0)
  (b) (3, -2, 0.5)
  (c) (-2, 1, 1)
  (d) (-1, -3, 0.2)

From D3.17 (p 184) of old book
Challenge Question: Forming Circular Polarization

- For what value of $\phi$, will the following fields add to produce right handed circular polarization:

$$E_1 = E_0 \cos(2\pi \times 10^8 t + 2\pi z + \pi/3) a_x,$$

$$E_2 = E_0 \cos(2\pi \times 10^8 t + 2\pi z + \phi) a_y$$

(a) $\pi/3$
(b) $-\pi/3$
(c) $\pi/6$
(d) $-\pi/6$
(e) $5\pi/6$
Writing linearly polarized as a sum of circular polarization

- Rewrite \( \mathbf{E} = E_0 \cos(\omega t + \beta z) \mathbf{a}_x \) as a sum of circular polarized fields.

From Problem 3.35a (p 203) of old book
Lecture 24 Summary

- Polarization describes the “tip” of $\mathbf{E}(t)$

- **Linear**
  - Direction is __________
  - Magnitude __________

- **Circular**
  - Direction __________
  - Magnitude is ________

- **Elliptical**
  - Direction __________
  - Magnitude __________

- Next up: Section 5.6: Wave reflection
Lectures 25-26*

Section 5.6

Reflection and Transmission
Standing Waves

* In spring semester, there is 1 fewer day of instruction so both of these lectures are covered in a single class period
Normal incidence plane wave

Medium 1
\( \sigma_1, \varepsilon_1, \mu_1 \)

Medium 2
\( \sigma_2, \varepsilon_2, \mu_2 \)

Incident (+)

\[
\tilde{E}_1(z) = \overline{E}_1^+ e^{-\eta_1 z} \hat{x} \\
\tilde{H}_1(z) = \overline{H}_1^+ e^{-\eta_1 z} \hat{y} = \frac{1}{\eta_1} \overline{E}_1^+ e^{-\eta_1 z} \hat{y}
\]

What should happen next?

\[ z < 0 \quad \text{z} > 0 \]

We will analyze the case \( J_s = 0 \) at \( z = 0 \) (no applied surface current).
For the case \( J_s \neq 0 \), use superposition of the \( J_s = 0 \) case with the answer for waves generated by a current sheet.
Reflected and Transmitted Waves are Generated

Medium 1
\[ \sigma_1, \varepsilon_1, \mu_1 \]

Incident (+)
\[ \widetilde{E}_1(z) = \overline{E}_1^+ e^{-\gamma_1z} \hat{x}, \quad \widetilde{H}_1(z) = \frac{1}{\eta_1} \overline{E}_1^+ e^{-\gamma_1z} \hat{y} \]

Reflected (-)
\[ \widetilde{E}_1(z) = \overline{E}_1^- e^{+\gamma_1z} \hat{x}, \quad \widetilde{H}_1(z) = \frac{-1}{\eta_1} \overline{E}_1^- e^{+\gamma_1z} \hat{y} \]

Medium 2
\[ \sigma_2, \varepsilon_2, \mu_2 \]

Transmitted (+)
\[ \widetilde{E}_2(z) = \overline{E}_2^+ e^{-\gamma_2z} \hat{x}, \quad \widetilde{H}_2(z) = \frac{1}{\eta_2} \overline{E}_2^+ e^{-\gamma_2z} \hat{y} \]

At \( z=0 \), \( \mathbf{J}_S = 0 \)
Apply Boundary Conditions

\[ \hat{a}_n \]

Medium 2

\[ J_s(\hat{X}) \]

Medium 1

\[ E_{t1} \]

\[ H_{t1} \]

\[ E_{t2} \]

\[ H_{t2} \]

\[ J_s = \hat{a}_n \times (\vec{H}_1 - \vec{H}_2) \]

\[ (\vec{H}_1 - \vec{H}_2)_t = J_s \times \hat{a}_n \]
Apply Boundary Conditions

Medium 1
\( \sigma_1, \varepsilon_1, \mu_1 \)

\[
\tilde{E}_{1x \text{ total}}(z) = \bar{E}_1^+ e^{-\bar{\gamma}_1 z} + \bar{E}_1^- e^{+\bar{\gamma}_1 z}
\]

\[
\tilde{H}_{1y \text{ total}}(z) = \frac{1}{\bar{\eta}_1} \left( \bar{E}_1^+ e^{-\bar{\gamma}_1 z} - \bar{E}_1^- e^{+\bar{\gamma}_1 z} \right)
\]

\( \mathbf{E}_{t1} = \mathbf{E}_{t2} \), \( \mathbf{H}_{t1} - \mathbf{H}_{t2} = \mathbf{J}_s = 0 \)

\[
\tilde{E}_{1x \text{ total}}(0^-) = \bar{E}_1^+ + \bar{E}_1^- = \bar{E}_2^+ = \tilde{E}_{2x \text{ total}}(0^+)
\]

\[
\tilde{H}_{1y \text{ total}}(0^-) = \frac{1}{\bar{\eta}_1} \left( \bar{E}_1^+ - \bar{E}_1^- \right) = \frac{1}{\bar{\eta}_2} \bar{E}_2^+ = \tilde{H}_{2y \text{ total}}(0^+)
\]

Medium 2
\( \sigma_2, \varepsilon_2, \mu_2 \)

\[
\tilde{E}_{2x \text{ total}}(z) = \bar{E}_2^+ e^{-\bar{\gamma}_2 z}
\]

\[
\tilde{H}_{2y \text{ total}}(z) = \frac{1}{\bar{\eta}_2} \bar{E}_2^+ e^{-\bar{\gamma}_2 z}
\]
Solve the Equations

\[ E_1^+ + E_1^- = E_2^+ \]
\[ \frac{\eta_2}{\eta_1} (E_1^+ - E_1^-) = E_2^+ \]

\[ E_1^+ + E_1^- = \frac{\eta_2}{\eta_1} (E_1^+ - E_1^-) \]

Reflection Coefficient:

\[ \therefore \Gamma = \frac{E_1^-}{E_1^+} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} \]

Transmission Coefficient:

\[ \tau = \frac{E_2^+}{E_1^+} = \frac{E_1^+ + E_1^-}{E_1^+} = 1 + \Gamma \]

\[ \therefore \tau = \frac{E_2^+}{E_1^+} = 1 + \Gamma = \frac{2\eta_2}{\eta_1 + \eta_2} \]
Challenge Question: Special cases

- If $\eta_2 = \eta_1$, we expect:

  (a) $\Gamma = 0, \tau = 1$ (0% reflected, 100% transmitted)
  (b) $\Gamma = 0, \tau = -1$ (0% reflected, 100% transmitted)
  (c) $\Gamma = 1, \tau = 0$ (100% reflected, 0% transmitted)
  (d) $\Gamma = -1, \tau = 0$ (100% reflected, 0% transmitted)
  (e) cannot be determined
Challenge Question: Special cases

- If medium 2 is a perfect conductor, we expect:

  (a) $\Gamma = 0$, $\tau = 1$ (0% reflected, 100% transmitted)
  (b) $\Gamma = 0$, $\tau = -1$ (0% reflected, 100% transmitted)
  (c) $\Gamma = 1$, $\tau = 0$ (100% reflected, 0% transmitted)
  (d) $\Gamma = -1$, $\tau = 0$ (100% reflected, 0% transmitted)
  (e) cannot be determined
Special Cases of Reflection/Transmission

\[ \Gamma \equiv \frac{\bar{E}_1^-}{\bar{E}_1^+} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \quad \tau \equiv \frac{\bar{E}_2^+}{\bar{E}_1^+} = 1 + \Gamma \]

- Impedance matched: \( \eta_2 = \eta_1 \), then \( \Gamma = 0 \), \( \tau = 1 \)
  - No reflection, entire wave is transmitted
- Perfect dielectrics: \( \sigma_1 = \sigma_2 = 0 \), then \( \Gamma \) and \( \tau \) are real since \( \eta_1 \) and \( \eta_2 \) are real
- Perfect conductor for medium 2: \( \eta_2 \rightarrow 0 \), then \( \Gamma = -1 \), \( \tau = 0 \)
  - No transmission, entire wave is reflected. If medium 1 is a perfect dielectric, a standing wave is set up there since reflected wave perfectly cancels input wave at many nodes: \( E(z=0) = 0 \) always, but if \( \sigma_1 = 0 \), then \( E(z=-m\lambda/2) = 0 \) also
Standing Waves

- The reflection from a PC gives $\Gamma=-1$
- If region 1 is a perfect dielectric, we get standing waves whereby the input and reflected wave add destructively at specific points in space (vs. travelling waves $f(t \pm z/v)$)

\[
\tilde{E}_{1x total}(z) = E_1^+ e^{-\beta_1 z} + E_1^- e^{+\beta_1 z} = E_1^+ (e^{-j\beta_1 z} - e^{+j\beta_1 z}) = E_1^+ (-2j \sin(\beta_1 z))
\]

\[
\tilde{H}_{1y total}(z) = \frac{1}{\eta_1} \left( E_1^+ e^{-\beta_1 z} - E_1^- e^{+\beta_1 z} \right) = \frac{E_1^+}{\eta_1} \left( e^{-j\beta_1 z} + e^{+j\beta_1 z} \right) = \frac{E_1^+}{\eta_1} 2 \cos(\beta_1 z)
\]

\[
\tilde{E}(z,t) = \hat{x} \text{Re}[E_1^+ (-2j \sin(\beta_1 z))e^{j\omega t}] = \hat{x} 2 \left| E_1^+ \right| \sin(\beta_1 z) \sin(\omega t + \angle E_1^+ )
\]

\[
\tilde{H}(z,t) = \hat{y} \text{Re}\left[ \frac{E_1^+}{\eta_1} 2 \cos(\beta_1 z)e^{j\omega t} \right] = \hat{y} \left| \frac{2}{\eta_1} E_1^+ \right| \cos(\beta_1 z) \cos(\omega t + \angle E_1^+ )
\]
Standing Waves: on average, the net energy transport is zero

Show $\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = 0$. Online notes use phasors. Here we’ll use $\mathbf{E}$ and $\mathbf{H}$ explicitly (still assume $\sigma_1=0$).

$$\tilde{E}(z,t) = 2|E_1^+| \sin(\beta_1 z) \sin(\omega t + \angle E_1^+) \hat{x}$$

$$\tilde{H}(z,t) = \frac{2}{\eta_1} |E_1^+| \cos(\beta_1 z) \cos(\omega t + \angle E_1^+) \hat{y}$$

$$\tilde{S}(z,t) = \frac{4}{\eta_1} |E_1^+|^2 \sin(\beta_1 z) \cos(\beta_1 z) \sin(\omega t + \angle E_1^+) \cos(\omega t + \angle E_1^+) \hat{z}$$

$$= \frac{1}{\eta_1} |E_1^+|^2 \sin(2\beta_1 z) \sin(2(\omega t + \angle E_1^+)) \hat{z} \quad \therefore \quad \langle \tilde{S}(z,t) \rangle = 0$$

since $\sin(2\omega t)$ is periodic

At any moment in time, there is energy transport, but it averages to zero.
Incident wave induces surface current on PC that radiates the reflected wave

$\vec{H}(z,t) = \hat{y} \frac{2}{\eta_1} |\vec{E}_1^+| \cos(\omega t + \angle \vec{E}_1^+) $

Thus, the BC's imply a surface current is induced at $z=0$:

$\vec{J}_S(t) = \hat{x} \frac{2}{\eta_1} |\vec{E}_1^+| \cos(\omega t + \angle \vec{E}_1^+) $

This surface current is now a source that can radiate a wave away from the plane in $-a_z$ (i.e. the reflected wave):

$\vec{E}_{\text{reflected}}(z,t) = (-\hat{x}) \frac{\eta_1}{2} J_s \left(t - \frac{z}{v_p}\right) = -\hat{x} |\vec{E}_1^+| \cos(\omega t + \beta z + \angle \vec{E}_1^+) $

which is precisely the wave we found when we used $\Gamma = -1$. The part that radiates into $+a_z$ perfectly cancels out the incident field for $z>0$ which gives us $E=0$ in the PC as expected.

Slight complication: $J$ was created by $J=\sigma E$, but $E=0$ at the PC. Note $\sigma=\infty$. 
Lectures 25-26 Summary

Medium 1
\[ \tilde{E}_1(z) = E_1^+ e^{-\eta_1 z} \hat{x} \]
\[ \tilde{H}_1(z) = \frac{1}{\eta_1} E_1^+ e^{-\eta_1 z} \hat{y} \]
\[ \tilde{E}_1(z) = \Gamma E_1^+ e^{+\eta_1 z} \hat{x} \]
\[ \tilde{H}_1(z) = \frac{-1}{\eta_1} \Gamma E_1^+ e^{+\eta_1 z} \hat{y} \]

Medium 2
\[ \tilde{E}_2(z) = \tau E_1^+ e^{-\eta_2 z} \hat{x} \]
\[ \tilde{H}_2(z) = \frac{1}{\eta_2} \tau E_1^+ e^{-\eta_2 z} \hat{y} \]

\[ \Gamma = \frac{E_1^-}{E_1^+} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} \]

\[ \tau = \frac{E_2^+}{E_1^+} = 1 + \Gamma = \frac{2\eta_2}{\eta_1 + \eta_2} \]

Next: Transmission lines: Section 6.5 & Chapter 7
ECE 329
Lectures 27-30
Section 6.5 and Chapter 7
Transmission Lines
Time Domain Analysis
Starting Point: Uniform Plane Wave

http://www.phy.ntnu.edu.tw/java/emWave/emWave.html
Starting Point: Uniform Plane Wave

• Consider $\mathbf{E}$ and $\mathbf{H}$ that are
  – Perpendicular to each other
  – Perpendicular to the direction of propagation
  – Magnitude is constant (“uniform”) in the plane perpendicular to the propagation direction
  – And for perfect dielectric media:
    • $\mathbf{E}$ and $\mathbf{H}$ are in phase
    • No attenuation in $z$-direction

\[ \vec{E} = E_x(z, t) \hat{a}_x \]
\[ \vec{H} = H_y(z, t) \hat{a}_y \]
Parallel Plate Transmission Line

Imagine a rectangular box made of perfect conductors on the upper and lower surfaces, filled by perfect dielectric medium.

- If we place conducting sheets in the path of the uniform plane wave, some of the wave enters the box and is guided by it.

\[ d \ll w \]

\[ z=0 \quad z=l \]
Parallel Plate Transmission Line

Cross section of the transmission line so wave is propagating into the page
Consider boundary conditions

The E and H fields inside the transmission line induce charge and current on the upper and lower surfaces. Apply BC’s on $D_n$ and $H_t$ to find $\rho_s$ and $J_s$ respectively.

$$|\rho_s| = |\varepsilon E_x|$$

$$|J_s| = |H_y|$$

Perfect conductor
$E_y=0$, $D_x=0$, $H_y=0$, $B_x=0$

Perfect dielectric
$E_y=0$, $D_x=\rho_s$, $H_y=J_s$, $B_x=0$
\[ V(z, t) = (d) E_x(z, t) \]
TL Current – Charges on both plates move to the right

\[ I(z, t) = wH_y(z, t) \]
TL Power

\[ P(z,t) = \int_{S} (\vec{E} \times \vec{H}) \cdot d\vec{S} = \int_{x=0}^{x=d} \int_{y=0}^{y=w} \frac{V}{d} \cdot \frac{I}{w} \, dx \, dy \]

\[ P(z,t) = V(z,t)I(z,t) \]

same as in circuit theory
Converting $E \leftrightarrow V$ and $H \leftrightarrow I$

Find the power crossing this transverse cross section if (a) $E = 300\pi$ V/m, (b) $H = 7.5$ A/m, (c) $V = 4\pi$ V, (d) $I = 0.5$ A

From D6.1 (p371) of old book
Another look at transverse plane of TL

\[ E_x(z,t) \]
\[ H_y(z,t) \]

\[ V(z,t) \]
\[ I(z,t) \]

\[ V(z,t) \text{ and } I(z,t) \text{ can be used to describe the state of the transmission line instead of } E_x(z,t) \text{ and } H_y(z,t) \]
Transmission Line Equations

For plane waves in perfect dielectric:

\[
\begin{align*}
\frac{\partial E_x}{\partial z} &= -\frac{\partial B_y}{\partial t} = -\mu \frac{\partial H_y}{\partial t} \\
\frac{\partial H_y}{\partial z} &= -\frac{\partial D_x}{\partial t} = -\varepsilon \frac{\partial E_x}{\partial t}
\end{align*}
\]

But in the transmission line:

\[
\begin{align*}
E_x(z,t) &= \frac{V(z,t)}{d} \\
H_y(z,t) &= \frac{I(z,t)}{w}
\end{align*}
\]
Transmission Line Equations

\[
\begin{align*}
\frac{\partial V}{\partial z} &= -\left( \frac{\mu d}{w} \right) \frac{\partial I}{\partial t} \\
\frac{\partial I}{\partial z} &= -\left( \frac{\varepsilon w}{d} \right) \frac{\partial V}{\partial t}
\end{align*}
\]

Substituting…

These are the transmission line equations!!

• They describe wave propagation along the TL in terms of currents and voltages
• It is just another way of stating Maxwell’s Eqns
Circuit Parameters

Rewrite the TL Equation using circuit parameters

\[ L = \frac{\text{Inductance}}{\text{Length}} = \frac{\text{Flux} / \text{Current}}{\text{Length}} = \frac{\psi}{I} \]

\[ C = \frac{\text{Capacitance}}{\text{Length}} = \frac{\text{Charge}/\text{Voltage}}{\text{Length}} = \frac{Q}{V} \]

\[ L = \frac{B_y z d / H_y w}{z} = \frac{B_y d}{H_y w} = \frac{\mu H_y d}{H_y w} = \frac{\mu d}{w} \]

\[ C = \frac{\rho z w / E_x d}{z} = \frac{\rho w}{E_x d} = \frac{\varepsilon E_x w}{E_x d} = \frac{\varepsilon w}{d} \]
Transmission Line Equation

\[ \frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t} \]

\[ \frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t} \]

For lossless transmission lines
- Perfect dielectric filling
- Perfect conductor outer shell

Don’t forget that L & C are related:

\[ LC = \mu \varepsilon \]
One small slice of the transmission line would have a finite inductance and capacitance.

If we put our small slices together, end to end, to make a whole transmission line, the inductance and capacitance would be distributed over the whole length of the TL.

How do we represent a distributed circuit element?
Flip the parallel plate for TL (ground the bottom plate)
Distributed Circuit

One slice of the TL:

\[ V(z + \Delta z) - V(z) = -L \frac{\partial I}{\partial t} \]
\[ I(z) - I(z + \Delta z) = C \frac{\partial V}{\partial t} \]

\[ \therefore \frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t} \]
\[ \therefore \frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t} \]

These are the same as the TL equations so this is the equivalent circuit for the TL.
Distributed Circuit

One slice of the TL:

The entire TL:
Characteristic Impedance

A more convenient way of representing the distributed circuit

\[ Z_0 = \sqrt{\frac{L}{C}} = \left| \frac{V(z,t)}{I(z,t)} \right| \]

\[ \nu_p = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{L C}} \]
Challenge Question: Transmission Lines

- For waves propagating down transmission lines, which of the following is **false**:
  (a) the impedances $Z_0$ and $\eta_0$ are equal
  (b) the propagation speed is $\leq c$
  (c) the propagation speed in two lines can be different even if the impedance $Z_0$ is the same
  (d) in steady state, the TL acts like plain wires
  (e) the fields store energy distributed among the inductive and capacitive segments of the TL
Designing coaxial cables

Design a 50$\Omega$ coaxial cable if $\varepsilon_r=2.56$ and $a=1\text{cm}$, i.e. find $b$. 

From D6.2a (p372) of old book
ECE 329
Lecture 28

TL Terminated by Resistive Load
Bounce Diagram
Solution to TL Equation

We can solve the TL Equations the same way that we solved the wave equation for uniform plane waves in free space:

\[ V(z,t) = Af \left( t - \frac{z}{v_p} \right) + Bg \left( t + \frac{z}{v_p} \right) \]

\[ I(z,t) = \frac{1}{Z_0} \left[ Af \left( t - \frac{z}{v_p} \right) - Bg \left( t + \frac{z}{v_p} \right) \right] \]

Solution is a superposition of traveling waves
- one going in +z direction
- one going in -z direction
Simplified shorthand notation

\[ V(z,t) = V^+(t - \frac{z}{v_p}) + V^-(t + \frac{z}{v_p}) \]

\[ I(z,t) = \frac{1}{Z_0} \left[ V^+(t - \frac{z}{v_p}) - V^-(t + \frac{z}{v_p}) \right] \]

Simplifying even more...

\[ V = V^+ + V^- \]

\[ I = \frac{1}{Z_0} (V^+ - V^-) = I^+ + I^- \]
Solution to TL equation

\[ V = V^+ + V^- \]

\[ I = \frac{1}{Z_0}(V^+ - V^-) = I^+ + I^- \]

\[ I^+ = \frac{V^+}{Z_0} \quad I^- = -\frac{V^-}{Z_0} \]

Solution to TL equation is summation of traveling waves propagating in the +z or -z directions
Example: TL + Resistive Load
Example, t=0

\[ V_0 - I^+ R_g - V^+ = 0 \]

\[ I^+ = \frac{V^+}{Z_0} \]

At t=0, a wave originates at z=0 and starts to travel in the +z direction.

Until the wave propagates to the end and reflects back, there is no \( V^- \) wave, and the load resistance \( R_L \) has no effect.

\[ V^+ = \tau_g V_0 \]

\[ V^+ = \frac{Z_0}{R_g + Z_0} V_0 \]

\[ I^+ = \frac{V_0}{R_g + Z_0} \]

\( \tau_g \) is the injection coefficient.
Example \( t=T \)

Time required for \( V^+ \) wave to reach load end of the TL

\[
T = \frac{l}{v_p}
\]

\[
V^+ + V^- = R_L (I^+ + I^-)
\]

Wave reflects to set up a “-” wave IN ADDITION TO the “+” wave

\[
V^- = V^+ \frac{R_L - Z_0}{R_L + Z_0}
\]

Equation 7.50a,b should say \( V^+ + V^- \) instead of \( V^+ - V^- \)
Definition: Reflection Coefficient

Voltage Reflection Coefficient

$$\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_0}{R_L + Z_0}$$

Current Reflection Coefficient

$$\frac{I^-}{I^+} = -\left(\frac{V^-}{Z_0}\right) = -\frac{V^-}{V^+} = -\Gamma$$
Example, $t=2T$

Reflected wave travels back towards the source, and gets there at $t=2T$

The wave gets RE-REFLECTED at the source end and travels back toward the load as a “+” wave

The re-reflection process continues forever until steady state conditions are reached
Special Case TL Terminations

Short-circuited line: \( R_L = 0 \)

\[
\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_0}{R_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1
\]

No voltage across \( R_L \)

Open-circuited line: \( R_L = \infty \)

\[
\Gamma = \frac{V^-}{V^+} = \frac{\infty - Z_0}{\infty + Z_0} = 1
\]

No current across \( R_L \)

Impedance-matched line: \( R_L = Z_0 \)

\[
\Gamma = \frac{V^-}{V^+} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0
\]

No reflection at \( R_L \)
First step: Calculate $V^+$, $I^+$, $\Gamma_{\text{load}}$, $\Gamma_{\text{source}}$

\[ V^+ = V_0 \frac{Z_0}{R_g + Z_0} = 100 \frac{60}{40 + 60} = 60V \]
\[ I^+ = \frac{V^+}{Z_0} = \frac{60}{60} = 1A \]
\[ \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{120 - 60}{120 + 60} = \frac{1}{3} \]
\[ \Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = \frac{40 - 60}{40 + 60} = -\frac{1}{5} \]

Second step: Construct 2 bounce diagrams (Voltage and Current)
**Voltage**

- $\Gamma = -1/5$
  - $t=0$
  - $V=0V$
  - $V^+=60V$
  - $V=60V$
  - $V=20V$
  - $V=80V$

- $\Gamma = 1/3$
  - $t=0$
  - $V=0V$
  - $I^+=1A$
  - $I=1A$
  - $I=2/3A$

**Current**

- $\Gamma = 1/5$
  - $t=0$
  - $I=0A$
  - $I^+=1A$
  - $I=-1/3A$

- $\Gamma = -1/3$
  - $t=0$
  - $I=0A$
  - $I^+=1A$
  - $I=-1/3A$
  - $I=2/3A$
Voltage

\[ Z_0 \]

\( z=0 \) \hspace{1cm} \( z=1 \)

\[ \Gamma=-1/5 \] \hspace{1cm} \[ \Gamma=1/3 \]

\( t=0 \)

\( \mu \text{sec} \)

1 \( \mu \text{sec} \)

2 \( \mu \text{sec} \)

3 \( \mu \text{sec} \)

4 \( \mu \text{sec} \)

5 \( \mu \text{sec} \)

6 \( \mu \text{sec} \)

\[ V=60V \] \hspace{1cm} \[ V=0V \]

\[ V=20V \] \hspace{1cm} \[ V=80V \]

\[ V^{-}=60V \] \hspace{1cm} \[ V^{-}=0V \]

\[ V^{-}=20V \] \hspace{1cm} \[ V^{-}=80V \]

\[ V^{-}=76V \] \hspace{1cm} \[ V^{-}=76V \]

\[ \Gamma=-1/5 \] \hspace{1cm} \[ \Gamma=1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ I=1A \] \hspace{1cm} \[ I^{-}=1A \]

\[ I=1A \] \hspace{1cm} \[ I^{-}=1A \]

\[ I=2/3A \] \hspace{1cm} \[ I^{-}=2/3A \]

\[ I=2/3A \] \hspace{1cm} \[ I^{-}=2/3A \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

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\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]

\[ \Gamma=1/5 \] \hspace{1cm} \[ \Gamma=-1/3 \]
Plotting the Line Voltage/Current

The bounce diagram is a 2D plot! Just have to sketch a cross sectional graph of the data in blue.
Question: What is $V(z)$ at $t=0.25 \mu\text{sec}$?
Question: What is $V(z)$ at $t=0.75 \, \mu\text{sec}$?
Question: What is $V(z)$ at $t=1.5 \ \mu\text{sec}$?
Question: What is $V(t)$ at $z=0.25\times l$ m?
Challenge Question: Bounce Diagram

- What is $V(L, 1.01 \mu s)$?
  (a) 10V
  (b) 15V
  (c) 20V
  (d) 25V
  (e) None of these

LG’s question
ECE 329
Lecture 29

Algebra of the Bounce Diagram
Steady State

If $\Gamma_s \Gamma_l < 1$, then eventually, the magnitude of the reflections will die down, and the voltage and current reach constant, steady state values

\[
\begin{align*}
V_{ss}^+ &= 60(1 - \frac{1}{15} + \frac{1}{15^2} + \ldots) \\
V_{ss}^- &= 20(1 - \frac{1}{15} + \frac{1}{15^2} + \ldots) \\
I_{ss}^+ &= 1(1 - \frac{1}{15} + \frac{1}{15^2} + \ldots) \\
I_{ss}^- &= -\frac{1}{3}(1 - \frac{1}{15} + \frac{1}{15^2} + \ldots)
\end{align*}
\]

Sum of all + waves = 56.25V  \\
Sum of all - waves = 18.75V  \\
Sum of all + waves = 0.9375A  \\
Sum of all – waves = -0.3125A

\[
\sum_{n=0}^{\infty} \left( -\frac{1}{15} \right)^n = \frac{1}{1 + 1/15} = \frac{15}{16}
\]
In SS, TL Looks Like a Wire

\[ V_{SS} = V_{SS}^+ + V_{SS}^- = 75V \]
\[ I_{SS} = I_{SS}^+ + I_{SS}^- = 0.625A \]

\[ V_{SS} = 100V \frac{120\Omega}{(40 + 120)\Omega} = 75V \]
\[ I_{SS} = \frac{100V}{(40 + 120)\Omega} = 0.625A \]
Algebra of the Bounce Diagram

Voltage divider:
\[
\tau_s = \frac{Z_0}{R_s + Z_0}
\]

\[ V^{-}(0,t) = \Gamma_L V^{+}(0,t-2T) \]

One-way transit time: \( T = \frac{l}{v_p} \)

\[
\Gamma_S = \frac{R_s - Z_0}{R_s + Z_0}
\]

\[
\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}
\]

Reflected wave is the forward wave at time 2T ago and multiplied by reflection coefficient \( \Gamma_L \)

\[
V(0,t) = V^{+}(0,t) + V^{-}(0,t) = f(t) - R_s I(0,t)
\]

\[
I(0,t) = I^{+}(0,t) + I^{-}(0,t) = \frac{V^{+}(0,t) - V^{-}(0,t)}{Z_0}
\]

New wave is source + old wave 2T ago multiplied by RT reflection coeff. \( \Gamma_L \Gamma_S \)

\[
V^{+}(0,t) = \tau_s f(t) + \Gamma_S \Gamma_L V^{+}(0,t-2T)
\]
Impulse Response at $z=0$

For $f(t) = \delta(t)$, the solution of:

$$V^+(0,t) = \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t - n2T) \equiv h^+(t)$$

$$V^-(0,t) = \Gamma_L V^+(0,t-2T) = \Gamma_L \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t - (n+1)2T) \equiv h^-(t)$$

Shown for: $\Gamma_S = 0.8$, $\Gamma_L = 0.9$
Writing Moving Delta Functions

\[ \delta(t) \quad \text{a pulse centered at } t = 0 \]

\[ \delta(t - \frac{z}{v}) \quad \text{a forward-moving pulse centered at } z = vt \]

\[ \delta([t - 2T] - \frac{z - 0}{v}) = \delta(t - \frac{z}{v} - 2T) \quad \text{a forward-moving pulse that passes through } (z, t) = (0, 2T) \]

\[ \delta(t + \frac{z}{v}) \quad \text{a backward-moving pulse centered at } z = -vt \]

\[ \delta([t - T] + \frac{z - L}{v}) = \delta(t + \frac{z}{v} - 2T) \quad \text{a backward-moving pulse that passes through } (z, t) = (L, T) \]

because \( T = L/v \)
Impulse Response for any (z,t)

\[ V^+(0,t) = \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t - n2T) \equiv h^+(t) \]

\[ V^-(0,t) = \Gamma_L \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t - (n+1)2T) \equiv h^-(t) \]

For arbitrary position z, replace t with \( t \pm \frac{z}{v_p} \)

\[ V^+(z,t) = \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t - \frac{z}{v_p} - n2T) \]

\[ V^-(z,t) = \Gamma_L \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t + \frac{z}{v_p} - (n+1)2T) \]

Note: the voltage is nonzero only on the bounce lines
General Solution for any Source

For $f(t)=\delta(t)$, the solution was:

$$V^+(z,t) = \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t - \frac{z}{v_p} - n2T)$$

$$V^-(z,t) = \Gamma_L \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t + \frac{z}{v_p} - (n+1)2T)$$

$$I = \frac{1}{Z_0} (V^+ - V^-)$$

For arbitrary $f(t)$, convolve the solution with $f$:

$$V^+(z,t) = \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n f(t - \frac{z}{v_p} - n2T)$$

$$V^-(z,t) = \Gamma_L \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n f(t + \frac{z}{v_p} - (n+1)2T)$$

$$I = \frac{1}{Z_0} (V^+ - V^-)$$

Note: the voltage can be nonzero in between the bounce lines depending on the function $f$
Challenge Question: Bounce Diagram

- What is \( I(L, \infty) \)?
  (a) 0.0A
  (b) 0.5A
  (c) 0.6A
  (d) 1.0A
  (e) None of these
Lectures 27-29 Summary

- **Transmission Line Equations**
  \[
  \frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t} \quad \frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t}
  \]

- **Characteristic impedance & Speed**
  \[
  Z_0 = \sqrt{\frac{L}{C}} = \left| \frac{V(z,t)}{I(z,t)} \right| \quad v_p = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{LC}}
  \]

- **Bounce Diagram**

- **Round Trip Equation**
  \[
  V^+(0, t) = \tau_s f(t) + \Gamma_s \Gamma_L V^+(0, t - 2T)
  \]
  \[
  V^+(z, t) = \tau_s \sum_{n=0}^{\infty} \left( \Gamma_s \Gamma_L \right)^n f(t - \frac{z}{v_p} - n2T)
  \]
  \[
  I = \frac{1}{Z_0} (V^+ - V^-)
  \]
  \[
  V^-(z, t) = \Gamma_L \tau_s \sum_{n=0}^{\infty} \left( \Gamma_s \Gamma_L \right)^n f(t + \frac{z}{v_p} - (n+1)2T)
  \]
ECE 329
Lecture 30

TL Discontinuities

(Optional) TL Circuits with Reactive Elements
TL Discontinuity

t=0, close

$V_0$ $R_g$ $Z_{01} \, v_{p1}$ $Z_{02} \, v_{p2}$ $R_L$

incident

transmitted

reflected
What happens at the boundary between two TL’s?

Assume that (+) wave hits the junction from the left

Assume that (+) wave hits the junction from the left
Before + wave hits boundary

\[ V^+, I^+ \]

\[ Z_{01} \quad \text{incident} \quad Z_{02} \]

As + wave hits boundary

\[ V^{++}, I^{++} \]

\[ V^+, I^- \]

\[ I^+, I^- \]

\[ Z_{01} \quad \text{incident} \quad Z_{02} \quad \text{transmitted} \]

\[ \text{reflected} \]
Calculation Space
Voltage Reflection Coeff

\[ \Gamma = \frac{V^-}{V^+} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \]

Fraction of incident voltage that gets reflected back

Voltage Transmission Coeff

\[ \tau_v = \frac{V^{++}}{V^+} = \frac{V^+ + V^-}{V^+} = 1 + \frac{V^-}{V^+} \]

\[ \tau_v = 1 + \Gamma \]

Fraction of incident voltage that gets transmitted through
Current Transmission Coeff

\[ \tau_c = \frac{I^{++}}{I^+} = \frac{I^+ + I^-}{I^+} = 1 + \frac{I^-}{I^+} \]

\[ \tau_c = 1 - \Gamma \]

Fraction of incident current that gets transmitted through
Power

\[ P_{\text{incident}} = V^+ I^+ \]

\[ P_{\text{reflected}} = V^- I^- \]

\[ P_{\text{transmitted}} = V^{++} I^{++} \]

Some of the incident power is reflected back and the rest is transmitted through to the second line.
TL Discontinuity Looks Like Load Resistor with $R=Z_{02}$

The reflection coefficient for the two configurations are the same, but power is not dissipated in the first case, rather it is transmitted down the line.

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$
Challenge Question: TL Discontinuity

- For a right moving wave (into the splitter), the fraction of power that is reflected is:
  (a) 1/9, (b) 1/4, (c) 1/3, (d) 1/2, (e) 2/3

Given your answer, what is the power transmitted down each line?
Example

\[ \delta(t) \] is a unit impulse function.

First step: Calculate \( V^+ \), \( I^+ \), and \( \Gamma \) at each location

Second step: Draw a bounce diagram to determine magnitude of pulses that come out the end of the third TL
Step 1: $V^+, I^+, \Gamma$

\[ V^+ = V_0 \frac{Z_0}{R_g + Z_0} = 1\delta \frac{50}{50 + 50} = 0.5\delta V \]

\[ I^+ = \frac{V^+}{Z_0} = \frac{0.5\delta}{50} = 0.01\delta A \]
\[ \begin{align*}
\delta(t) & \quad V^+ = 0.5 \delta \\
Z_{01} = 50 \Omega & \quad T = 2 \mu\text{sec} \\
Z_{02} = 100 \Omega & \quad T = 2 \mu\text{sec} \\
Z_{03} = 50 \Omega & \quad T = 2 \mu\text{sec}
\end{align*} \]
\[ V^+ = 0.5\delta \]

\[ V^+ = 0.67\delta \]
\( Z_{01} = 50 \Omega \)  
\( Z_{02} = 100 \Omega \)  
\( Z_{03} = 50 \Omega \)

\[ \delta(t) \]

\( t = 0 \mu\text{sec} \)
\( t = 2 \mu\text{sec} \)
\( t = 4 \mu\text{sec} \)
\( t = 6 \mu\text{sec} \)
\( t = 8 \mu\text{sec} \)
\( t = 10 \mu\text{sec} \)
\( t = 12 \mu\text{sec} \)

\( V^+ = 0.5\delta \)
\( V^+ = 0.67\delta \)
\( V^- = -0.22\delta \)
\( V^+ = 0.444\delta \)

\( \tau_v = 4/3 \)
\( \tau_v = 2/3 \)

\( \Gamma = 0 \)
\( \Gamma = 1/3 \)
\( \Gamma = -1/3 \)

\( T = 2 \mu\text{sec} \)
\[ t=0 \quad \mu\text{sec} \]
\[ t=2 \quad \mu\text{sec} \]
\[ t=4 \quad \mu\text{sec} \]
\[ t=6 \quad \mu\text{sec} \]
\[ t=8 \quad \mu\text{sec} \]
\[ t=10 \quad \mu\text{sec} \]
\[ t=12 \quad \mu\text{sec} \]

\[ Z_{01} = 50\Omega \quad Z_{02} = 100\Omega \quad Z_{03} = 50\Omega \]

\[ \delta(t) \quad \Gamma = 0 \quad \Gamma = 1/3 \quad \Gamma = -1/3 \quad \Gamma = -1/3 \quad \Gamma = 1/3 \quad \Gamma = 0 \]

\[ \tau_v = 4/3 \quad \tau_v = 2/3 \quad \tau_v = 2/3 \]

\[ V^+ = 0.5\delta \quad V^+ = 0.67\delta \quad V^- = -0.22\delta \quad V^+ = 0.444\delta \quad V^+ = 0.074\delta \]
Example 2

V^+ wave incident from the left
What is reflected voltage and current into line 1?
What is transmitted voltage and current into line 2?
Equivalent circuit “seen” by $V+$ when it gets to the end of line 1:

$$\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_{01}}{R_L + Z_{01}}$$

What is $R_L$ for this equivalent circuit?

$$R_L = (R_1) \parallel (R_2 + Z_{02})$$
How much voltage gets transmitted through to line 2?

\[
\tau_v = \frac{V^{++}}{V^+} = 1 + \Gamma
\]

\[
V^{++} = V^+ (1 + \Gamma)
\]

\[
V_{trans} = \frac{Z_{02}}{Z_{02} + R_2} V^{++}
\]
Time Domain Reflectometry

A powerful way to analyze discontinuities and resistive or reactive elements in transmission lines. Method: Send a very narrow impulse (pulse width << $T_0$) down the line and measure the reflected voltage waveform. Reverse the bounce diagram calculation to determine the nature of the discontinuities.

Transmission line or device under test

$v_p = 10^8 \text{m/s}$

$T = T_0 = 1 \text{ns}$

$Z_{01} = 50 \Omega$

$V(t)$

$1 \text{V}$
Time Domain Reflectometry

Transmission line or device under test $v_p = 10^8 \text{m/s}$

$V(z=0,t)$

$T = T_0 = 1\text{ns}$

$Z_0 = 50\Omega$

$50\Omega$

$V(t)$

$z = 0$

$V(t)$

$1\text{V}$

$V(z=0,t)$

$0.5\text{V}$

$0.25\text{V}$

$2\text{ns}$

$5\text{ns}$

$-0.375\text{V}$
Transmission line or device under test

\( v_p = 10^8 \text{m/s} \)

\( Z_{01} = 50 \Omega \)

\( T = T_0 = 1 \text{ns} \)

\( T = 0 \text{ns} \)

\( T = 1 \text{ns} \)

\( T = 2 \text{ns} \)

\( T = 3 \text{ns} \)

\( T = 4 \text{ns} \)

\( T = 5 \text{ns} \)

\( T = 6 \text{ns} \)

\( \Gamma = 0 \)

\( \Gamma = \frac{1}{2} \)

\( \Gamma = -\frac{1}{2} \)

\( \Gamma = ??? \)

\( \tau_v = \frac{3}{2} \)

\( \tau_v = \frac{1}{2} \)

\( V^+ = 0.5 \)

\( V^+ = 0.25 \)

\( V^+ = 0.75 \)

\( V^- = ??? \)

\( V^- = -0.375 \)

\( V^+ = ??? \)
Transmission line or device under test
\[ v_p = 10^8 \text{m/s} \]

- \[ Z_{01} = 50 \Omega \]
- \[ T = T_0 = 1 \text{ns} \]
- \[ V(t) \]
- \[ T \]
- \[ Z_{02} = 150 \Omega \]
- \[ \text{Short at } z = 0.15 \text{m} \]
Lecture 30 Summary

• TL discontinuity looks like a load resistor with $R = Z_{02}$ but no power is dissipated

\[ \Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \]

• TDR reverses bounce diagram calculation to infer TL properties

• Next class
  – Chapter 7 (TL in frequency domain)
ECE 329
Lecture 30b

TL Circuits with Reactive Elements
What happens?

• A “+” wave originates at \( t=0 \)
• The + wave travels down the TL until it reaches the end at time \( T \)
• The wave meets the inductor and a reflected “-” wave begins
• The \( V^- \) and \( I^- \) waves will be a function of time, because of the inductor
(Optional)

Inductive Termination

t=0+

\[ V^+ = \frac{V_0}{2} \]

\[ I^+ = \frac{V^+}{Z_0} = \frac{V_0}{2Z_0} \]
For the inductor:

\[ V = L \frac{dI}{dt} \]
(Optional)

Two ways to solve

• Rigorous mathematical method
  – Laplace Transform
    • We will do this first

• Shortcut method
Diff Eqn for the Inductor

\[ V = L \frac{dI}{dt} \quad \Rightarrow \quad (V^+ + V^-(t)) = L \frac{d(I^+ + I^-(t))}{dt} \]

\[ \frac{V_0}{2} + V^-(t) = L \frac{d}{dt} \left( \frac{V_0}{2Z_0} - \frac{V^-(t)}{Z_0} \right) \]

\[ \frac{V_0}{2} = - \frac{L}{Z_0} \frac{dV^-(t)}{dt} - V^-(t) \]

\[ \frac{dV^-(t)}{dt} + \frac{Z_0}{L} V^-(t) = - \frac{Z_0}{L} \frac{V_0}{2} \]

Laplace Transform \( V(t') \) to \( F^-(s) \)

\[ s \hat{V}^- (s) - V^-(0) + \frac{Z_0}{L} \hat{V}^- (s) = - \frac{Z_0}{L} \frac{V_0}{2} \frac{1}{s} \]
Laplace Transform for Inductor

**Optional**

**Laplace Transform for Inductor**

**Initial Condition**

At $t=T$, i.e. $t'=0$, inductor current $= 0$

Since inductor “looks” like an OPEN CIRCUIT

$$I = I^+ + I^- (T) = 0 \quad \Rightarrow \quad \frac{V_0}{2Z_0} - \frac{V^- (0)}{Z_0} = 0 \quad \Rightarrow \quad V^- (0) = \frac{V_0}{2}$$

$$s \hat{V}^- (s) - \frac{V_0}{2} + \frac{Z_0}{L} \hat{V}^- (s) = - \frac{Z_0}{L} \frac{V_0}{2} \frac{1}{s} \Rightarrow \hat{V}^- (s) = \frac{sL - Z_0}{sL + Z_0} \frac{V_0}{2s}$$

In s-space, we have $V^-(s) = \Gamma(s) V^+(s)$ with:

$$\Gamma(s) = \frac{Z(s) - Z_0}{Z(s) + Z_0} \quad \hat{V}^+ (s) = \frac{V_0}{2s} \quad Z(s) = sL \text{ for an inductor}$$
(Optional)

Invert Laplace Transform

\[ \hat{V}(s) = \hat{V}^+ + \hat{V}^- = \frac{V_0}{2s} + \frac{V_0}{2s} \frac{sL - Z_0}{sL + Z_0} = \frac{V_0}{2s} \frac{2sL}{sL + Z_0} = \frac{V_0}{s + Z_0 / L} \]

\[ V(t) = V_0 e^{-(Z_0 / L)t} = V_0 e^{-(Z_0 / L)(t-T)} \quad \text{for } t > T \]

\[ V(t) \text{ [volts]} \]

\[ \text{Graph: } V_0 \text{ at } t = T, \text{ decaying exponential } \]

Goddard/Cunningham

ECE329 Lectures 27-30

87
(Optional)
Shortcut Method

\[ V(t) = V_0 e^{-(Z_0 / L)(t-T)} \quad \text{t>T} \]

\( V(t) \) [volts]

At \( t=T \) the inductor “looks” like an OPEN CIRCUIT

At \( t=\infty \) the inductor “looks” like a CLOSED CIRCUIT
(Optional)
Shortcut Method

\( t = T^+ \)

\[ V^+ + V^-(T) = 0 \]

\( t = \infty \)

\[ V^+ + V^-(t) = 0 \]

\[ I^+ + I^-(T) = 0 \]

\[ I^+ + I^-(t) \]
Shortcut Method

What happens “in between”?

We know \( V(t) \) has exponential decay with time constant of \( L/Z_0 \)

At \( t=T \) the inductor “looks” like an open ckt

At \( t=\infty \) the inductor “looks” like a closed ckt
Summary of Shortcut Method

• Solve for voltage and current with inductive or reactive elements in their initial uncharged state
  – Inductor: Open
  – Capacitor: Short

• Solve for voltage and current with elements in their final charged state
  – Inductor: Short
  – Capacitor: Open

• Solve for circuit time constant for exponential function that occurs between initial and final states

• **Warning**: Method can only be used for reactive elements charged by a DC source with no back reflections
(Optional)

Shortcut Method

\[ \tau = Z_{eq} C \]

\[ \tau = \frac{L}{Z_{eq}} \]
Shortcut Method
Inductor Voltage / Capacitor Current

\[ Ae^{-t/\tau} \]

For \( t > T \)

\[ Ae^{-(t-T)/\tau} \]

For \( t > T \)
Shortcut Method
Inductor Current / Capacitor Voltage

\[ A(1 - e^{-t/\tau}) \]

For \( t > T \)

\[ A(1 - e^{-(t-T)/\tau}) \]
No initial charge or current. Determine the unknown single element (R, L, or C) and the ratio $Z_{02}/Z_{01}$. 

Measured voltage at $z=0$.
• Reactive terminations and discontinuities require solving differential equations or studying start/end behavior & time constants
  – Reflection coefficient $\Gamma$ varies in time

\[
\tau = Z_{eq} C
\]

\[
\tau = \frac{L}{Z_{eq}}
\]

• Next class
  – Chapter 7 (TL in frequency domain)
ECE 329
Lectures 31-34

Sections 7.A, 7.3, 7.1
(Section 31-34 in Online Notes)

Line Terminated by an Arbitrary Load
Smith Charts
Short and Open Circuited TLs
Half and Quarter Wave Transformers
Example

• Find $\Gamma_L$ for a TL with $Z_0=60\Omega$ that is terminated with an RLC series combo: $R=30\Omega$, $L=1\mu\text{H}$, $C=100\text{pF}$ at the following radian frequencies (a) $\omega=10^8$ and (b) $\omega=2\times10^8$
  - Hint: $Z_c=1/(j\omega C)$ and $Z_L=j\omega L$
TL’s in Co-sinusoidal steady state

\[ I(d) \]

\[ V(d) \]

\[ Z_0 \]

\[ G = V_g \]

\[ Z_g \]

\[ F = V_g \]

\[ z = -l \]

\[ z = 0 \]

\[ d = l \]

\[ d = 0 \]

Distributed circuits are best handled with phasor and Fourier techniques and with Smith charts.

\[ f(t) = \text{Re} [\tilde{F} e^{j\omega t}] \]

\[ V(z,t) = \text{Re} [\tilde{V}(z)e^{j\omega t}] \]

\[ I(z,t) = \text{Re} [\tilde{I}(z)e^{j\omega t}] \]
Phasors satisfy usual TL equations

\[
\frac{\partial V}{\partial z} = -\mathcal{L} \frac{\partial I}{\partial t} \quad \frac{\partial I}{\partial z} = -\mathcal{C} \frac{\partial V}{\partial t}
\]

\[
\begin{align*}
\frac{d\tilde{V}}{dz} &= -\mathcal{L}(j\omega \tilde{I}) \\
\frac{d\tilde{I}}{dz} &= -\mathcal{C}(j\omega \tilde{V})
\end{align*}
\]

\[
\frac{d^2\tilde{V}}{dz^2} = -\mathcal{L}\mathcal{C}\omega^2 \tilde{V} \quad \frac{d^2\tilde{I}}{dz^2} = -\mathcal{L}\mathcal{C}\omega^2 \tilde{I}
\]

\[
\tilde{V} = V^\pm e^{\pm j\beta z}
\]

\[
\beta = \omega \sqrt{\mathcal{L}\mathcal{C}}
\]

\[
\tilde{V} = V^+ e^{j\beta d} + V^- e^{-j\beta d}
\]

\[
\tilde{I} = \frac{1}{Z_0} \left( V^+ e^{j\beta d} - V^- e^{-j\beta d} \right)
\]

\[
\tilde{I} = \pm \frac{V^\pm}{Z_0} e^{\pm j\beta z}
\]

\[
Z_0 = \sqrt{\mathcal{L}/\mathcal{C}}
\]
Boundary Condition at Load

\[ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \]

\[ V^- = \Gamma_L V^+ \]

\[ \tilde{V}(d) = V^+ e^{j\beta d} + V^- e^{-j\beta d} = V^+ e^{j\beta d} (1 + \Gamma_L e^{-2j\beta d}) \]

\[ \tilde{I}(d) = \frac{1}{Z_0} (V^+ e^{j\beta d} - V^- e^{-j\beta d}) = \frac{V^+}{Z_0} e^{j\beta d} (1 - \Gamma_L e^{-2j\beta d}) \]
Key Definition: Line Impedance

\[
Z(d) \equiv \frac{\tilde{V}(d)}{\tilde{I}(d)}
\]

\[
Z(d) = Z_0 \frac{1 + \Gamma L e^{-2 j \beta d}}{1 - \Gamma L e^{-2 j \beta d}}
\]

\[
\tilde{V}(l) = \tilde{F} \frac{Z(l)}{Z(l) + Z_g} = V^+ (e^{j \beta l} + \Gamma L e^{-j \beta l})
\]

Allows you to solve for \( V^+ \) and thus get \( V(d,t) \) and \( I(d,t) \)
**Challenge Question: Line Impedance**

\[ Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} \]

\[ Z(d) = Z_0 \frac{1 + \Gamma_L e^{-2j\beta d}}{1 - \Gamma_L e^{-2j\beta d}} \]

- What is the smallest distance \( d \) from the load for which input impedance = load impedance?

(a) \( d_{\text{min}} = \lambda/8 \)
(b) \( d_{\text{min}} = \lambda/4 \)
(c) \( d_{\text{min}} = \lambda/3 \)
(d) \( d_{\text{min}} = \lambda/2 \)
(e) \( d_{\text{min}} = \lambda \)
Challenge Question: Line Impedance

\[
Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)}
\]

\[
Z(d) = Z_0 \frac{1 + \Gamma L e^{-2j\beta d}}{1 - \Gamma L e^{-2j\beta d}}
\]

- What is the smallest distance \(d\) from the load for which input voltage = load voltage?

(a) \(d_{\text{min}} = \lambda/8\)
(b) \(d_{\text{min}} = \lambda/4\)
(c) \(d_{\text{min}} = \lambda/3\)
(d) \(d_{\text{min}} = \lambda/2\)
(e) \(d_{\text{min}} = \lambda\)
Key Definition: Generalized Reflection Coefficient

\[ \Gamma(d) \equiv \frac{\tilde{V}^-(d)}{\tilde{V}^+(d)} \]

\[ \Gamma(d) = \frac{V^- e^{-j\beta d}}{V^+ e^{j\beta d}} = \Gamma_L e^{-2j\beta d} \]

Allows you to find the backwards wave if forward wave is known
Key Definitions: Admittance and Normalized Impedance

Characteristic Admittance

\[ Y_0 \equiv \frac{1}{Z_0} \]

Normalized Impedance

\[ z(d) \equiv \frac{Z(d)}{Z_0} \]

\[ z(d) = \frac{1 + \Gamma(d)}{1 - \Gamma(d)} \]

Normalized Admittance

\[ y(d) \equiv \frac{1}{z(d)} \]

\[ y(d) = \frac{1 - \Gamma(d)}{1 + \Gamma(d)} \]

\[ \Gamma(d) = \Gamma_L e^{-2j\beta d} \]
Converting between Impedance and Reflection

\[ \Gamma(d) = \Gamma_L e^{-2j\beta d} \]

\[ z(d) = \frac{1 + \Gamma(d)}{1 - \Gamma(d)} \]

Invert to solve for \( \Gamma(d) \):

\[ \Gamma(d) = \frac{z(d) - 1}{z(d) + 1} \]
Smith Chart

\[
\Gamma = \frac{z - 1}{z + 1}
\]

\(z = r + jx\)

\[r \geq 0\]

\[
\Gamma = \Gamma_r + j\Gamma_i
\]

\[
\Gamma_r = \frac{r^2 + x^2 - 1}{(r + 1)^2 + x^2}
\]

\[
\Gamma_i = \frac{2x}{(r + 1)^2 + x^2}
\]

\[|\Gamma| \leq 1\]
Short and Open Circuited Lines
Standing Waves for SC Line

\[ \tilde{V}(d) = V^+ (e^{j\beta d} - e^{-j\beta d}) = 2jV^+ \sin(\beta d) \]

\[ \tilde{I}(d) = \frac{V^+}{Z_0} (e^{j\beta d} + e^{-j\beta d}) = 2Y_0V^+ \cos(\beta d) \]

\[ Z(d) \equiv \frac{\tilde{V}(d)}{\tilde{I}(d)} = jZ_0 \tan \beta d \]
Standing Waves for SC Line

\[ V(d,t) = \text{Re}[\tilde{V}(d)e^{j\omega t}] = \text{Re}[2jV^+ \sin(\beta d)e^{j\omega t}] \]

\[ = \text{Re}[2e^{j\pi/2}|V^+|e^{j\theta} \sin(\beta d)e^{j\omega t}] \]

\[ = 2|V^+|\sin(\beta d)\text{Re}[e^{j(\omega t+\theta+\pi/2)}] \]

\[ = 2|V^+|\sin(\beta d)\cos(\omega t+\theta+\pi/2) \]

\[ = -2|V^+|\sin(\beta d)\sin(\omega t+\theta) \]

\[ I(d,t) = 2Y_0|V^+|\cos(\beta d)\cos(\omega t+\theta) \]
Standing Waves for SC Line

\[ V(d, t) = -2V^+ \sin(\beta d) \sin(\omega t + \theta) \]

\[ I(d, t) = 2Y_0 V^+ \cos(\beta d) \cos(\omega t + \theta) \]

V(0,t)=0 always (voltage null)
I(0,t) varies (current maxima)

Dependence is different than traveling wave: \( \omega t \pm \beta z \)
Input Impedance for SC Line

\[ Z_{in} = Z(l) = jZ_0 \tan \beta l = jZ_0 \tan \frac{2\pi fl}{v_p} \]

\[ Z_0 \hspace{1cm} \text{Is equivalent to:} \hspace{1cm} Z_{in} \]
SC Line can act as an inductor or a capacitor depending on $\beta l$

$$Z_{in} = jZ_0 \tan \beta l$$

If $\tan(\beta l) > 0$, shorted
TL is inductive

If $\tan(\beta l) < 0$, shorted
TL is capacitive

e.g. $\beta l < \pi/2$
or $l < \lambda/4$, TL is inductive

Equivalent Circuit:

TL looks like an open at $\lambda/4$
Shorted Load
TL looks like a short at $\lambda/2$
Show that the equivalent inductance is \( L = \frac{\mathcal{L}}{l} \) if \( \lambda >> 4l \)

\[
Z_{\text{in}} = j Z_0 \tan \beta l \\
Z_L = j \omega L
\]
For \( l = \text{even } \lambda/4 \), TL is a short
For \( l = \text{odd } \lambda/4 \), TL is an open

\[
Z_{\text{in}} = jZ_0 \tan \beta l =
\begin{cases}
0 & \text{a short for } \beta l = n\pi, \ n = 0,1,2,\ldots \\
\infty & \text{an open for } \beta l = (n+1/2)\pi
\end{cases}
\]

If \( Z_{\text{in}} = 0 \), voltage drop is zero, just like a short
If \( Z_{\text{in}} = \infty \), current is zero, just like an open
Example

• A TL is shorted and your job is to find the location to fix it. You are equipped with a tunable frequency generator and an ammeter. Compare this to TDR.
Exercise

Locate $z(0)$ and $\Gamma(0)$ of the shorted line on the S.C., and observe how $z(d)$ and $\Gamma(d)$ vary as $d$ increases, noting in particular to what happens at $d = \lambda/4$ (open conditions are reached) and at $d = \lambda/2$ (back to short conditions).
Standing Waves for OC Line

\[ \tilde{V}(d) = V^+ (e^{j\beta d} + e^{-j\beta d}) = 2V^+ \cos(\beta d) \]

\[ \tilde{I}(d) = \frac{V^+}{Z_0} (e^{j\beta d} - e^{-j\beta d}) = 2jY_0 V^+ \sin(\beta d) \]

\[ Y(d) \equiv \frac{1}{Z(d)} = \frac{\tilde{I}(d)}{\tilde{V}(d)} = jY_0 \tan \beta d \]
Standing Waves for OC Line

Same phasor algebra as before with current & voltage reversed!

\[ I(d,t) = -2Y_0 V^+ \sin(\beta d) \sin(\omega t + \theta) \]

\[ V(d,t) = 2 V^+ \cos(\beta d) \cos(\omega t + \theta) \]

I(0,t)=0 always (current null)
V(0,t) varies (voltage maxima)
Input Admittance for OC Line

\[ Y_{in} = Y(l) = jY_0 \tan \beta l = jY_0 \tan \frac{2\pi fl}{v_p} \]

\[ Z_0 \]

Is equivalent to:

\[ Z_{in} = \frac{1}{Y_{in}} \]
OC Line can act as an inductor or a capacitor depending on $\beta l$

$$Y_{in} = jY_0 \tan \beta l$$

If $\tan(\beta l)<0$, shorted TL is inductive

If $\tan(\beta l)>0$, shorted TL is capacitive

e.g. $\beta l<\pi/2$
or $l<\lambda/4$, TL is capacitive

Equivalen Circuit:

$$\text{TL looks like a short at } \lambda/4$$

$$\text{TL looks like an open at } \lambda/2$$

Input Admittance / $Y_0$ (1/ohms)

$\beta \cdot L$ in units of $2\pi$

Open Load
Show that the equivalent capacitance is $C = \varepsilon l$ if $\lambda >> 4l$

\[ Y_{in} = jY_0 \tan \beta l \]
\[ Y_C = j\omega C \]
For $l=\text{even } \lambda/4$, TL is an open
For $l=\text{odd } \lambda/4$, TL is a short

\[ Y_{in} = jY_0 \tan \beta l = \begin{cases} 
0 = \text{an open for } \beta l = n\pi, & n = 0,1,2,\ldots \\
\infty = \text{a short for } \beta l = (n + 1/2)\pi
\end{cases} \]

If $Y_{in}=0$, current is zero, just like an open
If $Y_{in}=\infty$, voltage drop is zero, just like a short
Exercise

Locate $z(0)$ and $\Gamma(0)$ of the open circuited line on the S.C., and observe how $z(d)$ and $\Gamma(d)$ vary as $d$ increases, noting in particular to what happens at $d = \lambda/4$ (short conditions are reached) and at $d = \lambda/2$ (back to open conditions).
y(d) is z(d) shifted by $\lambda/4$

• Proof:

$$z(d \pm \lambda/4) = \frac{1 + \Gamma(d \pm \lambda/4)}{1 - \Gamma(d \pm \lambda/4)} = \frac{1 + \Gamma(d)e^{\pm 2j\beta\lambda/4}}{1 - \Gamma(d)e^{\pm 2j\beta\lambda/4}}$$

$$= \frac{1 + \Gamma(d)e^{\mp j\pi}}{1 - \Gamma(d)e^{\mp j\pi}} = \frac{1 - \Gamma(d)}{1 + \Gamma(d)} = \frac{1}{z(d)}$$

$$y(d) = z(d \pm \lambda/4)$$

Thus, the Smith Chart can be used for both z and y

Opens and shorts exchange with each other every $\lambda/4$
Lectures 31-32 Summary

- As $d$ increases by $\lambda/4$, SC and OC TL switch from being a short ($V=0$) to an open ($I=0$)
- Smith Chart is a bilinear transformation of the half plane $z(d)$ ($r \geq 0$) onto the unit circle $|\Gamma| \leq 1$

$$\Gamma(d) = \frac{z(d) - 1}{z(d) + 1}$$
$$z(d) = \frac{1 + \Gamma(d)}{1 - \Gamma(d)}$$

- How to Use Smith Chart
  1. Calculate $z(0)$ and find it on chart using $r$ and $x$
  2. Find $\Gamma(0)$ as the distance and angle from origin
  3. Move CW along circle of radius $|\Gamma(0)|$ to obtain $\Gamma(d)$
  4. Read off $z(d)$ by looking at grid location $(r, x)$
  5. If needed, find $y(d)$ on the same circle, $180^\circ$ away
ECE 329
Lecture 33

Microwave Resonators
Natural Resonances

- A lossless T.L. of any length $l$ with open and/or short terminations on either ends can be considered a “microwave resonator”
  - It can sustain unforced voltage and current standing-wave oscillations at a set of discrete resonance frequencies $\omega_n$
  - Example, for SC TL, $z_{in}=\infty$ (open) or 0 (short) has standing waves for the set of $d = \lambda_n/4$ satisfying $l=n\lambda_n/4$
Parallel & Series Resonances

- We can find $\lambda_n$ and $\omega_n$ by applying the appropriate BCs at both ends.

- Series resonance if $z_{in}(l)=0$
  - Analogous to an LC circuit in series and requires a short placed across TL at $d=l$.
  - Like a short input.

- Parallel resonance: $y_{in}(l)=0$
  - Analogous to an LC circuit in parallel and requires an open across the TL at $d=l$.
  - Like an open input.

\[ \beta = \frac{2\pi}{\lambda} \]

\[ d = \frac{\lambda}{4} \]
Parallel & Series Resonances
4 simple cases

<table>
<thead>
<tr>
<th>Shorted Load</th>
<th>Open Load</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Z_0</strong></td>
<td><strong>Z_0</strong></td>
</tr>
<tr>
<td><strong>d=l</strong></td>
<td><strong>d=l</strong></td>
</tr>
<tr>
<td><strong>Z_L=0</strong></td>
<td><strong>Z_L=∞</strong></td>
</tr>
<tr>
<td><strong>d=0</strong></td>
<td><strong>d=0</strong></td>
</tr>
</tbody>
</table>

**L=\lambda/4, 3\lambda/4, 5\lambda/4, ...**
Input is like an open
➔ **Parallel** resonance

**L=\lambda/2, \lambda, 3\lambda/2, ...**
Input is like a short
➔ **Series** resonance

**L=\lambda/4, 3\lambda/4, 5\lambda/4, ...**
Input is like a short
➔ **Series** resonance

**L=\lambda/2, \lambda, 3\lambda/2, ...**
Input is like an open
➔ **Parallel** resonance
Examples

a) Find the 3 lowest frequencies for parallel resonances if the load is shorted and $v=c$, $l=3m$.

b) Find the resonance frequencies for a 10m long TL that is open circuited at both ends if $v = 2/3 \, c$. 
ECE 329
Lecture 34

Half-wave and quarter-wave transformers
Half-wave transformers

- Given a fixed drive frequency $\omega$, there is a length of line $l = \lambda/2$ such that:

- Note: Current and voltage both invert their algebraic signs
Half-wave transformers

• Proof:

\[ e^{\pm j\beta \lambda / 2} = e^{\pm j\pi} = -1 \]

\[ \tilde{V}(d) = V^+ e^{j \beta d} + V^- e^{-j \beta d} \]

\[ \therefore \tilde{V}_{in} = \tilde{V}(\lambda / 2) = -V^+ - V^- = -\tilde{V}(0) = -\tilde{V}_L \]

\[ \tilde{I}_{in} = \frac{V^+ e^{j \beta d} - V^- e^{-j \beta d}}{Z_0} = \frac{-V^+ + V^-}{Z_0} = -\tilde{I}_L \]
Challenge Question: Half-wave transformers

What is the input impedance of a half-wave transformer?

(a) $Z_{in} = Z_0$
(b) $Z_{in} = Z_0 + Z_L$
(c) $Z_{in} = 1/(1/Z_0 + 1/Z_L)$
(d) $Z_{in} = Z_L$
(e) $Z_{in} = Z_0^2/Z_L$

LG’s question
Quarter-wave transformers

- Given a fixed drive frequency $\omega$, there is a length of line $l=\lambda/4$ such that:

- Note: The current through the load does not depend on $Z_L$ (current-forcing)
Quarter-wave transformers

- **Proof:**

\[
\begin{align*}
\pm j\beta \lambda/4 &= \pm j\pi/2 = \pm j \\
\tilde{V}(d) &= V^+ e^{j\beta d} + V^- e^{-j\beta d} \\
\tilde{V}_i &= jV^+ - jV^- = j\tilde{I}_L Z_0 \Rightarrow \tilde{I}_L = -j\tilde{V}_i / Z_0 \\
\tilde{I}_i &= \frac{jV^+ - (\pm jV^-)}{Z_0} = j \frac{\tilde{V}_L}{Z_0} \Rightarrow \tilde{V}_L = -j\tilde{I}_i Z_0
\end{align*}
\]
Challenge Question: Quarter-wave transformers

What is the input impedance of a quarter-wave transformer?

(a) $Z_{in} = Z_0$
(b) $Z_{in} = Z_0 + Z_L$
(c) $Z_{in} = 1/(1/Z_0 + 1/Z_L)$
(d) $Z_{in} = Z_L$
(e) $Z_{in} = Z_0^2/Z_L$
Verifying an Identity

- Verify the Lecture 32 result: $y(d) = z(d \pm \lambda/4)$
Half-wave transformer as 2 quarter-wave transformer

\[ \begin{align*}
I_{in} & \quad I_L = -jV_{in}/Z_0 \quad I_L = -I_{in} \\
V_{in} & \quad V_L = -jI_{in}Z_0 \quad V_L = -V_{in} \\
d = \lambda/2 & \quad d = \lambda/4 \quad d = 0
\end{align*} \]
Examples

• For $Z_L=50+50j\Omega$, find $Z_{in}$ for a $\lambda/4$ transformer with $Z_0=50\Omega$.

• Check your answer with a Smith Chart.

• Find $V_{in}$ if a source with open circuit voltage $V_g=100V$ and Thevenin impedance $Z_g=j25\Omega$ is connected.

• Find $I_L$ and $<P_L>$. 
Examples

• For $Z_L = 100\Omega$, find $Z_{in}$ for a $3\lambda/4$ TL if $Z_0 = 50\Omega$.

• Check your answer with a Smith Chart.

• Find $V_L$ and $I_L$ if a source with open circuit voltage $V_g = j10V$ and Thevenin impedance $Z_g = 25\Omega$ is connected.
Lecture 34 Summary

• TL transformers can be used to change the load impedance $Z_L$ to a new value as seen at the input port $Z_{in}$ and thus adjust $V_L$ and $I_L$

• If the TL is $\lambda/4$, then we have current forcing:

$$\tilde{I}_L = -j\tilde{V}_{in} / Z_0$$

$$\tilde{V}_L = -j\tilde{I}_{in}Z_0$$

$$Z_LZ_{in} = Z_0^2$$

• If the TL is $\lambda/2$, then $I$ and $V$ both change their own signs:

$$\tilde{I}_L = -\tilde{I}_{in}$$

$$\tilde{V}_L = -\tilde{V}_{in}$$

$$Z_L = Z_{in}$$
ECE 329
Lectures 35-36
Rao - Sections 7.2, 7.3
Online Notes – 35-36

Line Terminated by an Arbitrary Load
Standing Wave Parameters
Smith Charts
Imposed condition on load end will result in some form of standing wave oscillation:

\[ f(t) = \text{Re}[\tilde{F}e^{j\omega t}] \]
\[ V(z, t) = \text{Re}[\tilde{V}(z)e^{j\omega t}] \]
\[ I(z, t) = \text{Re}[\tilde{I}(z)e^{j\omega t}] \]
Summary of Equations

\[ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \]

\[ V^- = \Gamma_L V^+ \]

\[ \Gamma(d) = \Gamma_L e^{-2j\beta d} \]

\[ \tilde{V}(d) = V^+ e^{j\beta d} (1 + \Gamma_L e^{-2j\beta d}) = V^+ e^{j\beta d} (1 + \Gamma(d)) \]

\[ \tilde{I}(d) = \frac{V^+ e^{j\beta d}}{Z_0} (1 - \Gamma_L e^{-2j\beta d}) = \frac{V^+ e^{j\beta d}}{Z_0} (1 - \Gamma(d)) \]

\[ Z(d) \equiv \frac{\tilde{V}(d)}{\tilde{I}(d)} = Z_0 \frac{1 + \Gamma(d)}{1 - \Gamma(d)} \]

\[ z(d) = \frac{1 + \Gamma(d)}{1 - \Gamma(d)} \]

\[ \Gamma(d) = \frac{z(d) - 1}{z(d) + 1} \]
Standing Wave Parameters

Amplitude of the standing waves:

\[
|\tilde{V}(d)| = |V^+| \|1 + \Gamma(d)\|
\]

\[
|\tilde{I}(d)| = Y_0 |V^+| \|1 - \Gamma(d)\|
\]

Where would \(V_{\text{min}}\) be?
What about \(I_{\text{min}}\) and \(I_{\text{max}}\)?
Standing Wave Parameters

\[ |\vec{V}(d)| = |V^+| |1 + \Gamma(d)| \]
\[ |\vec{I}(d)| = Y_0 |V^+| |1 - \Gamma(d)| \]

\[ \Gamma = 0.75e^{j\pi/4} \]

Unlike SC or OC where \(|\Gamma| = 1\), now we have imperfect nulls for voltage and current b/c \(|\Gamma| < 1\)

\[ d_{\text{max}} = \frac{\lambda}{4\pi} \theta \]

\[ VSWR = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \]
Standing Wave Parameters

\[ \Gamma = 0.75e^{j\pi/4} \quad VSWR = \frac{V_{\text{max}}}{V_{\text{min}}} = 7 \]

\[ \Gamma = 1.0e^{j\pi/4} \quad VSWR = \frac{V_{\text{max}}}{V_{\text{min}}} = \infty \]

Peak positions move in time because it is a standing wave plus a travelling wave

Peak positions stay constant because it is purely a standing wave

\( \omega t = 0 \) to \( \pi \)
Pure Standing Wave Animation
\[ \Gamma = 1, \ VSWR = \infty, \ \text{max} \ V \ \text{at load} \]
Pure Travelling Wave Animation

$\Gamma = 0$, $\text{VSWR} = 1$, $V_{\text{max}} = V_{\text{min}}$

Animations courtesy of Dr. Mojtaba Fallahpour, ECE 329 (Spring 2015)
Partial Standing Wave Animation
\[ \Gamma = 0.25e^{j\pi/4}, \text{VSWR}=1.667, \text{max} \, V \text{ at } \lambda/16 \]

Animations courtesy of Dr. Mojtaba Fallahpour, ECE 329 (Spring 2015)
Challenge Question: VSWR

- If $Z_L = Z_0 = 50\, \Omega$ and $Z_g = 100\, \Omega$, what will be the VSWR?

(a) $\infty$
(b) 2
(c) 1
(d) $1/2$
(e) 0

LG’s question
Standing Wave Useful Facts

\[ VSWR = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = z(d_{\text{max}}) \]

Thus,

\[ VSWR = z(d_{\text{max}}) = y(d_{\text{min}}) \]

because

\[ y(d) = z(d \pm \lambda/4) \]
Lecture 35 Summary

- As \( d \) increases, amplitude \(|V(d)|\) varies like \(|1+\Gamma|\) while \(|I(d)|\) varies like \(|1-\Gamma|\).

\[
VSWR = \frac{V_{\text{max}}}{V_{\text{min}}} = z(d_{\text{max}})
\]

\[
d_{\text{max}} = \frac{\lambda}{4\pi} \theta
\]

\[
VSWR = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1+|\Gamma|}{1-|\Gamma|}
\]
ECE 329
Lecture 36
Sections 7.2

More Standing Wave Parameters and
More Practice with Smith Charts
Example: VSWR measurements

• Find $Z_L$ if VSWR measurements are made on a line with $Z_0=60\,\Omega$:
  (a) $\text{SWR} = 1.5$ and $d_{\text{min}} = 0$

  (b) $\text{SWR} = 3.0$ and $d_{\text{min}} = 3$ and 9 cm

  (c) $\text{SWR} = 2.0$ and $d_{\text{min}} = 3$ and 7 cm
Calculation Space
Example: Input Impedance

- Find $Z_{in}$ if $Z_L = 45 + 60j\Omega$ and $Z_0 = 75\Omega$ and $v_p = c$ for the following cases:
  
  (a) $f = 15\text{MHz}$, $l = 5\text{m}$

  (b) $f = 50\text{MHz}$, $l = 3\text{m}$

  (c) $f = 37.5\text{MHz}$, $l = 5\text{m}$

From D7.6 (p 463) of old book
Calculation Space
Example

\[ Z_0 = 50 \Omega \]
\[ R_L = 30 \Omega \]
\[ R_g = 10 \Omega \]
\[ L_g = 100 \text{nH} \]
\[ C_L = 1 \text{nF} \]
\[ V_g = 100 \angle 0^\circ \text{V} \]
\[ l = 1.5\pi \text{m} \]
\[ \omega = 10^8 \text{rad/s} \]

- Using SC, determine: (a) \( z(0) \), (b) \( \Gamma(0) \), (c) VSWR, (d) locations of \( V_{\text{max}} \)
Calculation Space
Example

- Continue the problem and using SC, determine: (e) $\Gamma(l)$, (f) $Z_{in}$, (g) $V(l)$, (h) $I(l)$, (i) $<P>$
Calculation Space
ECE 329
Lectures 37-39
Sections 7.3
Online Notes: 37-39

Average Power
Quarter Wave Transformer Matching
Single Stub Matching
(Optional) Double Stub Matching
Distribution Networks
Lossy Line
Average Power

- In a lossless TL circuit,

\[ \langle P_{\text{in}} \rangle = \langle P(d) \rangle = \langle P_L \rangle \]

Note that

\[ \langle P(d) \rangle = \frac{1}{2} \text{Re} \left[ \tilde{V}(d) \tilde{I}^*(d) \right] \]

\[ = \ldots = \frac{1}{2} \left( \frac{|V^+|^2}{Z_0} - \frac{|V^-|^2}{Z_0} \right) = \frac{1}{2} \frac{|V^+|^2}{Z_0} \left( 1 - |\Gamma_L|^2 \right) \]

and so \( |\Gamma_L|^2 \) represents the power reflection coefficient.
Impedance matching

• When $Z_L \neq Z_0$, power is reflected back to the generator and $VSWR > 1$ (bad)

• Impedance matching achieves $VSWR = 1$ by adjusting the input impedance, $Z_{in}$, to be equal to the TL characteristic impedance, $Z_0$
Quarter Wave Matching Transformer

Before:

\[ V_{\text{max}} \] \quad \text{and} \quad \frac{\Gamma_{\text{max}}}{\Gamma_{\text{min}}} = \frac{1+|\Gamma|}{1-|\Gamma|} \]

After inserting a quarter-wave transformer, \( Z_1 = Z_0 \) so no more reflections and VSWR=1

Adjust \( Z_q \) and \( d_q \) for match
Quarter Wave Matching Transformer

Key Observations:
1. $Z_0$ and $Z_q$ are real since TL’s are lossless
2. $Z_1 = Z_0$ for a match so $Z_1$ must be real
3. $Z_1Z_2 = Z_q^2$ since we have a QW transformer
4. $Z_2$ must be real from equation #3
5. $d_q$ must be at a voltage max or min from #4
Quarter Wave Matching Transformer

Solution: \[ Z_q = \sqrt{Z_0 Z_2} = Z_0 \sqrt{z_2} \]

\[
d_q = \begin{cases} 
  d_{\text{max}} &= \frac{\lambda \theta}{4\pi} \\
  d_{\text{min}} &= \frac{\lambda \theta}{4\pi} + \frac{\lambda}{4} \mod \frac{\lambda}{2}
\end{cases}
\]

\[ \theta = \text{angle}(\Gamma_L) \]
Example: Design a QWT (2 choices)

\[ Z_0 = 50\Omega; \text{ load is } R = 30\Omega \text{ in series with } L = 1\text{nH for } \lambda = 10\text{mm, } v_p = c/\pi \]

Hints:
Find \( z(0) \)
Get \( |\Gamma_L| \) & \( \theta \)
Find \( d_{\text{max}}, \ d_{\text{min}} \)
Get both \( Z_2 \)'s
Similar to Optics

- Anti-reflection coatings:
  - thickness=$\lambda/4$ and $n_q=\sqrt{n_1n_2}$

http://hyperphysics.phy-astr.gsu.edu/Hbase/phyopt/antiref.html

Anti-reflection coatings work by producing two reflections that interfere destructively with each other.
For QWT matching, which is **false**:

(a) VSWR=1 in segment 1

(b) segment 1 can have any length

(c) there is a voltage min or max at both the left and right edges of segment 2

(d) $\Gamma(d)=1$ in segment 3

(e) $d_q$ can be increased by integer multiples of $\lambda/2$ without affecting the matching
QWT Matching

For $z_L = 4$ and the QWT located at the nearest voltage minimum, draw the Smith Chart circles in each of the three segments.
Workspace

\[ z_L = 4 \]
\[ d_q = \frac{\lambda}{4} \]
Single Stub Matching

Before:

After inserting a shorted stub in parallel, $y_{in}=1$ so no more reflections and $VSWR=1$

More convenient than QWT since stub has same impedance

Adjust $d_s$ and $l_s$ for match
Key Observations:

1. $y_{in}=1$ for a match because $Z_{in}=Z_0 \rightarrow y_{in}=Z_0/Z_{in}=1$

2. $y_{in}=y'_1+y_s$ since admittance adds for parallel elements

3. $y_s=1/(j \tan \beta l_s)=jb$ is purely imaginary for SC line
   a) See Lect 32, slide 18: $Z_{in}=jZ_0\tan(\beta l_s)$
   b) Needed amount of susceptance, $b$, depends on $|\Gamma_L|_{13}$
Single Stub Matching

\[ y'_1 = y_{in} - y_s = 1 -jb \]

\[ \Gamma'_1 = \frac{1-y'_1}{1+y'_1} = \frac{jb}{2-jb} = \Gamma_L e^{-2j\beta d_s} \Rightarrow |\Gamma_L| = \frac{|b|}{\sqrt{4+b^2}} \]

\[ \therefore b = \pm \frac{2|\Gamma_L|}{\sqrt{1-|\Gamma_L|^2}} \]
(Optional)

Single Stub Matching

\[ \theta = \angle \Gamma_L = \angle \left( \frac{jb}{2 - jb} e^{2j\beta d_s} \right) = \left( \angle \frac{jb}{2 - jb} \right) + 2\beta d_s \]

\[ \angle \frac{jb}{2 - jb} = \angle \frac{jb(2 + jb)}{4 + b^2} = \angle jb + \angle (2 + jb) = \pm \frac{\pi}{2} + \tan^{-1} \left( \frac{b}{2} \right) \text{ if } b > 0 \]

\[ \text{if } b < 0 \]

\[ \therefore d_s = \frac{\lambda}{4\pi} \left[ \theta \mp \frac{\pi}{2} - \tan^{-1} \left( \frac{b}{2} \right) \right] \text{ mod } \lambda / 2 \]
(Optional)

Single Stub Matching

\[ y_s = jb = \frac{1}{j \tan(\beta l_s)} \Rightarrow \tan(\beta l_s) = -\frac{1}{b} \]

\[ \therefore l_s = \frac{\lambda}{2\pi} \tan^{-1}\left(-\frac{1}{b}\right) \mod \frac{\lambda}{2} \]
Key Observations:
1. $y_{in}=1$ for a match, $y_s=jb$ is purely imaginary for the SC stub
2. Thus $y_1'=y_{in}-y_s=1-jb$ must be on the unit conductance circle
Unit Conductance
Circle: $y_1' = 1 - j\beta$
Smith Chart for Single Stub

1. Find $z(0)$ on S.C.
2. Go $\frac{1}{2}$ revolution around regular $|\Gamma(0)|$ circle to find $y(0)$
3. From $y(0)$, move clockwise towards generator until intersection with unit conductance circle (distance is $d_s$)
4. Read off $y_1'=y(d_s)$; $b=-\text{Im}(y_1')$
5. Need a stub with susceptance $+b$ so on a clean S.C., find $l_s$ such that $y(l_s)=jb$ starting from short: $y(0)=\infty$
Repeat D7.8a (p 472 old book):

\[ z(0) = 0.5 \]

Thus, we need a stub with \( y_s(l_s) = 0.72j \) at a distance of 0.098\( \lambda \) from load.
Find length for shorted stub:
\[ y(l_s) = 0.72j \]
\[ z(0) = 0 \]
\[ y(0) = \infty \]

\[ l_s = (0.25 + 0.098)\lambda \]
\[ l_s = 0.348\lambda \]
Repeat D7.8b
\[ z(0) = 0.2 - 0.4j \]

We are already exactly on unit conductance circle
Thus, we need a stub with
\[ y(l_s) = -2j \] at \( d_s = 0\lambda \) from load
Find length for shorted stub: $y(l_s) = -2j$

$z(0) = 0$

$y(0) = \infty$

$l_s = (0.324 - 0.25) \lambda$

$l_s = 0.074 \lambda$
Challenge question: Single stub tuning

- If the stub is left open instead of being shorted, which is **false**:

  (a) the same splice position $d_s$ will work
  (b) the needed admittance $y_s$ is unchanged
  (c) the admittance $y_1'$ is unchanged
  (d) the length $l_s$ must be increased by $\lambda/4 \pmod{\lambda/2}$
  (e) the positions of any voltage max or min along the stub are unchanged
Quarter wave transformer matching inserts a length of $l = \lambda/4$ of a specific impedance $Z_q = \sqrt{Z_0Z_2}$ at a distance $d_q$ from the load, where voltage is min or max.

Pros: Easy design
Cons: Have to redo QWT for each $f$
• Single stub matching inserts a shorted stub of the same impedance, $Z_0$, but with a specific length, $l_s$, and distance, $d_s$, from the load (not necessarily at $V_{\text{max}}$ or $V_{\text{min}}$)

Pros: $Z=Z_0$ on all lines
Cons: Adjusting $d_s$ may be inconvenient

• Find $d_s$ by moving CW from $y(0)$ to the unit conductance circle $y_1' = 1 - jb$; then find $l_s$ by going CW from $y_s(0) = \infty$ to $y_s(l_s) = +jb$

$$\therefore l_s = \frac{\lambda}{2\pi} \tan^{-1} \left( -\frac{1}{b} \right) \mod \frac{\lambda}{2}$$
ECE 446: Principles of Experimental Research

Principles of Experimental Research is an inter-disciplinary course designed for first year graduate students and advanced undergraduates. The course counts as an ECE lab elective (B.S.) or Professional Development (M.Eng.), yet students from any engineering or science department are encouraged to attend. The course focuses on: (1) design of experiments, (2) prevalent experimental techniques, (3) data collection, organization, and statistical analysis techniques, (4) oral and written presentation of scientific material, and (5) scientific computing languages and software. The main course objective is for students to develop the basic skills needed for pursuing a career or an advanced degree involving experimental research.

Prof. Goddard • 4 Credit Hours for Grad and Undergrad Students
(An ECE Lab Elective or MEng Professional Development Course)
ECE 329
Lecture 38(b)

Double Stub Matching
**Double Stub Matching**

More convenient than SSM since stub locations are fixed

---

After inserting two shorted stubs in parallel, $y_{in} = 1$ so no more reflections and $VSWR = 1$

$$VSWR = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

---

Adjust $l_1$ and $l_2$ for match.
(Optional)

Double Stub Matching

- Derivation proceeds similar to SSM, but algebra gets complicated quickly (p468)

\[ y_2' = y_{in} - y_{s2} = 1 - jb_2 \]

\[ \Gamma_2' = \frac{1 - y_2'}{1 + y_2'} = \frac{jb_2}{2 - jb_2} = \Gamma_1 e^{-2j\beta d_{12}} \Rightarrow \Gamma_1 = \frac{jb_2}{2 - jb_2} e^{2j\beta d_{12}} \]

\[ y_1 = \frac{1 - \Gamma_1}{1 + \Gamma_1} \]

Plug in \( \Gamma_1 \) from above

\[ y_1' = y_1 - y_{s1} = y_1 - jb_1 \]

Plug in \( y_1 \) from above

\( y_1' \) is known: start with \( y(0) \) and move CW by \( d_1 \) (predetermined); so just have to solve for \( b_1 \) and \( b_2 \) satisfying the real/imag parts of \( y_1' = y_1 - jb_1 \)
Double Stub Matching

• $y_1$ depends on $b_2$ and not $b_1$ so can solve for $b_2$ from real part of $y_1' = y_1 - jb_1$

$$g' = \text{Re}[y_1'] = \text{Re}[y_1] = \text{function of } b_2$$

$$b_2 = \frac{\cos \beta d_{12} \pm \sqrt{1/g' - \sin^2 \beta d_{12}}}{\sin \beta d_{12}}$$

Note that there will be no solution if $g' > 1/\sin^2 \beta d_{12}$
(Optional)

Double Stub Matching

- With $b_2$ solved, can find $b_1$ from the imaginary part of $y_1' = y_1 - jb_1$

$$b' \equiv \text{Im}[y_1'] = \text{Im}[y_1] - b_1 \Rightarrow$$

$$b_1 = \text{Im}[y_1] - b' = \text{function of } b_2$$

$$b_1 = \frac{b_2^2 \sin 2\beta d_{12} - 2b_2 \cos 2\beta d_{12}}{2 - 2b_2 \sin 2\beta d_{12} + 2b_2^2 \sin^2 \beta d_{12}} - b'$$

Finally, given $b_1$ and $b_2$, we can find $l_1$ and $l_2$ using:

$$l_s = \frac{\lambda}{2\pi} \tan^{-1} \left( -\frac{1}{b} \right) \mod \lambda / 2$$
Example: Design DSM

- $Z_0 = 50\Omega$, Termination is $Z = 30 - 40j\Omega$ and we choose to fix $d_1 = 0$ and $d_{12} = 0.375\lambda$
Key Observations:

1. For a match, $y_2'$ must be on unit conductance circle

2. Thus, $y_1$ is on the auxiliary circle
   a) Auxiliary circle is UCC pivoted CCW towards the load by $d_{12}$
Unit Conductance

Circle: $y_2' = 1 - j\beta$

Auxiliary Circle $y_1$ is UCC pivoted $d_{12}$ CCW towards load

$d_{12} = 0.375\lambda$
Repeat DSM: \( d_1 = 0 \) (Optional)
\( d_{12} = 0.375\lambda \)
\( z(0) = 0.6 - 0.8j \)

To get \( y_2' \) on UCC, need \( y_1 \) on AUX so draw AUX first using given \( d_{12} \)
Repeat DSM: $d_1 = 0$ (Optional)

$d_{12} = 0.375\lambda$

$z(0) = 0.6 - 0.8j$

$y(0) = 0.6 + 0.8j$

Since $d_1 = 0$, $y_1' = y(0)$
Find susceptance to go from \( y_1' \) along constant conductance circle to \( y_1 \) on AUX

\[
b_1 = -0.89 \quad \text{or} \quad b_1 = -2.72
\]

\[
l_1 = 0.134 \lambda \quad \text{or} \quad l_1 = 0.056 \lambda
\]

\[
l_s = \frac{\lambda}{2\pi} \tan^{-1} \left( -\frac{1}{b} \right)
\]
Find corresponding $y_2'$ by going from AUX to UCC

$y_2' = 1 - 0.53j$ or $y_2' = 1 + 2.5j$

$b_2 = 0.53$ or $b_2 = -2.5$

$$l_s = \frac{\lambda}{2\pi} \tan^{-1}\left(-\frac{1}{b}\right)$$

$l_2 = 0.327\lambda$ or $l_2 = 0.061\lambda$
Lecture 38(b) Summary

- Double stub matching inserts two shorted stubs of impedance $Z_0$ with specific lengths, $l_1$ and $l_2$, at fixed spots, $d_1$ and $d_1 + d_{12}$, from the load.

Pros: $Z = Z_0$ on all lines, fixed $d_1$, $d_{12}$
Cons: Many calculations

- Go from $y(0)$ to $y_1'$ a distance $d_1$ along $|\Gamma(0)|$ circle. Find $b_1$ by going from $y_1'$ along constant conductance circle to $y_1$ which is on the AUX circle. Find $y_2'$ by pivoting AUX by $d_{12}$ to UCC and read off $b_2 = -\text{Im}(y_2')$.

- Calculate $l_1$ and $l_2$ using:

$$l_s = \frac{\lambda}{2\pi} \tan^{-1}\left(-\frac{1}{b}\right)$$
ECE 329
Lecture 39
Online Notes – 38, 39

Distribution Networks
Lossy Line
Corporate Ladder

- A network for combining 4 identical loads $Z_L$ into an equivalent single load $Z_{in} = Z_L$

By symmetry, the average power delivered to each load is identical.
Corporate Ladder

- Verify $Z_{in} = Z_L$ by calculating $Z_1$ and $Z_2$
Calculation Space

\[ z_L = y(\lambda/4) \]
\[ y_L = z(\lambda/4) \]

\[ y_{tot} = y(\lambda/4) + y(\lambda/4) = 2z_L \]

\[ \lambda/4 \]
\[ Z_1 \]
\[ Z_L \]

\[ Z_1 \]
\[ Z_L \]

so \( z_2 = y_{tot} = 2z_L \) and thus \( y_2 = y_L/2 \)

Hence, \( y_{in} = y_L \) and \( z_{in} = z_L \)
Corporate Ladder

- If $Z_L = Z_0$, then the TL segment lengths can be freely varied without affecting $Z_{in}$ (phased array antennas)

What would the VSWR here be?
Challenge question: Corporate ladder

- For the corporate ladder, which is **true:**
  (a) there are no reflections anywhere in the ladder
  (b) VSWR=1 in segment 1
  (c) all the generator power is dissipated at the loads
  (d) the power is dissipated evenly among the loads
  (e) all segments can be made $\lambda/2$ instead of $\lambda/4$
Distributed Circuit of Lossless TL

One slice of the TL:

\[ \frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t} \quad \frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t} \]

The entire TL:
Distributed Circuit of Lossy TL

Model the Ohmic losses in conducting wires and leakage losses in the imperfect dielectric in between

One slice of the lossy TL:

\[ \frac{\partial V}{\partial z} = -(j\omega L + R)I \]

\[ \frac{\partial I}{\partial z} = -(j\omega C + G)V \]
Solution of Lossy TL

\[ \ddot{V}(d) = V^+ e^{\ddot{\gamma}d} + V^- e^{-\ddot{\gamma}d} \]
\[ \ddot{I}(d) = \frac{V^+ e^{\ddot{\gamma}d}}{\ddot{Z}_0} - \frac{V^- e^{-\ddot{\gamma}d}}{\ddot{Z}_0} \]
\[ \ddot{\gamma} = \alpha + j\beta = \sqrt{(j\omega L + R)(j\omega C + G)} \]
\[ \ddot{Z}_0 = \sqrt{\frac{j\omega L + R}{j\omega C + G}} \]

These reduce to the lossless results as \( R \) and \( G \to 0 \)

\[ \ddot{\gamma} = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC} = j\frac{\omega}{v_p} = j\beta \]
\[ \ddot{Z}_0 = \frac{j\omega L}{\sqrt{j\omega C}} = \sqrt{\frac{L}{C}} \]
Lossy TL (high-frequency limit)

\[ \bar{\gamma} = \sqrt{(j\omega L + R)(j\omega C + G)} \]
\[ = j\omega \sqrt{LC} \sqrt{1 + \frac{R}{j\omega L}} \left(1 + \frac{G}{j\omega C}\right) \]
\[ \approx j\omega \sqrt{LC} + \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right) \]

\[ \beta \approx \omega \sqrt{LC} \]

\[ \bar{Z}_0 = \sqrt{\frac{j\omega L + R}{j\omega C + G}} \approx \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}} \]

\[ \alpha \approx \frac{1}{2} \left( \frac{\mathcal{R}}{Z_0} + GZ_0 \right) \]

Similar to lossless case
Signals are attenuated
50Ω and 75Ω coax are so popular because ...

- The shunt conductance of the imperfect dielectric is small compared to the series resistance of the conductor, so:

\[
\alpha \approx \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right) \approx \frac{1}{2} \frac{R}{Z_0}
\]

Plugging in the formula for \( R \) and \( Z_0 \) of a coax with inner and outer radii \( a \) and \( b \), you can show \( \alpha \) is minimized when \( b/a = 3.59 \) (for fixed \( b \)), which works out to 50Ω and 75Ω for a dielectric and air filled coax, respectively. Loss decreases with \( b \), so use thicker coax to reduce loss.
### Suggested courses that build on ECE329

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How do I figure out which variables are important?

What is the data telling me?

There must be a more accurate technique …

Compile !?!?!

I get to design my own experiments in this class!

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Good luck on the final!

• It was a pleasure teaching ECE329 this semester
  – Thank you for studying so hard 😊
ECE 329

Review for Final Exam

(page and chapter #’s are for the old book)
Coulomb’s Law

Electric Field Around a Point Charge

\[ \vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2} \hat{a}_R \]

Field strength is proportional to the density of field lines.
Calculating the Electric Field

Point Charge at position \((x_1, y_1, z_1)\)

\[
R = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]

Unit vector pointing along direction from \(Q\) to Point \(R\)

\[
\hat{a}_R = \frac{(x_2 - x_1)\hat{a}_x + (y_2 - y_1)\hat{a}_y + (z_2 - z_1)\hat{a}_z}{R}
\]

\[
\vec{E} = \frac{Q}{4\pi \varepsilon_0 R^2} \hat{a}_R
\]

Position where we want to calculate electric field at Position \((x_2, y_2, z_2)\)

Unit vector pointing along direction from \(Q\) to Point \(R\)

Use superposition for extended charges
* Important

Patented 5-Step Program for Problem Solving

1. MAKE A **LARGE CLEAR DRAWING**
   a. Also draw cross-sections if the problem is in 3D
   b. Pick a coordinate system that is appropriate for the symmetry of the problem

2. Divide charge distributions into tiny pieces

3. Find $d\mathbf{E}$ of one tiny piece

4. Use SYMMETRY to eliminate any components that cancel (i.e. add to ZERO)

5. INTEGRATE to add contribution of ALL the tiny pieces
Example: F due to line of charge (1)

What is the small amount of force, \( dF \), applied by a small sliver of the rod?

\[
dF = \left[ \frac{Q}{L} \right] \frac{dx}{4\pi \varepsilon_0 x^2} q \hat{a}_x
\]

Differential charge in one small sliver (coul)

Differential force applied to q
Ampere’s Force Law

\[ d\vec{F}_1 = I_1 d\vec{l}_1 \times d\vec{B}_1 \]

Force on current 1 due to current 2
Lorentz Force Equation

If a region of space contains BOTH an $\mathbf{E}$ field and a $\mathbf{B}$ field, a moving charge will experience force from both at the same time…

$$\vec{F}_{TOTAL} = \vec{F}_E + \vec{F}_M$$

$$\vec{F}_{TOTAL} = q\vec{E} + q\vec{v} \times \vec{B}$$
Application: Mass Spectrometers

• Part I: Velocity Selector
  – Particles with a specific velocity in crossed EM fields are undeflected

\[
\vec{E} = E_0 \hat{a}_z \\
\vec{B} = -B_0 \hat{a}_y \\
\vec{v} = v_0 \hat{a}_x \\
\vec{F}_{TOTAL} = q(E_0 - v_0 B_0) \hat{a}_z = 0 \text{ iff } v_0 = E_0 / B_0
\]
Surface Integrals

- Flux = # of arrows that pass thru a surface; it depends on:
  - The density of vectors
  - The angle of the surface
  - The area of the surface

\[ \text{Flux} = \vec{B} \cdot d\vec{S} \]
“Closed” Bug Catching Net

Bug Density
Vectors
(Bugs/m²)

\[ \vec{B} \]

\[ \sum \vec{B} \]

\( \frac{d\rho}{dx} \)

\( \frac{d\rho}{dy} \)

\( dS \) (normal to net section)
Gauss’ Law for B Fields

Net flux of magnetic field lines through any closed surface MUST be zero.

\[ \iiint_S \vec{B} \cdot d\vec{S} = 0 \]
* Important

Gauss’ Law for E-fields

• Field lines begin on + charges, end on – ones
  – Electric flux out = Net charge enclosed, regardless of shape or location of charges

\[ \psi_E = \iiint_S \vec{D} \cdot d\vec{S} = Q_{\text{enclosed}} \]

FLUX OUT = CHARGE ENCLOSED
Use Symmetry to Find Flux

6-sided cube with Q at the center:
Flux out of entire box = Q
Flux out of one side = Q/6

6-sided cube with Q NOT at the center:
Flux out of entire box = Q
Flux out of one side = ???

Hemisphere and bottom disc with Q at the center:
Flux out of top hemisphere = Q/2
Flux out of bottom disc = 0
(E is perpendicular to dS)
Faraday’s Law

\[ \int_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_S \vec{B} \cdot d\vec{S} \]

- The EMF generated in the loop is the NEGATIVE of the rate of change of the magnetic flux enclosed in the loop.
- Right Hand curls around C so thumb points in direction of \( d\vec{S} \)

* Important
Induced emf around rectangular loop in a time-varying $\mathbf{B}$ field

Rectangular wire loop
In the xz-plane

$\mathbf{B} = B_0 \cos \omega t \hat{\mathbf{y}}$

Steps:
1. Write down Faraday's law $\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}$
2. Write down expression for $d\mathbf{S}$. DIRECTION!!
3. Perform dot product $\mathbf{B}^* d\mathbf{S}$
4. Solve double integral over limits of the loop
5. Take time derivative of result. Put in “−” sign!
Ampere's Law

There are TWO sources of MMF:
1. Flow of charges due to current
2. Time-varying electric field

Called (by Maxwell) “Displacement Current”

\[
\oint \vec{H} \cdot d\vec{l} = \iint_S \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iint_S D \cdot d\vec{S}
\]

**Important**

MMF (Amps) = “Regular” Current (Amps) + Displacement Current (Amps)
Displacement Current

• Current flow changes amount of charge \( I_{in} = \frac{dQ}{dt} \)
  - Since the charge changes, the electric flux out of the surface changes, i.e. a displacement current

\[
\bar{E}(t) = \frac{Q(t)}{4\pi\varepsilon_0 R^2} \hat{a}_R
\]

\[
\psi_E = \iint_S \varepsilon_0 \vec{E} \cdot d\vec{S} = \varepsilon_0 E(Surf\ Area)
\]

\[
= \varepsilon_0 \frac{Q(t)}{4\pi\varepsilon_0 R^2} \left(4\pi R^2\right) = Q(t)
\]

\[
I_d = \frac{d\psi_E}{dt} = \frac{dQ}{dt}
\]

\[\therefore I_d = \frac{dQ}{dt} = I_{in}\] so displacement current out = regular current in
B for an infinitely long solid cylindrical conductor

The path with radius \( r < a \) “encloses” only a portion of the entire current.

No changing D-field so Ampere’s Law, with \( d\psi_E/dt = 0 \):

\[
\oint_C \vec{H} \cdot d\vec{l} = I_{\text{enclosed}} = \iint_S \vec{J} \cdot d\vec{S}
\]

\[
2\pi r H_\phi = \frac{I}{\pi a^2} \pi r^2 \quad \text{inside or} \quad = I \quad \text{outside}
\]
* Important

Conservation of Charge

In general, we can pour charges in from more than one direction, or take some out from other parts of the container.

Net Rate of Current flow OUT = Net Rate of Charge DECREASE

\[ \oint \oint_{S} \vec{J} \cdot d\vec{S} = - \frac{dQ_{enc}}{dt} = - \frac{d}{dt} \iiint_{V} \rho dV \]
Curl measures circulation

No rotation!

Anti-clockwise rotation.

Clockwise rotation.
Divergence measures

# of field lines created

Flux in = flux out so no sources or sinks inside V.

Flux out > flux in Positive divergence. Must be a source inside V.

Flux out < flux in Negative divergence. Must be a sink or drain inside V.
Curl and Divergence

\[ E_x \]

\[ \frac{dE_x}{dx} \neq 0 \]

YES DIVERGENCE
NO CURL
Varies ALONG

\[ E_y \]

\[ \frac{dE_y}{dx} \neq 0 \]

NO DIVERGENCE
YES CURL
Varies ACROSS
Curl and Divergence

\[ \int \int (\nabla \times \vec{v}) \cdot d\vec{S} = \oint \vec{v} \cdot d\vec{l} \]
Stokes’ Theorem

\[ \int \int \nabla \cdot \vec{v} \; dV = \int \int \nabla \cdot \vec{v} \; d\vec{S} \]
Divergence Theorem
Maxwell’s Equations

Faraday’s Law
\[ \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint \mathbf{B} \cdot d\mathbf{S} \]
\[ \nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \]

Ampere’s Law
\[ \oint \mathbf{H} \cdot d\mathbf{l} = \iint \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \iint \mathbf{D} \cdot d\mathbf{S} \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt} \]

Gauss’ Law
\[ \iint \mathbf{B} \cdot d\mathbf{S} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]

Gauss’ Law
\[ \iiint \mathbf{D} \cdot d\mathbf{S} = \iiint \rho dV \]
\[ \nabla \cdot \mathbf{D} = \rho \]

Continuity Eq.
\[ \iint \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \iiint \rho dV \]
\[ \nabla \cdot \mathbf{J} = -\frac{d\rho}{dt} \]
Realizable Fields

• Is it time dependent (d/dt)?
• Is there any free charge $\rho$ or current density $\mathbf{J}$?
• Apply Maxwell’s equations

$$\nabla \cdot \vec{D} = \rho$$  
$$\nabla \cdot \vec{B} = 0$$  
$$\nabla \cdot \vec{J} = -\frac{d\rho}{dt}$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$  
$$\nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$$

$$\vec{D} = \varepsilon \vec{E}$$  
$$\vec{B} = \mu \vec{H}$$

*Important*
Field Source

- Infinite plane sheet of current at $z=0$

\[ \vec{J}_s = -J_s(t)\hat{a}_x \]

At $z=0$

$J_s$ = surface current in Amps per meter

Free space $z<0$

Free space $z>0$
Solution

\[ \vec{E}(z,t) = \frac{\eta_0}{2} J_s (t \mp \frac{z}{v_p}) \hat{a}_x \]

\[ \vec{H}(z,t) = \pm \frac{1}{2} J_s (t \mp \frac{z}{v_p}) \hat{a}_y \]

\( z \geq 0 \)
Two definitions

\[ v_p = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \text{speed of light} = 3 \times 10^8 \ (m/s) \]

\[ \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \ (\text{ohms}) \quad \text{Intrinsic impedance of free space} \]

\[ |E| = \eta_0 |H| \]
* Important

Web Demo

http://www.phy.ntnu.edu.tw/java/emWave/emWave.html
* Important

Sinusoidal Plane Waves

\[
\vec{J}_s = -J_{s0} \cos(\omega t) \hat{a}_x \\
\vec{E}(z,t) = \frac{\eta_0 J_s}{2} (t + \frac{z}{v_p}) \hat{a}_x \\
\vec{H}(z,t) = \pm \frac{1}{2} J_s (t + \frac{z}{v_p}) \hat{a}_y
\]

\[
\vec{E}(z,t) = \frac{\eta_0 J_{s0}}{2} \cos(\omega t + \beta z) \hat{a}_x \\
\vec{H}(z,t) = \pm \frac{J_{s0}}{2} \cos(\omega t + \beta z) \hat{a}_y
\]

\[
\beta = \frac{\omega}{v_p} \\
z \geq 0
\]
Wave Parameters

Electric Field

\[ \vec{E}(z,t) = \frac{\eta_0 J_{s0}}{2} \cos(\omega t + \beta z) \hat{a}_x \] (V/m)

Phase

\[ \phi = \omega t + \beta z \] (radians)

Angular Frequency

\[ \omega = \frac{\partial \phi}{\partial t} \] (radians/sec)

Linear Frequency

\[ f = \frac{\omega}{2\pi} \] (1/sec)

Phase Constant

\[ \beta = \left| \frac{\partial \phi}{\partial z} \right| \] (radians/m)

Wavelength

\[ \lambda = \frac{2\pi}{\beta} \] (m)

Phase Velocity

\[ v_p \equiv \frac{\omega}{\beta} = \lambda f = c \] (m/sec)

Impedance

\[ \eta_0 = \left| \frac{\vec{E}}{\vec{H}} \right| \] (Ω)

**Very Important**
Polarization

**Linear**
- **DIRECTION:** Constant
- **MAGNITUDE:** Varies
  - $\vec{F}_1 = F_1 \cos(\omega t + \phi) \hat{a}_x$
  - $\vec{F}_2 = \pm F_2 \cos(\omega t + \phi) \hat{a}_y$
- The vectors are IN PHASE
- $\vec{F}_1 + \vec{F}_2$

**Circular**
- **DIRECTION:** Varies
- **MAGNITUDE:** Constant
  - $\vec{F}_1 = F_0 \cos(\omega t + \phi) \hat{a}_x$
  - $\vec{F}_2 = F_0 \sin(\omega t + \phi) \hat{a}_y$
- The vectors must be:
  - EQUAL MAGNITUDE
  - OUT OF PHASE by $\pi/2$
  - PERPENDICULAR

**Elliptical**
- **DIRECTION:** Varies
- **MAGNITUDE:** Varies
- Most general:
  - If it is not linear or circular

*Important*

"Left-Hand" Polarized CCW as seen by source

"Right-Hand" Polarized CW as seen by source
Writing Fields in Free Space

• Sinusoidal field propagating in $\hat{a}_z$ has left circular polarization, $\lambda = 3\, \text{m}$, $E(0,0) = E_0 \hat{a}_y$

• Answer: $\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{3} \, \text{rad/m}$;
• $\lambda f = c \rightarrow f = 1 \times 10^8 \, \text{Hz} \rightarrow \omega = 2\pi f = 2\pi \times 10^8 \, \text{rad/s}$
• $E_y$ is max first then $\frac{1}{4}$ period later $E_x$ is max

• $E = E_0 \cos(\omega t - \beta z) \, \hat{a}_y + E_0 \sin(\omega t - \beta z) \, \hat{a}_x$
• $H = E_0/\eta_0 \cos(\omega t - \beta z) (-\hat{a}_x) + E_0/\eta_0 \sin(\omega t - \beta z) \, \hat{a}_y$
*Important*

**Definition: Poynting Vector**

The \( \mathbf{E} \) and \( \mathbf{H} \) fields are carrying power with them as they propagate

\[
\vec{S} = \vec{E} \times \vec{H}
\]

Definition for the Power Flow Density of an EM Field

Units for \( \mathbf{S} \): Watts/m\(^2\)

\[
\iiint \vec{S} \cdot d\vec{S} = \iiint (\vec{E} \times \vec{H}) \cdot d\vec{S}
\]

Power flow **out** of a CLOSED surface (units = Watts)
Poynting’s Theorem

\[-\frac{\partial}{\partial t} (u_m + u_e) = \nabla \cdot \vec{S} + \vec{E} \cdot \vec{J}\]

Integrate over the volume and apply Divergence Theorem:

\[-\frac{\partial}{\partial t} \iiint_V (u_m + u_e) dV = \oiint_S \vec{S} \cdot d\vec{S} + \iiint_V \vec{E} \cdot \vec{J} dV\]

Rate the fields

LOSE energy

\[\vec{E} = \text{Re}[\tilde{E}(z)e^{j\omega t}], \quad \vec{H} = \text{Re}[\tilde{H}(z)e^{j\omega t}]\]

\[\langle \vec{S} \rangle = \text{Re}[\frac{1}{2} \vec{E} \times \vec{H}^*]\]

Power flow OUT of surface

Rate of work done BY the fields

\[\vec{E} \cdot \vec{J} \text{ is non-negative when the fields move charge (resistive load):}\]

\[J = \sigma \vec{E} \text{ so } \vec{E} \cdot \vec{J} = \sigma |\vec{E}|^2 \geq 0\]

\[\text{Zero only if } \sigma = 0 \text{ (perfect dielectric)}\]

\[\vec{E} \cdot \vec{J} \text{ is negative when the applied current injects energy into the fields (the current sheet at } z=0 \text{ is a power source)}\]
## 3 types of materials

<table>
<thead>
<tr>
<th>Conductors</th>
<th>Dielectrics</th>
<th>Magnetic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Free electrons</strong></td>
<td><strong>Polarized atoms/molecules</strong></td>
<td><strong>Magnetic moments</strong></td>
</tr>
<tr>
<td>+e-</td>
<td>+Q</td>
<td>B_{int}</td>
</tr>
<tr>
<td>-e-</td>
<td>-Q</td>
<td>I</td>
</tr>
<tr>
<td><strong>Bound electrons</strong></td>
<td><strong>Bound electrons</strong></td>
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</tr>
<tr>
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<td>I</td>
</tr>
<tr>
<td>-e-</td>
<td>-Q</td>
<td>I</td>
</tr>
</tbody>
</table>

### Conductor Properties
- $E = 0$ inside
- $\rho = 0$ inside
- $\rho = \rho_s$ only surface charge
- $V$ is same throughout
- $E_{outside} \perp$ to surface

### Dielectric Properties
- $E \neq 0$ inside but it is reduced
- $E_{tot} = E_a + E_s$
- $D = \varepsilon E_{tot} = P + \varepsilon_0 E_{tot}$

### Magnetic Properties
- $B_{tot} = B_a + B_s$
- $B_{tot} = \mu H = \mu_0(H + M)$

---

*Important Chapter 4*
Connection of Concepts for Electrostatics

- **Integral**
  - $\vec{P} = \varepsilon_0 \chi_e \vec{E}$
  - $\vec{E} = -\nabla V$
  - $V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l}$

- **Proportional**
  - $\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$
  - $\nabla \cdot \vec{D} = \rho_{vol}$
  - $D_{1n} - D_{2n} = \rho_{s0}$

- **Derivative**
  - $Q = \rho_{s0} A$ or $Q = \rho_{vol} V$
  - $Q = \int \rho_{s0} dS$ or $Q = \int \rho_{vol} dV$
  - $\rho$

- **Exremely Important**

---

Goddard/Cunningham
ECE329 Cumulative Review
Connection of Concepts for Electrodynamics

\[ \Phi \text{ and } \vec{A} \text{ scalar and vector potential} \]

\[ \vec{B} = \nabla \times \vec{A} \]

\[ \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \]

\[ \vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P} \]

\[ \vec{B} = \mu \vec{H} = \mu_0 (\vec{H} + \vec{M}) \]

\[ \nabla \cdot \vec{D} = \rho_{vol} \]

\[ D_{1n} - D_{2n} = \rho_{s0} \]

\[ \vec{H} = \vec{J}_S \times \hat{a}_n \]

\[ \vec{J} = \int \vec{J}_S(x) \, dx \text{ or } \vec{J} = \int \vec{J} \cdot d\vec{S} \]

Integral

Derivative

Proportional

polarization (material response)

magnetization (material response)

magnetic flux density

electric field

displacement flux density

magnetic field

charge density

current density

Derivative

Proportional

polarization (material response)

magnetization (material response)

magnetic flux density

electric field

displacement flux density

magnetic field

charge density

current density

Goddard/Cunningham
ECE329 Cumulative Review
Conducting Slab D4.2 (p217)

- Neutral slab \( \rho_1 = -\rho_2 \)
- \( E_{\text{inside}} = 0 \) \( \Rightarrow \) \( E_z = (\rho_B + \rho_2 - \rho_A - \rho_1)/2\varepsilon_0 = 0 \)

\[ \rho_1 = (\rho_B - \rho_A)/2, \quad \rho_2 = (\rho_A - \rho_B)/2 \]
Dielectric Slab

\[ E_{\text{Secondary}} = \frac{-P_0}{\varepsilon_0} \hat{a}_z \]

\[ E_{\text{Applied}} = \frac{\rho_{s0}}{\varepsilon_0} \hat{a}_z \]

\[ E_{\text{Total}} = \frac{P_0}{\varepsilon_0 \chi_e} = \frac{\rho_{s0}/\varepsilon_0}{1 + \chi_e} \]

\[ D = \varepsilon E_{\text{Total}} = \varepsilon_0 (1 + \chi_e) E_{\text{Total}} = \rho_{s0} \]

- **E-field strength reduced by** \((1 + \chi_e)\)
- **D-field strength is same as free space because charge is fixed**

If instead voltage were fixed, then **E-field would be same as free space**
Magnetic Sheets D4.6 (p238)

- Hint (Single infinite current sheet)

\[ \mathbf{H} = \frac{\mathbf{J}}{2} \times \hat{a}_n = -\left( \frac{\mathbf{J}}{2} \right) \hat{a}_x \]

\[ \mathbf{H} = \frac{\mathbf{J}}{2} \times \hat{a}_n = \left( \frac{\mathbf{J}}{2} \right) \hat{a}_x \]

Answer:

\[ \mathbf{H} = 0 \]

\[ \mu = 100\mu_0 \]

\[ \mathbf{H} = \mathbf{J} \hat{a}_x \]

\[ \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \]

To solve for \( \mathbf{M} = 99 \mathbf{H} \)
Inside a material, Maxwell’s Equations become:

\[
\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}
\]

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\]

\[
\nabla \cdot \vec{D} = \rho
\]

\[
\nabla \cdot \vec{B} = 0
\]

\[
\n\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}
\]

\[
\n\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}
\]

\[
\n\nabla \cdot (\varepsilon \vec{E}) = \rho
\]

\[
\n\nabla \cdot \vec{H} = 0
\]
Solve PDEs with Phasors

- Technique simplifies the algebra

\[ E_x(z,t) = \text{Re}[\tilde{E}_x(z)e^{j\omega t}] \]

\[ \frac{\partial E_x}{\partial t} = \text{Re}[j\omega\tilde{E}_x(z)e^{j\omega t}] \]

\[ \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \]

\[ \frac{\partial H_y}{\partial z} = -\sigma E_x - \varepsilon \frac{\partial E_x}{\partial t} \]

\[ H_y(z,t) = \text{Re}[\tilde{H}_y(z)e^{j\omega t}] \]

\[ \frac{\partial }{\partial t} \equiv j\omega \]

\[ \frac{\partial \tilde{E}_x}{\partial z} = -\mu (j\omega) \tilde{H}_y \]

\[ \frac{\partial \tilde{H}_y}{\partial z} = -\sigma \tilde{E}_x - \varepsilon (j\omega) \tilde{E}_x \]
Final Solution for $E_x$ and $H_y$

\[
E(z,t) = \frac{|\eta| J S_0}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z + \tau) \hat{a}_x
\]

\[
H(z,t) = \frac{\pm J S_0}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z) \hat{a}_y
\]

Strength of fields drops exponentially according to the attenuation constant $z \geq 0$

Magnitudes $|E|$ and $|H|$ related through magnitude of the complex impedance, $|\eta|$

$E$ and $H$ are out of phase by the phase of the complex impedance, $\tau = \arg(\eta)$
* Important

\[ \bar{E}(z,t) = \frac{|\eta| J_{S0}}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z + \tau) \hat{a}_x \]

\[ \bar{H}(z,t) = \frac{\pm J_{S0}}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z) \hat{a}_y \]

\[ \Delta z = \frac{\tau}{\beta} \]
*Important*

**Complex Propagation**

**Constant and Impedance**

\[
\frac{\partial E_x^2}{\partial z^2} = \bar{\gamma}^2 E_x
\]

\[
\bar{\gamma} = \sqrt{j \omega \mu (\sigma + j \omega \varepsilon)} = \alpha + j \beta = |\bar{\gamma}| e^{j\psi}
\]

\[
\bar{\eta} = \sqrt{\frac{j \omega \mu}{\sigma + j \omega \varepsilon}} = |\bar{\eta}| e^{j\tau}
\]

\[
\frac{\bar{\gamma} \bar{\eta}}{\bar{\eta}} = j \omega \mu
\]

\[
\Rightarrow \tau + \psi = \pi / 2
\]

\[
\bar{\gamma} / \bar{\eta} = \sigma + j \omega \varepsilon
\]

Very useful!

\[
\Rightarrow 45^\circ \leq \psi \leq 90^\circ
\]

\[
\therefore \beta \geq \alpha > 0
\]

\[
\therefore \bar{\gamma}^2 \text{ is in Quadrant II}
\]

\[
\Re[\bar{\gamma}^2] < 0, \Im[\bar{\gamma}^2] > 0
\]
## Dielectrics vs Conductors

<table>
<thead>
<tr>
<th>Perfect Dielectric</th>
<th>Imperfect Dielectric</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition:</strong></td>
<td>$\sigma = 0$</td>
</tr>
<tr>
<td><strong>Attenuation:</strong></td>
<td>$\alpha = 0$</td>
</tr>
<tr>
<td><strong>Speed:</strong></td>
<td>$v_p = c / \sqrt{\mu_r \varepsilon_r} \leq c$</td>
</tr>
<tr>
<td><strong>$E, H$ In Phase:</strong></td>
<td>$\tau = 0$</td>
</tr>
<tr>
<td><strong>Impedance:</strong></td>
<td>$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Good Conductor</th>
<th>Perfect Conductor</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition:</strong></td>
<td>$\sigma / \omega \varepsilon &gt;&gt; 1$</td>
</tr>
<tr>
<td><strong>Attenuation:</strong></td>
<td>$\alpha \approx \sqrt{\omega \mu \sigma} / 2$</td>
</tr>
<tr>
<td><strong>Speed:</strong></td>
<td>$v_p \approx \sqrt{2 \omega / \sigma \mu}$</td>
</tr>
<tr>
<td><strong>$E, H$ $45^\circ$ Phase:</strong></td>
<td>$\tau \approx \pi / 4$</td>
</tr>
<tr>
<td><strong>Impedance:</strong></td>
<td>$</td>
</tr>
</tbody>
</table>

<p>| | |</p>
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<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td><strong>Definition:</strong></td>
<td>$\sigma \rightarrow \infty$</td>
</tr>
<tr>
<td><strong>Attenuation:</strong></td>
<td>$\alpha \rightarrow \infty$</td>
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<tr>
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</tr>
<tr>
<td><strong>Impedance:</strong></td>
<td>$</td>
</tr>
</tbody>
</table>

**Very Important**

\[
\begin{align*}
\sigma & = 0 \\
\alpha & = 0 \\
v_p & = c / \sqrt{\mu_r \varepsilon_r} \leq c \\
\tau & = 0 \\
|\eta| & = \eta_0 \sqrt{\mu_r / \varepsilon_r}
\end{align*}
\]
** Very Important

Boundary Conditions

- **Never** use the differential form of Maxwell’s equations at a boundary – only use integral form.

\[
\hat{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \times \hat{a}_n
\]

\[
(\vec{H}_1 - \vec{H}_2)_t = \vec{J}_s \times \hat{a}_n
\]
Example: Potentials for a Point Charge

\[ \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r \]

\[ \vec{E} = -\nabla V \]

\[ V(r) = -\int_{r_1}^{\infty} \vec{E} \cdot d\vec{l} \]

“Absolute” potential at \( r_1 \) using zero potential at \( r = \infty \)

\[ d\vec{l} = -|dr| \hat{a}_r = dr\hat{a}_r \]

Since \( dr < 0 \) going from \( r = \infty \)

\[ V(r) = -\int_{r_1}^{\infty} \frac{Q}{4\pi\varepsilon_0 r^2} \, dr \]

\[ = \frac{Q}{4 \pi \varepsilon_0} \left( \frac{1}{r_1} - \frac{1}{\infty} \right) = \frac{Q}{4 \pi \varepsilon_0 r_1} \]
Superposition Example: Potential of a Line Charge

\[ \rho_l = 10^{-10} \, (C/m) \]

\[ a = 1 \, m \]

Find Potential at point “P”, 1 m away from the line

\[ V_P = \frac{1}{4 \pi \varepsilon_0} \int_{-a}^{a} \frac{dQ}{r} \]

\[ V_P = \frac{1}{4 \pi \varepsilon_0} \int_{-a}^{a} \frac{\rho_l \, dz'}{r} \]

\[ V_P = \frac{1}{4 \pi \varepsilon_0} \int_{-a}^{a} \frac{\rho_l \, dz'}{\sqrt{z'^2 + 1}} \]

\[ V_P = \frac{\rho_l}{4 \pi \varepsilon_0} \ln \left( \frac{z' + \sqrt{z'^2 + 1}}{1} \right) \bigg|_{-a}^{a} \]
Example 5.5 (p-n junction)

\[ \nabla^2 V = -\frac{\rho}{\varepsilon} \]

\[ \vec{E} = -\nabla V \]

\[
\frac{dE}{dx} = \frac{\rho}{\varepsilon} \\
E = \int_{-\infty}^{x} \frac{\rho}{\varepsilon} dx \\
V = -\int_{-\infty}^{x} E dx
\]
**Very Important**

**Steps to Find Capacitance**

- Laplace Equation \( \nabla^2 V = 0 \)
- Find \( V \) using boundary conditions
- Find \( \mathbf{E} \) using \( \mathbf{E} = -\nabla V \)
- Find \( \mathbf{D} \) using \( \mathbf{D} = \varepsilon \mathbf{E} \)
- Get surface charge density on one conductor using BC
- Charge
- Capacitance

\[
\rho_s = \mathbf{a}_n \cdot (D_{n1} - D_{n2})
\]

\[
\rho = \varepsilon V_0 / d
\]

\[
Q = (Area)(\rho_s)
\]

\[
C = Q / V_0
\]

\[
V(x) = V_0 \frac{x}{d}
\]
Coaxial Cable

\[ \nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) = 0 \]

\[ \Rightarrow r \frac{\partial V_r}{\partial r} = c_1 \Rightarrow V_r = c_1 \ln r + c_2 \]

\[ V(b) = 0 \Rightarrow c_2 = -c_1 \ln b \]

\[ V(a) = V_0 \Rightarrow c_1 = \frac{V_0}{\ln(a/b)} \]

\[ V(r) = -V_0 \frac{\ln(r/b)}{\ln(b/a)} \]

\[ E = -\frac{dV}{dr} = \frac{V_0}{r \ln(b/a)} \]

\[ \rho = \begin{cases} 
\varepsilon V_0/(a \ln(b/a)), & r = a \\
-\varepsilon V_0/(b \ln(b/a)), & r = b 
\end{cases} \]

\[ C = \frac{2\pi\varepsilon L}{\ln(b/a)} \]
Now, instead of applying a voltage across the inner and outer conductor, a current, $I$, flows down the length of the outer conductor and returns in the opposite direction through the inner conductor.

Results in magnetic field

$$H_\phi = \frac{I}{2\pi r}$$

in between the coax.
Inductance of Coaxial Cable

\[ \vec{B} = \mu \vec{H} = \frac{\mu I}{2\pi r} \hat{a}_\phi \]

\[ \psi = \int B \cdot dS = \int_{r=0}^{b} \int_{z=0}^{h} \left( \frac{\mu I}{2\pi r} \right) (drdz) \]

\[ \psi = \frac{\mu I z}{2\pi} \ln(b/a) \]
Inductance

\[ L = \frac{\psi}{I} \quad \text{Units: Henry (H)} \]

\[ L = \frac{\mu z}{2\pi} \ln(b/a) \]

\[ L = \frac{L}{z} = \frac{\mu}{2\pi} \ln(b/a) \quad \text{Inductance/Length (H/m)} \]
Relationships between Capacitance, Conductance & Inductance

Notice in the above examples,

\[ \mathcal{C} = \varepsilon \cdot \text{GeometricalFactor} \]
\[ \mathcal{L} = \mu / \text{GeometricalFactor} \]
\[ \mathcal{G} = \sigma \cdot \text{GeometricalFactor} \]

This is true in general and so we have the following:

\[ \mathcal{L} \mathcal{C} = \mu \varepsilon \quad \mathcal{G} / \mathcal{C} = \sigma / \varepsilon \]

If you know one (L, C, or G), you can find the other two from the material parameters.
$V(z,t)$ and $I(z,t)$ can be used to describe the state of the transmission line instead of $E_x(z,t)$ and $H_y(z,t)$.
Transmission Line Equations

\[ \frac{\partial V}{\partial z} = -\left( \frac{\mu d}{w} \right) \frac{\partial I}{\partial t} \]
\[ \frac{\partial I}{\partial z} = -\left( \frac{\varepsilon w}{d} \right) \frac{\partial V}{\partial t} \]

These are the transmission line equations!!
- They describe wave propagation along the TL in terms of currents and voltages
- It is just another way of stating Maxwell’s Eqns
First step: Calculate $V^+$, $I^+$, $\Gamma_{load}$, $\Gamma_{source}$

$$V^+ = V_0 \frac{Z_0}{R_g + Z_0} = 100 \frac{60}{40 + 60} = 60V$$

$$I^+ = \frac{V^+}{Z_0} = \frac{60}{60} = 1A$$

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{120 - 60}{120 + 60} = \frac{1}{3}$$

$$\Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = \frac{40 - 60}{40 + 60} = -\frac{1}{5}$$

Second step: Construct 2 bounce diagrams (Voltage and Current)
**Very Important**

Voltage

\[ Z_0 \]

\[ z=0 \quad z=1 \]

\[ \Gamma = -\frac{1}{5} \quad \Gamma = \frac{1}{3} \]

\[ t=0 \]

\begin{align*}
V &= 0V \\
V^+ &= 60V \\
V &= 60V \\
V^- &= 20V \\
V &= 20V \\
V^+ &= -4V \\
V &= 80V \\
V^- &= -4V \\
V &= 76V \\
\end{align*}

\begin{align*}
1 \mu sec \\
2 \mu sec \\
3 \mu sec \\
4 \mu sec \\
5 \mu sec \\
6 \mu sec \\
\end{align*}

Current

\[ Z_0 \]

\[ z=0 \quad z=1 \]

\[ \Gamma = \frac{1}{5} \quad \Gamma = -\frac{1}{3} \]

\[ t=0 \]

\begin{align*}
I &= 0A \\
I^+ &= 1A \\
I &= 1A \\
I^- &= -\frac{1}{3}A \\
I &= -\frac{1}{3}A \\
I^+ &= -\frac{1}{15}A \\
I &= \frac{2}{3} - \frac{1}{15} \quad = \frac{3}{5} \\
\end{align*}

\[ \text{Goddard/Cunningham} \]

ECE329 Cumulative Review
**Very Important**

In SS, TL Looks Like a Wire

\[ V_{SS} = V_{SS}^+ + V_{SS}^- = 75V \]
\[ I_{SS} = I_{SS}^+ + I_{SS}^- = 0.625A \]

\[ V_{SS} = 100V \frac{120\Omega}{(40+120)\Omega} = 75V \]
\[ I_{SS} = \frac{100V}{(40+120)\Omega} = 0.625A \]
**Very Important**

**Algebra of the Bounce Diagram**

For \( f(t) = \delta(t) \), the solution of:

\[
V^+(0, t) = \tau_s \delta(t) + \Gamma_s \Gamma_L V^+(0, t - \frac{2l}{v_p})
\]

is an impulse response:

\[
V^+(0, t) = \tau_s \sum_{n=0}^{\infty} \left(\Gamma_s \Gamma_L\right)^n \delta(t - n \frac{2l}{v_p}) \equiv h^+(t)
\]

\[
V^-(0, t) = \Gamma_L V^+(0, t - \frac{2l}{v_p})
\]

\[
= \Gamma_L \tau_s \sum_{n=0}^{\infty} \left(\Gamma_s \Gamma_L\right)^n \delta(t - (n + 1) \frac{2l}{v_p}) \equiv h^-(t)
\]

For arbitrary position \( z \), replace \( t \) with \( t \pm z/v_p \):

\[
V^+(z, t) = \tau_s \sum_{n=0}^{\infty} \left(\Gamma_s \Gamma_L\right)^n \delta(t - \frac{z}{v_p} - n \frac{2l}{v_p})
\]

\[
V^-(z, t) = \Gamma_L \tau_s \sum_{n=0}^{\infty} \left(\Gamma_s \Gamma_L\right)^n \delta(t + \frac{z}{v_p} - (n + 1) \frac{2l}{v_p})
\]

Note: the voltage is nonzero only on the bounce lines

\[
I = \frac{1}{Z_0} (V^+ - V^-)
\]
**Very Important**

### Algebra of the Bounce Diagram

For $f(t) = \delta(t)$, the solution was:

\[
V^+(z, t) = \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t - \frac{z}{v_p} - n\frac{2l}{v_p})
\]

\[
V^-(z, t) = \Gamma_L \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n \delta(t + \frac{z}{v_p} - (n+1)\frac{2l}{v_p})
\]

\[
I = \frac{1}{Z_0} (V^+ - V^-)
\]

For arbitrary $f(t)$, convolve the solution with $f$:

\[
V^+(z, t) = \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n f(t - \frac{z}{v_p} - n\frac{2l}{v_p})
\]

\[
V^-(z, t) = \Gamma_L \tau_s \sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n f(t + \frac{z}{v_p} - (n+1)\frac{2l}{v_p})
\]

\[
I = \frac{1}{Z_0} (V^+ - V^-)
\]

Note: the voltage can be nonzero in between the bounce lines depending on the function $f$.
**Writing Moving Delta Functions**

\[ \delta(t) \]

- a pulse centered at \( t = 0 \)

\[ \delta(t - \frac{z}{v}) \]

- a forward-moving pulse centered at \( z = vt \)

\[ \delta([t - 2T] - \frac{z - 0}{v}) = \delta(t - \frac{z}{v} - 2T) \]

- a forward-moving pulse that passes through \((z, t) = (0, 2T)\)

\[ \delta(t + \frac{z}{v}) \]

- a backward-moving pulse centered at \( z = -vt \)

\[ \delta([t - T] + \frac{z - L}{v}) = \delta(t + \frac{z}{v} - 2T) \]

- a backward-moving pulse that passes through \((z, t) = (L, T)\)

because \( T = L/v \)

*Important*
Example 2, Step 1: $V^+$, $I^+$, $\Gamma$

$$V^+ = V_0 \frac{Z_0}{R_g + Z_0} = 1 \frac{50}{50 + 50} = 0.5\delta$$

$$I^+ = \frac{V^+}{Z_0} = \frac{0.5}{50} = 0.01\delta$$
Transmission Line Junction

Equivalent circuit “seen” by V+ when it gets to the end of line 1:

\[ V^+ + V^- \quad V^{++} R_2 \]

\[ \Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_{01}}{R_L + Z_{01}} \]

\[ R_L = (R_1) \parallel (R_2 + Z_{02}) \]

What is R_L for this equivalent circuit?
How much voltage gets transmitted through to line 2?

\[
\tau_v = \frac{V^{++}}{V^+} = 1 + \Gamma
\]

\[
V^{++} = V^+ (1 + \Gamma)
\]

\[
V_{\text{trans}} = \frac{Z_{02}}{Z_{02} + R_2} V^{++}
\]
Inductive Termination

For the inductor:

\[ V = L \frac{dI}{dt} \]
**Diff Eqn for the Inductor**

\[
V = L \frac{dI}{dt} \quad \Rightarrow \quad (V^+ + V^-(t)) = L \frac{d(I^+ + I(t))}{dt}
\]

\[
\frac{V_0}{2} + V^-(t) = L \frac{d}{dt} \left( \frac{V_0}{2Z_0} - \frac{V^-(t)}{Z_0} \right)
\]

\[
\frac{V_0}{2} = -\frac{L}{Z_0} \frac{dV^-(t)}{dt} - V^-(t)
\]

\[
\frac{dV^-(t)}{dt} + \frac{Z_0}{L} V^-(t) = -\frac{Z_0}{L} \frac{V_0}{2}
\]

**Laplace Transform \( V(t') \) to \( F(s) \)**

\[
s\hat{V}^-(s) - V^-(0) + \frac{Z_0}{L} \hat{V}^-(s) = -\frac{Z_0}{L} \frac{V_0}{2} \frac{1}{s}
\]
Optional

Laplace Transform for Inductor

Initial Condition

At \( t=T, \) i.e. \( t'=0, \) inductor current = 0
since inductor “looks” like an OPEN CIRCUIT

\[
I = I^+ + I^-(T) = 0 \quad \Rightarrow \quad \frac{V_0}{2Z_0} - \frac{V^-(0)}{Z_0} = 0 \quad \Rightarrow \quad V^-(0) = \frac{V_0}{2}
\]

\[
s\hat{V}^- (s) - \frac{V_0}{2} + \frac{Z_0}{L} \hat{V}^- (s) = -\frac{Z_0}{L} \frac{V_0}{2} \frac{1}{s} \quad \Rightarrow \quad \hat{V}^- (s) = \frac{sL - Z_0}{sL + Z_0} \frac{V_0}{2s}
\]

In s-space, we have \( V^-(s) = \Gamma(s) \ V^+(s) \) with:

\[
\Gamma(s) = \frac{Z(s) - Z_0}{Z(s) + Z_0} \quad \hat{V}^+(s) = \frac{V_0}{2s} \quad Z(s) = sL \quad \text{for an inductor}
\]
Optional

Invert Laplace Transform

\[ \hat{V}(s) = \hat{V}^+ + \hat{V}^- = \frac{V_0}{2s} + \frac{V_0}{2s} \frac{sL - Z_0}{sL + Z_0} = \frac{V_0}{2s} \frac{2sL}{sL + Z_0} = \frac{V_0}{s + Z_0 / L} \]

\[ V(t) = V_0 e^{-\left(\frac{Z_0}{L}\right)t'} = V_0 e^{-\left(\frac{Z_0}{L}\right)(t-T)} \quad \text{t}>T \]

V(t) [volts]

\[ V_0 \]

\[ \text{t} \]
Shortcut Method

What happens “in between”?

We know $V(t)$ has exponential decay with time constant of $L/Z_0$

At $t=T$ the inductor “looks” like an open ckt

At $t=\infty$ the inductor “looks” like a closed ckt

$$\tau = Z_{eq} C$$

$$\tau = \frac{L}{Z_{eq}}$$

Optional
Optional

Initial distribution = Forward + Backward waves

The voltage (made up of charges on the TL) will begin to spread by forming + and – waves.
Example. TL w/ $Z_0 = 50\Omega$

\[ V^+(z,0) = \frac{V(z,0) + 50I(z,0)}{2} \]

\[ V^-(z,0) = \frac{V(z,0) - 50I(z,0)}{2} \]
Phasors satisfy usual TL equations

\[
\frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t}
\]
\[
\frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t}
\]

\[
\frac{d\bar{V}}{dz} = -L(j \omega \bar{I})
\]
\[
\frac{d\bar{I}}{dz} = -C(j \omega \bar{V})
\]

\[
\bar{V} = V^\pm e^{\mp j \beta z}
\]

\[
\bar{I} = \pm \frac{V^\pm}{Z_0} e^{\mp j \beta z}
\]

\[
\beta = \omega \sqrt{L/C}
\]

\[
\bar{V} = V^+ e^{j \beta d} + V^- e^{-j \beta d}
\]

\[
\bar{I} = \frac{1}{Z_0} (V^+ e^{j \beta d} - V^- e^{-j \beta d})
\]

\[
Z_0 = \sqrt{L/C}
\]
** Very Important

Key Definition: Line Impedance

\[ Z(d) \equiv \frac{\tilde{V}(d)}{\tilde{I}(d)} \]

\[ Z(d) = Z_0 \frac{1 + \Gamma_L e^{-2j\beta d}}{1 - \Gamma_L e^{-2j\beta d}} \]

\[ \tilde{V}(l) = \tilde{F} \frac{Z(l)}{Z(l) + Z_g} = V^+(e^{j\beta l} + \Gamma_L e^{-j\beta l}) \]

Allows you to solve for \( V^+ \) and thus get \( V(d,t) \) and \( I(d,t) \)
**Very Important**

Key Definition: Generalized Reflection Coefficient

\[ \Gamma(d) \equiv \frac{\tilde{V}^{-}(d)}{\tilde{V}^{+}(d)} \]

\[ \Gamma(d) = \frac{V^{-}e^{-j\beta d}}{V^{+}e^{j\beta d}} = \Gamma_L e^{-2j\beta d} \]

Allows you to find the backwards wave if forward wave is known.
Key Definitions: Admittance and Normalized Impedance

** Very Important

Characteristic Admittance

\[ Y_0 \equiv \frac{1}{Z_0} \]

Normalized Impedance

\[ z(d) \equiv \frac{Z(d)}{Z_0} \]

\[ z(d) = \frac{1 + \Gamma(d)}{1 - \Gamma(d)} \]

Normalized Admittance

\[ y(d) \equiv \frac{1}{z(d)} \]

\[ y(d) = \frac{1 - \Gamma(d)}{1 + \Gamma(d)} \]

\[ \Gamma(d) = \Gamma_L e^{-2j\beta d} \]
*** Extremely Important

Summary of TL Equations

\[
\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}
\]

\[
V^- = \Gamma_L V^+
\]

\[
\Gamma(d) = \Gamma_L e^{-2j\beta d}
\]

\[
\bar{V}(d) = V^+ e^{j\beta d} (1 + \Gamma_L e^{-2j\beta d}) = V^+ e^{j\beta d} (1 + \Gamma(d))
\]

\[
\bar{I}(d) = \frac{V^+ e^{j\beta d}}{Z_0} (1 - \Gamma_L e^{-2j\beta d}) = \frac{V^+ e^{j\beta d}}{Z_0} (1 - \Gamma(d))
\]

\[
Z(d) \equiv \frac{\bar{V}(d)}{\bar{I}(d)} = Z_0 \frac{1 + \Gamma(d)}{1 - \Gamma(d)}
\]

\[
y(d) \equiv \frac{1}{z(d)} = z(d \pm \frac{\lambda}{4})
\]

\[
z(d) = \frac{1 + \Gamma(d)}{1 - \Gamma(d)}
\]

\[
\Gamma(d) = \frac{z(d) - 1}{z(d) + 1}
\]
Smith Chart

\[ \Gamma = \frac{z - 1}{z + 1} \]

\[ z = r + jx \]

S.C. is a map between the half-plane \( r \geq 0 \) and the disc \( |\Gamma| \leq 1 \)
Standing Waves for SC Line

\[ V(d) = Z_0 \]

\[ I(d) \]

\[ \Gamma_L = -1 \]

\[ \bar{V}(d) = V^+(e^{j\beta d} - e^{-j\beta d}) = 2jV^+ \sin(\beta d) \]

\[ \bar{I}(d) = \frac{V^+}{Z_0}(e^{j\beta d} + e^{-j\beta d}) = 2Y_0V^+ \cos(\beta d) \]

\[ Z(d) \equiv \frac{\bar{V}(d)}{\bar{I}(d)} = jZ_0 \tan \beta d \]
Standing Waves for SC Line

\[
V(d, t) = -2|V^+| \sin(\beta d) \sin(\omega t + \theta)
\]

\[
I(d, t) = 2Y_0|V^+| \cos(\beta d) \cos(\omega t + \theta)
\]

V(0,t)=0 always (voltage null)

I(0,t) varies (current maxima)

Dependence is different than traveling wave: \(\omega t \pm \beta z\)
Input Impedance for SC Line

\[ Z_{in} = Z(l) = jZ_0 \tan \beta l \]
SC Line can act as an inductor or a capacitor depending on $\beta l$

$$Z_{in} = jZ_0 \tan \beta l$$

If $\tan(\beta l) > 0$, shorted TL is inductive

If $\tan(\beta l) < 0$, shorted TL is capacitive

e.g. $\beta l < \pi/2$ or $l < \lambda/4$, TL is inductive

* Important

Equivalent Circuit:
For $\beta l = 0, \pi, 2\pi, \ldots$ TL is a short
For $\beta l = \pi/2, 3\pi/2, \ldots$ TL is an open

\[
Z_{in} = jZ_0 \tan \beta l = \begin{cases} 
0 = & \text{a short for } \beta l = n\pi, \ n = 0, 1, 2, \ldots \\
\infty = & \text{an open for } \beta l = (n + 1/2)\pi 
\end{cases}
\]

If $Z_{in} = 0$, voltage drop is zero, just like a short
If $Z_{in} = \infty$, current is zero, just like an open

\[
\beta l = n\pi \\
l = \text{even } \lambda/4
\]

\[
\beta l = (n+1/2)\pi \\
l = \text{odd } \lambda/4
\]
Standing Waves for OC Line

Same phasor algebra as before with current & voltage reversed!

\[ I(d,t) = -2Y_0 |V^+| \sin(\beta d) \sin(\omega t + \theta) \]

\[ V(d,t) = 2|V^+| \cos(\beta d) \cos(\omega t + \theta) \]

\[ \beta = \frac{2\pi}{\lambda} \]

\[ d = \frac{\lambda}{4} \]

Oscillates in time

\[ I(0,t) = 0 \text{ always (current null)} \]

\[ V(0,t) \text{ varies (voltage maxima)} \]
For $\beta l=0, \pi, 2\pi, \ldots$ TL is an open
For $\beta l=\pi/2, 3\pi/2, \ldots$ TL is a short

$$Y_{in} = jY_0 \tan \beta l = \begin{cases} 
0 = \text{an open for } \beta l = n\pi, \ n = 0, 1, 2, \ldots \\
\infty = \text{a short for } \beta l = (n+1/2)\pi \end{cases}$$

If $Y_{in}=0$, current is zero, just like an open
If $Y_{in}=\infty$, voltage drop is zero, just like a short
Parallel & Series Resonances

- We can find $\lambda_n$ and $\omega_n$ by applying the appropriate BCs at both ends.

- Series resonance if $z_{in}(l)=0$
  - Analogous to an LC circuit in series and requires a short placed across TL at $d=l$
  - Like a short input

- Parallel resonance: $y_{in}(l)=0$
  - Analogous to an LC circuit in parallel and requires an open across the TL at $d=l$
  - Like an open input

*Important*
Parallel & Series Resonances

*Important*

**Shorted Load**

- \( L = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots \)
- Input is like an open
  - \( \rightarrow \) **Parallel** resonance

**Open Load**

- \( L = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots \)
- Input is like a short
  - \( \rightarrow \) **Series** resonance
**Very Important**

Standing Wave for Arbitrary Load

\[ |V(d)| = |V^+| |1 + \Gamma(d)| \]
\[ |I(d)| = Y_0 |V^+| |1 - \Gamma(d)| \]

\[ \Gamma = 0.75 e^{j \pi/4} \]

Unlike SC or OC where \(|\Gamma|=1\), now we have imperfect nulls for voltage and current b/c \(|\Gamma|<1\)

\[ d_{max} = \frac{\lambda}{4\pi} \theta \]

\[ VSWR = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \]

\[ VSWR = z(d_{max}) = y(d_{min}) \]
Quarter Wave Transformers

Key Observations:
1. $Z_0$ and $Z_q$ are real since TL’s are lossless
2. $Z_1 = Z_0$ for a match so $Z_1$ must be real
3. $Z_1 Z_2 = Z_q^2$ since we know $y(d) = z(d \pm \lambda/4)$
4. $Z_2$ must be real from equation #3
5. $d_q$ must be at a voltage max or min from #4

Adjust $Z_q$ and $d_q$ for match
Quarter Wave Transformers

- Quarter wave transformer matching inserts a length of $l = \frac{\lambda}{4}$ of a specific impedance $Z_q = \sqrt{Z_0 Z_2}$ at a distance $d_q$ from the load, where voltage is min or max.

**Very Important**

Pros: Easy design
Cons: Have to redo QWT for each $f$
**Very Important**

Smith Chart for Single Stub

Key Observations:
1. $y_{in} = 1$ for a match, $y_s = jb$ is purely imaginary for the SC stub
2. Thus $y_1' = y_{in} - y_s = 1 - jb$ must be on the unit conductance circle
   a. Needed amount of susceptance, $b$, depends on $|\Gamma_L|$
** Very Important

Single Stub Matching

• Single stub matching inserts a shorted stub of the same impedance, $Z_0$, but with a specific length, $l_s$, and distance, $d_s$, from the load (not necessarily at $V_{\text{max}}$ or $V_{\text{min}}$)

Pros: $Z=Z_0$ on all lines
Cons: Adjusting $d_s$ may be inconvenient

• Find $d_s$ by moving CW from $y(0)$ to the unit conductance circle $y_1'=1-jb$; then find $l_s$ by going CW from $y_s(0)=\infty$ to $y_s(l_s)=+jb$ or using formula:

$$l_s = \frac{\lambda}{2\pi} \tan^{-1}\left(-\frac{1}{b}\right) \mod \lambda/2$$
Repeat D7.8a (p 472):

\( z(0) = 0.5 \)

Thus, we need a stub with \( y_s(l_s) = 0.72j \) at a distance of
\( 0.098\lambda \) from load.

\( y_1' = 1 - jb \)

\( y_1 = 1 - 0.72j \)

so \( b = +0.72 \)
Find length for shorted stub: \( y(l_s) = 0.72j \)

\( z(0) = 0 \)
\( y(0) = \infty \)

\[ l_s = (0.25 + 0.098)\lambda \]
\[ l_s = 0.348\lambda \]

or use:

\[ l_s = \frac{\lambda}{2\pi} \tan^{-1}\left( -\frac{1}{b} \right) \mod \frac{\lambda}{2} \]

\[ l_s = -0.151\lambda = 0.349\lambda \]
Optional

Double Stub Matching w/ S.C.

Key Observations:
1. For a match, $y_2'$ must be on unit conductance circle
2. Thus, $y_1$ is on the auxiliary circle
   a) **AUXILIARY CIRCLE** is UCC pivoted **CCW** towards the **load** by $d_{12}$
Repeat DSM: \( d_1 = 0 \),
\( d_{12} = 0.375 \lambda \)
z(0) = 0.6 - 0.8j

To get \( y_2' \) on UCC, need \( y_1 \) on AUX, so draw AUX first using given \( d_{12} \)

\[ \text{UCC: } y_2' = 1 - jb \]

\[ \text{AUX = pivoted UCC by } d_{12} \text{ CCW (to load)} \]
Repeat DSM: $d_1=0$, $d_{12}=0.375\lambda$

$z(0)=0.6-0.8j$

$y(0)=0.6+0.8j$

Since $d_1=0$, $y_1'=y(0)$
Find susceptance to jump from $y_1'$ along constant conductance circle to $y_1$ on AUX

$b_1 = -0.89$ or $b_1 = -2.72$

$$l_s = \frac{\lambda}{2\pi} \tan^{-1} \left( -\frac{1}{b} \right)$$

$l_1 = 0.134\lambda$ or $l_1 = 0.056\lambda$
Find corresponding $y_2'$ by going from AUX to UCC

$y_2' = 1 - 0.53j$ or $y_2' = 1 + 2.5j$

$b_2 = 0.53$ or $b_2 = -2.5$

$$l_s = \frac{\lambda}{2\pi} \tan^{-1} \left( -\frac{1}{b} \right)$$

$l_2 = 0.327\lambda$ or $l_2 = 0.061\lambda$
Double Stub Matching

• Double stub matching inserts two shorted stubs of impedance $Z_0$ with specific lengths, $l_1$ and $l_2$, at fixed spots, $d_1$ and $d_1+d_{12}$, from the load.

Pros: $Z=Z_0$ on all lines, fixed $d_1$, $d_{12}$
Cons: Many calculations

• Go from $y(0)$ to $y_1'$ a distance $d_1$ along $|\Gamma(0)|$ circle. Find $b_1$ by going from $y_1'$ along constant conductance circle to $y_1$ which is on the AUX circle. Find $y_2'$ by pivoting AUX by $d_{12}$ to UCC and read off $b_2=-\text{Im}(y_2')$.

• Calculate $l_1$ and $l_2$ using:

$$l_s = \frac{\lambda}{2\pi} \tan^{-1}(-\frac{1}{b}) \mod \lambda/2$$
Corporate Ladder

- A network for combining 4 identical loads $Z_L$ into an equivalent single load $Z_{in} = Z_L$

By symmetry, the average power delivered to each load is identical.
Distributed Circuit of Lossy TL

Model the Ohmic losses in conducting wires and leakage losses in the imperfect dielectric in between.

One slice of the lossy TL:

\[
\frac{\partial V}{\partial z} = -(j \omega L + R)I
\]

\[
\frac{\partial I}{\partial z} = -(j \omega C + G)V
\]
Solution of Lossy TL

\[ \tilde{V}(d) = V^+ e^{\gamma d} + V^- e^{-\gamma d} \]

\[ \gamma = \alpha + j\beta = \sqrt{(j\omega L + R)(j\omega C + G)} \]

\[ \tilde{I}(d) = \frac{V^+ e^{\gamma d}}{\tilde{Z}_0} - \frac{V^- e^{-\gamma d}}{\tilde{Z}_0} \]

\[ \bar{Z}_0 = \frac{\sqrt{j\omega L + R}}{\sqrt{j\omega C + G}} \]

These reduce to the lossless results as \( R \) and \( G \to 0 \)

\[ \bar{\gamma} = \sqrt{(j\omega L)(j\omega C)} = j\omega \sqrt{LC} = j \frac{\omega}{v_p} = j\beta \]

\[ \bar{Z}_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}} \]
Good luck on the final!

- It was a pleasure teaching ECE329 this semester
  - Thank you for studying so hard 😊