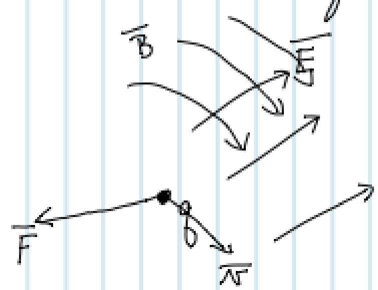
SUMMARY

We are now poised to started solving real problems. Most electromagnetic analysis is based on the solution of the differential form of Maxwell's equations and we have accumulated (almost) enough mathematical tools to begin learning how to use them. BUT FIRST — REVIEW

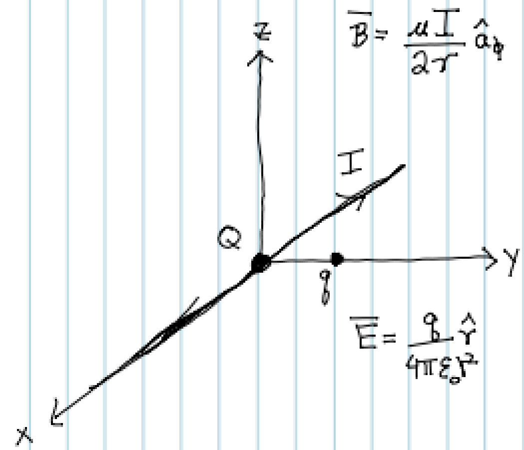
I) LORENTZ equation $\bar{F} = q\bar{E} + q\bar{v} \times \bar{B}$ ← Magnetic field

force on charge q · charge · electric field · velocity



Math Tools:

- vector algebra
- vector notation
- concept of fields



visualization exercise

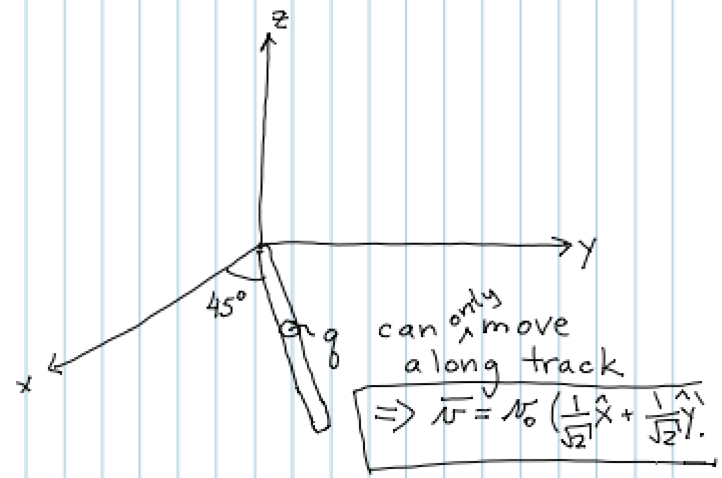
find the force on q
if its velocity is:

a) $\bar v = 0$ at $(1,1,0)$

b) $\bar v = -1\hat x$ $(0,1,0)$

c) $\bar v = 1\hat z$ $(0,1,0)$

d) $\bar v = 1\hat z$ $(0,0,1)$



Given that a external electric and magnetic field exist (but are unknown), design an experiment to determine

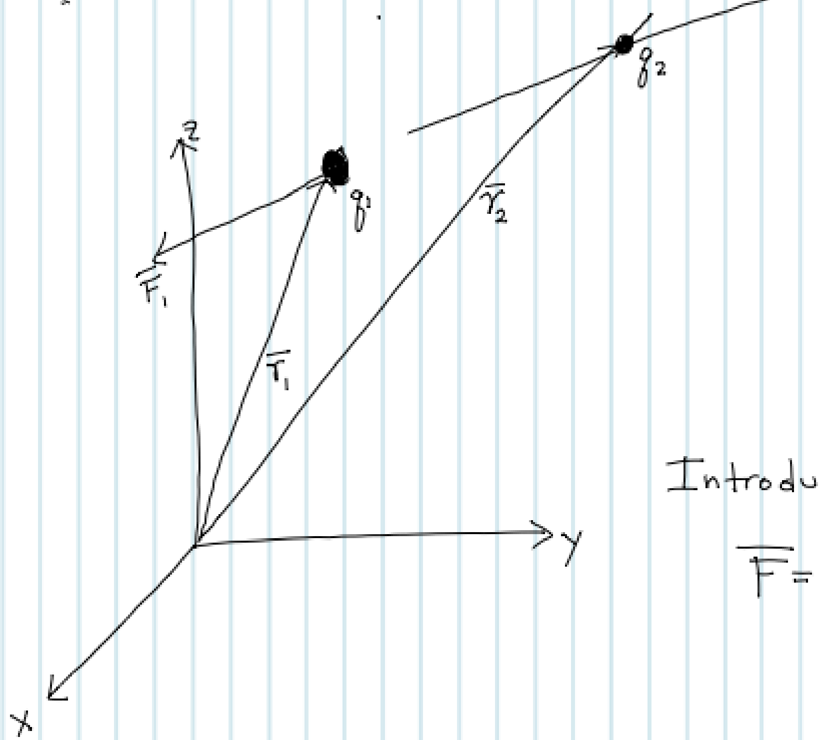
$\bar E = E_x\hat x + E_y\hat y + E_z\hat z$

and

$\bar B = B_x\hat y + B_y\hat y + B_z\hat z$.

Can you find all components?

II) COULOMB's Law



$$\bar{F}_2 = \frac{q_1 q_2}{4\pi\varepsilon_0 |\bar{r}_2 - \bar{r}_1|^2} \cdot \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|}$$

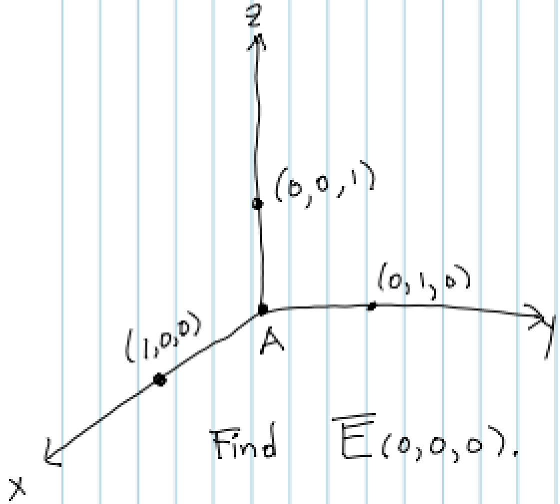
written in terms of position vectors

$$\bar{r}_1 = x_1\hat{x} + y_1\hat{y} + z_1\hat{z}$$
$$\bar{r}_2 = x_2\hat{x} + y_2\hat{y} + z_2\hat{z}$$

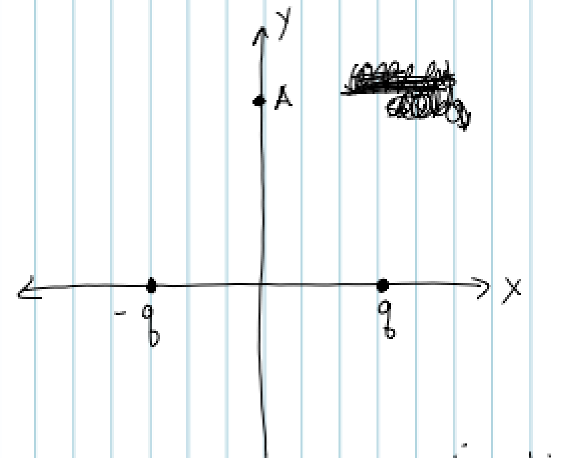
$\bar{F}_1 = ?$

Introduced the electric field $\bar{E}$ ← V/m

$\bar{F} = q\bar{E}$ ← force per unit charge.

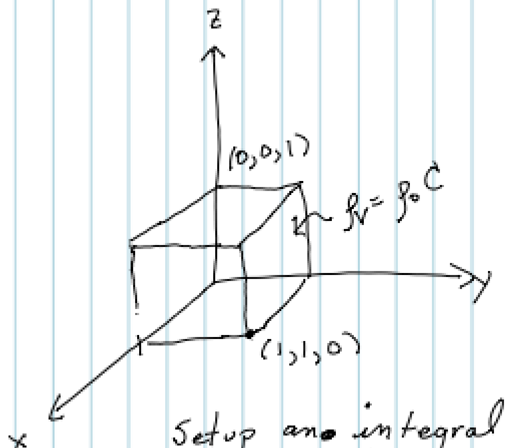


Find $\bar{E}(0,0,0)$.



i) what is the direction of the force at A?

ii) Its magnitude.



Setup an integral that computes $\bar{E}$ everywhere for this cube of charge

III. GAUSS' LAW (INTEGRAL FORM)

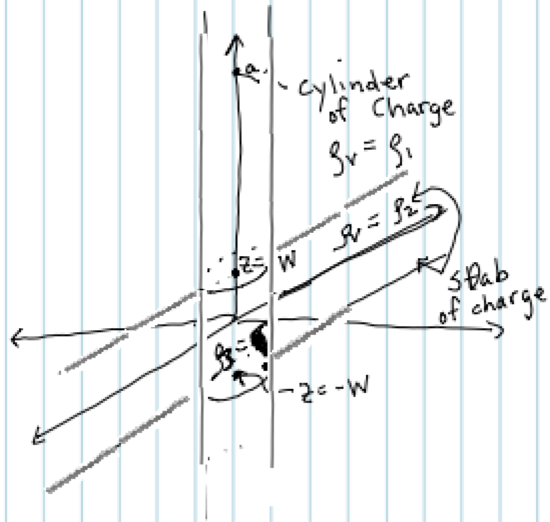
$$\underbrace{\oint \varepsilon_0 \bar{E} \cdot d\bar{s}}_{\text{the electrical flux through any closed surface equals...}} = \oint \bar{D} \cdot d\bar{s} = \underbrace{Q_{in}}_{\text{... the charge enclosed}}$$

the electrical flux ... the charge enclosed

through <u>any</u> closed surface equals...

MATH TOOLS:

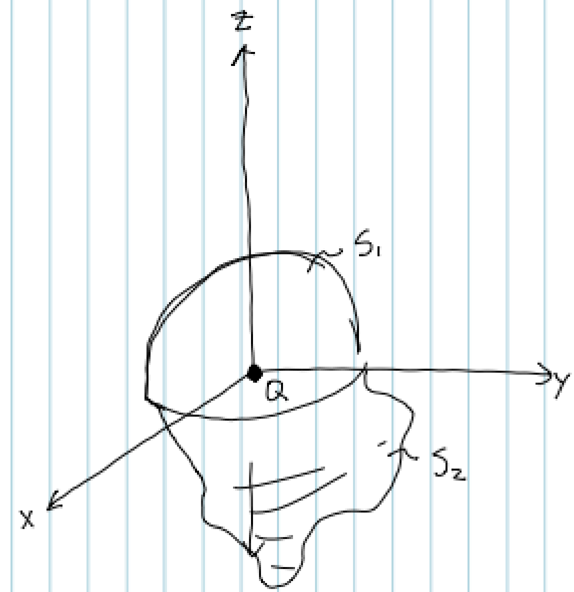
- Surface integrals
- Divergence theorem



i) can the be solved using the relation - $\oint \varepsilon_0 \bar{E} \cdot d\bar{s} = Q$?

ii) if so what must $\rho_3 = ?$

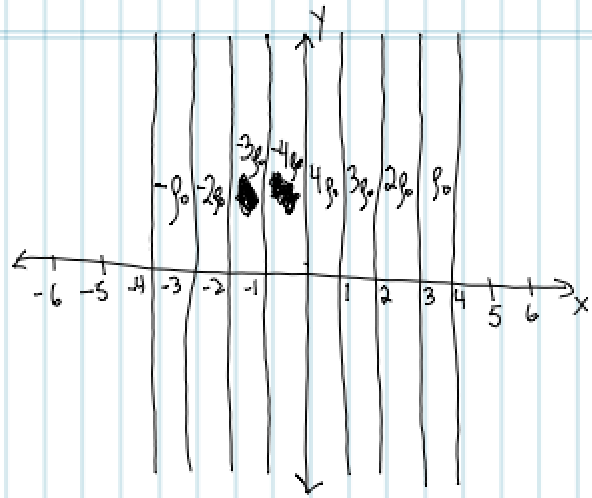
iii) if so find $\bar{E}$ everywhere.



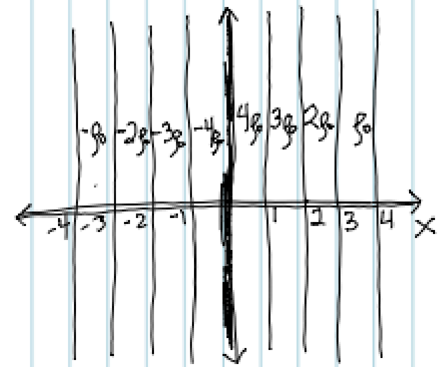
i) find $\int \bar{E} \cdot d\bar{S}$ through the hemisphere S_1

ii) find $\int \bar{E} \cdot d\bar{s}$ through S_2

iii) design a symmetric charge dist. + surf



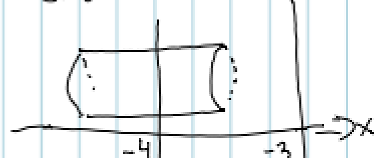
Find $\bar{E}$ everywhere –



METHOD II: Gauss' Law (tediously)

By symmetry $\overline{E} = E_x(x)\hat{x}$

Choose Surface of integration



Choose any cylindrical surface with sides parallell to field.

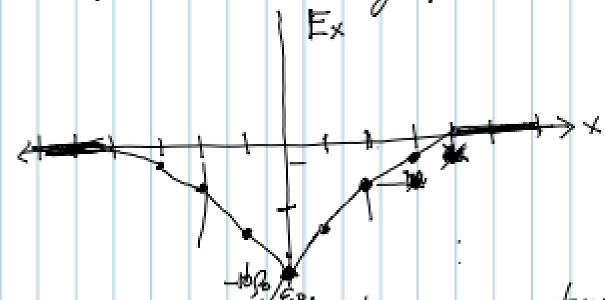
$$\oint \overline{D}\cdot d\overline{s} = \oint \varepsilon_0 \overline{E}\cdot d\overline{s} = \int_{left} \varepsilon_0 \overline{E}\cdot ds + \int_{right} \varepsilon_0 \overline{E}\cdot ds$$

$$\Rightarrow d\overline{s} = \pm dy\,dz\,\hat{x}$$

$$\int \varepsilon_0 \overline{E}\cdot d\overline{s} = \int \varepsilon_0 E_x\,dx\,dy$$

Find $\overline{E}$ everywhere:

Method I - the graphical approach.



i) Argue that by symmetry $\overline{E} = E_x(x)\hat{x}$

ii) Argue that $\overline{E} = 0$ for $x>4$ and $x<-4$. WHY?

iii) Find $\overline{E}$ at $x=3$

to the right of All slabs EXCEPT the one between $x=3$ and $x=4$.

$$\overline{E} = \overline{E}_{right} + \overline{E}_{left}$$

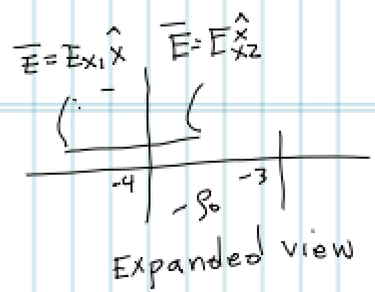
$$= \left[-\frac{\rho_0\cdot 1}{2\varepsilon_0} + -\frac{\rho_0\cdot 1}{2\varepsilon_0}\right]\hat{x} = -\frac{\rho_0}{\varepsilon_0}\hat{x}$$

iv) Find $\overline{E}$ at $x=2$.

$$\overline{E} = \left[\underbrace{\frac{-3\rho_0}{2\varepsilon_0}}_{\text{REPEAT}} + \frac{-3\rho_0\cdot 1}{2\varepsilon_0}\right]\hat{x} = -\frac{3\rho_0\cdot 1}{\varepsilon_0}\hat{x}$$

METHOD II: (cont...)

$$\oint \varepsilon_0 \bar{E}\cdot d\bar{s} = -\int_{\text{left}} \underbrace{\varepsilon_0 E_{x1}(x)}_{\text{field outside slab}}\,dydz + \int_{\text{right}} \underbrace{\varepsilon_0 E_{x2}(x)}_{\text{field in slab}}\,dydz$$



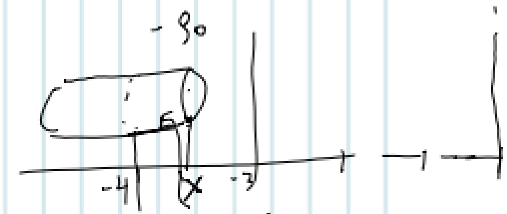
Now we must argue that

$E_{x1} = 0$

So that $\oint \varepsilon_0 \bar{E}\cdot d\bar{s} = +\varepsilon_0\int_{\text{right}} E_{x2}(x)\,dydz$.

E_{x2} is a constant value on the face so the integral evaluates to:

$$\boxed{\varepsilon_0 E_{x2}(x)\cdot \underbrace{\pi a^2}_{\text{area of cylinder's cap}}}$$



find volume

$$\underbrace{\iint \int_{-4}^{x} (-\rho_0)\,dxdydz}_{\pi a^2\cdot(4+x)(-\rho_0)}.$$

Now find the amount of charge enclosed by the surface.

EQUATE: $\varepsilon_0 E_{x2}(x)\cancel{\pi a^2} = \cancel{\varepsilon_0 \pi a^2}\,(4+x)(-\rho_0)$

$$E_{x2}(x) = \frac{\rho_0(x+4)}{\varepsilon_0}$$

at $x=-4$ $E_{x2}=0$
$x=-3$ $E_{x2} = -\rho_0$

IV Gauss' Law (differential form)

$$\nabla \cdot \overline{D} = \rho$$

$\Uparrow$

electric flux density C/m^2

- $\nabla \cdot \overline{D} = 0$

MATH TOOLS:
- divergence theorem
- operator notation
- concept of divergence

Suppose - $\overline{E} = \frac{x}{z^2}\hat{x} + \frac{y}{z^2}\hat{y}$

determine the charge density ρ that generated $\overline{E}$.

Suppose $\rho = \rho_0 - \frac{\rho_0 x}{W}$ $0 < x < W$

find $\overline{D}$ using differential form.

Suppose - $\overline{E} = \frac{x}{z^2}\hat{x} + \frac{y}{z^2}\hat{y}$

$\boxed{\text{Determine } \rho \text{ -}}$

By Gauss' LAW

$$\nabla \cdot \overline{E} = \rho$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} =$$

$$\frac{\partial}{\partial x}\left(\frac{x}{z^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{z^2}\right) + \frac{\partial}{\partial z}(0)$$

$$= \frac{1}{z^2} + \frac{1}{z^2} = \frac{2}{z^2} = \rho \, ?$$

REALLY.

$$\nabla \times \overline{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{z^2} & \frac{y}{z^2} & 0 \end{vmatrix} = +\frac{2y}{z^3}\hat{x} + \frac{-2x}{z^3}\hat{y}$$

$$\neq 0$$

Suppose $\rho = \rho_0 - \frac{\rho_0 x}{W}$ $\quad 0 < x < W$

$\boxed{\text{find } D \text{ using the differential form}}$

$$\nabla \cdot \underline{D} = \rho$$

symmetry $\overline{D} = D_x(x)\hat{x}$

$$\Rightarrow \nabla \cdot \overline{D} = \frac{\partial D_x}{\partial x}$$

In region I and III:

$$\frac{\partial D_x}{\partial x} = 0 \Rightarrow D_x = \text{const.}$$

In region II:

$$\frac{\partial D_x}{\partial x} = \rho_0 - \frac{\rho_0 x}{W}$$

$$D_x = \rho_0 x - \frac{\rho_0 x^2}{2W} + \text{const.}$$

V. Maxwell's Eq. for Static fields (Electric field)

$$\nabla \cdot \overline{D} = \rho \iff \oint \overline{D} \cdot d\overline{s} = Q$$

$$\nabla \times \overline{E} = 0 \iff \oint \overline{E} \cdot d\overline{l} = 0$$

MATH TooLs:

- Faraday's Law (integral form)
  - line integral
  - STOKES theorem
- Faraday's Law (differential form)
  - Curl
  - Conservative field

Given a vector field $\overline{F} = (y+z)\,\hat{x} + (x+z)\,\hat{y} + (x+y)\,\hat{z}$

i) Is it conservative?

ii) Can a potential function be found such that

$$\overline{F} = \nabla \phi \,?$$

Repeat for $\overline{F} = (x+2z)\,\hat{x} + (x+z)\,\hat{y} + (x+y)\,\hat{z}$

Solutions:

Given the vector field $\bar F = (y+z)\hat x + (x+z)\hat y + (x+y)\hat z$

i) is $\bar F$ a conservative field?

METHOD 1: take the curl –

$$\nabla \times \bar F = \begin{vmatrix} \hat x & \hat y & \hat z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+z & x+z & x+y \end{vmatrix} = \cancelto{0}{\left[\underbrace{\frac{\partial}{\partial y}(x+y)}_{=1} - \underbrace{\frac{\partial}{\partial z}(x+z)}_{=1}\right]}\hat x +$$

$$\cancelto{0}{\left[\frac{\partial}{\partial z}(y+z) - \frac{\partial}{\partial x}(x+y)\right]}\hat y +$$

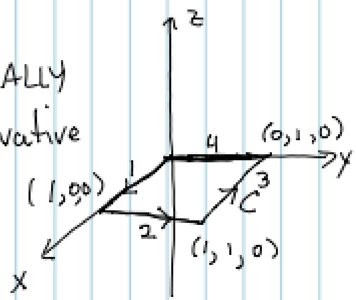
$$\cancelto{0}{\left[\frac{\partial}{\partial x}(x+z) - \frac{\partial}{\partial y}(y+z)\right]}\hat z$$

$\Rightarrow$ yes $\bar F$ describes a conservative field

METHOD 2: find the line integral around closed path.

METHOD 2: (cont. ...)

drawback – to REALLY, REALLY prove that $\overline{F}$ is conservative must repeat infinitely many times



pick a simple path (contour C) and solve

$$\oint_C \overline{E}\cdot d\overline{l} \overset{?}{=} 0$$

Leg 1: $\overline{E} = (y+z)\,\hat{x} + (x+z)\,\hat{y} + (x+y)\,\hat{z}$

$d\overline{l} = dx\,\hat{x}$

$$\int \overline{E}\cdot d\overline{l} = \int_0^1 (y+z)\,dx$$

Leg 2: $d\overline{l} = dy\,\hat{y}$

$$\int E\cdot d\overline{l} = \int_0^1 (x+z)\,dy$$

Leg 3: $$\int \overline{E}\cdot d\overline{l} = \int_1^0 (y+z)\,dx$$

Leg 4: $$\int \overline{E}\cdot d\overline{l} = \int_1^0 (x+z)\,dy$$

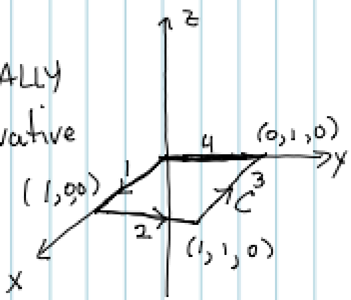
cancel

cancel

did not need to know location of path to find solution.

METHOD 2: (cont. ...)

drawback – to REALLY, REALLY prove that $\overline{F}$ is conservative must repeat infinitely many times



pick a simple path (contour C) and solve

$$\oint_C \overline{E}\cdot d\overline{l} \overset{?}{=} 0$$

Leg 1: $\overline{E} = (y+z)\,\hat{x} + (x+z)\,\hat{y} + (x+y)\,\hat{z}$

$d\overline{l} = dx\,\hat{x}$

$$\int \overline{E}\cdot d\overline{l} = \int_0^1 (y+z)\,dx$$

Leg 2: $d\overline{l} = dy\,\hat{y}$

$$\int E\cdot d\overline{l} = \int_0^1 (x+z)\,dy$$

Leg 3: $$\int \overline{E}\cdot d\overline{l} = \int_1^0 (y+z)\,dx$$

Leg 4: $$\int \overline{E}\cdot d\overline{l} = \int_1^0 (x+z)\,dy$$

cancel

cancel

did not need to know location of path to find solution.

METHOD 3:

In general $\oint \bar{E} \cdot d\bar{l} = \oint E_x dx + E_y dy + E_z dz$

foa our problem –

$$= \oint (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$$

can we find an exact differential?

$$= \oint d\phi(x,y,z)$$

finding this special function leads into the next part.

VI Voltage –

Because one of Maxwell's Eq. for static electric fields constrains the CURL of $\overline{E}$ to be zero $\Rightarrow \nabla \times \overline{E} = 0$

We can use this property to define a scalar function V to replace $\overline{E}$

$$\overline{E} = -\nabla V$$

$$\boxed{\begin{array}{l}\text{MATH TOOLS:} \\ \text{– gradient} \\ \text{– potential functions} \\ \text{– exact derivative}\end{array}}$$

If $V(x,y,z) = \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9}$

find $\overline{E}$. Sketch

$V(x,y,z)$ in the x-y plane

and y-z plane -

Draw $\overline{E}$ super posed

on the graph of V

If $\overline{E} = E_o \hat{x}$ find V

Method 1:

Know relationship between

V and $\overline{E}$

$$\overline{E} = -\nabla V$$

write out components

$$\overline{E} = E_o \hat{x}$$

$$-\nabla V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

equate components:

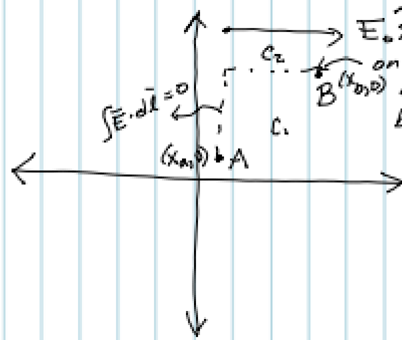
$$E_o = -\frac{\partial V}{\partial x} \Rightarrow V = -E_o x + v(y,z)$$

$$0 = -\frac{\partial V}{\partial y} \Rightarrow V = C_1 + f(x,z)$$

$$0 = -\frac{\partial V}{\partial z} \Rightarrow V = C_2 + f(x,y)$$

need to know something else about solution to determine final form.

METHOD 2: $\int_A^B \overline{E} \cdot d\overline{l} = V(A) - V(B)$



only depends on x distance between $A+B$

$$\int_A^B \bar{E}\cdot d\bar{l} = -\int_{x_a}^{x_b} E_0\,dx = -E_0 x\Big|_{x_a}^{x_b}$$

$$= E_0(x_a - x_b)$$

choose x_b to be at the origin where we know $V=0$ and x_a to be any arbitrary x

$$\boxed{V(x) = E_0 x}$$

have a general form for V

And as a lead-in for the next section –
Look what we can do with the differential form
of the equations –