

Maxwell's Equations

static electric fields

$$\nabla \cdot \vec{D} = \rho$$

relates E to its source

$$\oint \vec{D} \cdot d\vec{S} = Q$$

$$\nabla \times \vec{E} = 0$$

E is a conservative field

$$\oint \vec{E} \cdot d\vec{l} = 0$$

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relates E to its source

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This equation gives us Coulomb's law:

Use the integral form of Gauss' Law to find the field of a point charge Q centered at the origin

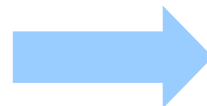
$$\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} D_r(r) r^2 \sin(\theta) d\theta d\phi = Q$$

remember $D_r(r) = \epsilon_o E_r(r)$

integrates to 4π

$$D_r(r) r^2 \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \sin(\theta) d\theta d\phi = Q$$

solving for $D_r(r)$ gives Coulomb's Law



$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r}$$

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$$\nabla \cdot \vec{D} = \rho \quad \text{relates E to its source} \quad \oint \vec{D} \cdot d\vec{S} = Q$$

This equation gives us Coulomb's law:

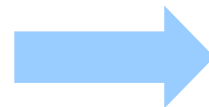
Use the differential form of Gauss' Law to find the field of a point charge Q centered at the origin

$$\nabla \cdot \vec{D} = \frac{1}{r^2} \frac{\partial (r^2 D_r(r))}{\partial r} = 0$$

remember $D_r(r) = \epsilon_0 E_r(r)$

$$\frac{1}{r^2} \frac{\partial (r^2 D_r(r))}{\partial r} = 0 \quad \text{or} \quad \frac{\partial (r^2 D_r(r))}{\partial r} = 0$$

solving for $D_r(r)$ gives Coulomb's Law



$$\vec{D} = \frac{A}{r^2} \hat{r}$$

with some hand waving we can show that $A = Q/4\pi$

Maxwell's Equations

static electric fields

$$\nabla \cdot \vec{D} = \rho$$

relates E to its source

$$\oint \vec{D} \cdot d\vec{S} = Q$$

But solving only this equation may not be enough – we want to find the electric field that satisfies this equation, but also satisfies the other constraint.

Suppose you were told that a solution to this equation $\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0$

is $\vec{D} = y \hat{x} - x \hat{y}$

can this be correct?

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Suppose you were told that a solution to this equation $\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0$

is $\vec{D} = y\hat{x} - x\hat{y}$ can this be correct?

NO

because

$$\nabla \times \vec{E} \neq 0$$

$$\oint \vec{E} \cdot d\vec{l} \neq 0$$

$$\vec{D} = \epsilon_0 \vec{E}$$

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$$\nabla \cdot \vec{D} = \rho$$

relates E to its source

$$\oint \vec{D} \cdot d\vec{S} = Q$$

But solving only this equation may not be enough – we want to find the electric field that satisfies this equation, but also satisfies the other constraint.

Now suppose you were told that a solution to this equation $\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0$

is $\vec{D} = y \hat{x} + x \hat{y}$ can this be correct?

YES

because

$$\nabla \times \vec{E} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

Maxwell's Equations

static electric fields

$$\nabla \cdot \vec{D} = ?$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \neq 0$$

$$\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial 0}{\partial z} = y + x$$

for this to be a valid electric field
then charge distribution generating
the field must be $\rho_v = y + x$

$$\nabla \times \vec{E} = ?$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{\epsilon_o} & \frac{x}{\epsilon_o} & 0 \end{vmatrix} = 0$$

$$\vec{D} = y \hat{x} + x \hat{y}$$

will in general be *path dependent* except for when \mathbf{E} is curl-free.

Example 2: The field $\mathbf{E} = \hat{x}y \pm \hat{y}x$ is curl-free with the $+$ sign, but not with $-$ as verified below by computing $\nabla \times \mathbf{E}$. Calculate the line integral of \mathbf{E} (for both signs, \pm) from point $o = (0, 0, 0)$ to point $p = (1, 1, 0)$ for two different paths C going through points $u = (0, 1, 0)$ and $l = (1, 0, 0)$, respectively (see margin).

Solution: First we note that

$$\nabla \times (\hat{x}y \pm \hat{y}x) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & \pm x & 0 \end{vmatrix} = \hat{z}(\pm 1 - 1)$$

which confirms that $\mathbf{E} = \hat{x}y \pm \hat{y}x$ is curl-free with with $+$ sign, but not with $-$. In either case, the integral to be performed is

$$\int_o^p \mathbf{E} \cdot d\mathbf{l} = \int_o^p (E_x dx + E_y dy + E_z dz) = \int_o^p (y dx \pm x dy).$$

For the first path C_u going through $u = (0, 1, 0)$, we have

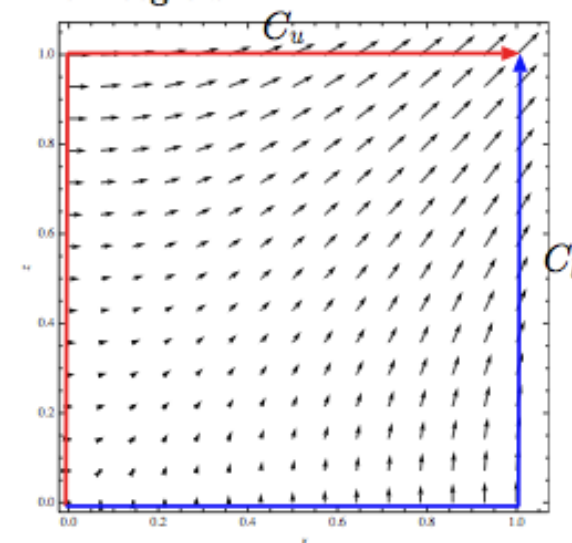
$$\int_o^p (y dx \pm x dy) = \int_{y=0}^1 (\pm x) dy|_{x=0} + \int_{x=0}^1 y dx|_{y=1} = 0 + 1 = 1.$$

For the second path C_l going through $l = (1, 0, 0)$, we have

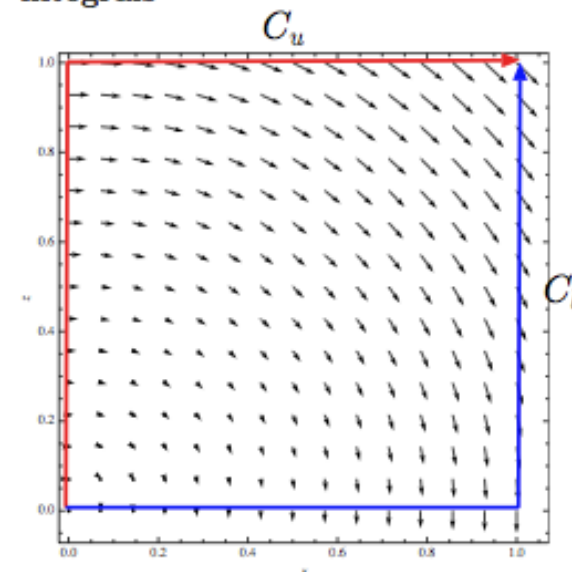
$$\int_o^p (y dx \pm x dy) = \int_{x=0}^1 y dx|_{y=0} \pm \int_{y=0}^1 x dy|_{x=1} = 0 \pm 1 = \pm 1.$$

Clearly, the result shows that the line integral $\int_o^p \mathbf{E} \cdot d\mathbf{l}$ is *path independent* for $\mathbf{E} = \hat{x}y + \hat{y}x$ which is curl-free, and path dependent for $\mathbf{E} = \hat{x}y - \hat{y}x$ in which case $\nabla \times \mathbf{E} \neq 0$.

Curl-free: path-independent line integrals



“Curly”: path-dependent line integrals



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static electric fields

$$\nabla \times \vec{E} = 0 \quad \text{E is a conservative field} \quad \oint \vec{E} \cdot d\vec{l} = 0$$

Because of the very special constraints on the electric field we can use a time honored mathematical tool called *potential functions*.

write out components for both integral and differential form

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= 0 \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= 0 \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= 0 \end{aligned}$$

$$\begin{aligned} \oint E_x dx + E_y dy + E_z dz &= 0 \\ d\vec{l} &= dx \hat{x} + dy \hat{y} + dz \hat{z} \end{aligned}$$

Maxwell's Equations

static electric fields

$$\nabla \times \vec{E} = 0$$

E is a conservative field

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$\oint E_x dx + E_y dy + E_z dz = 0$$

Suppose a function V can be found such that:

$$E_x = \frac{\partial V}{\partial x}$$

$$E_y = \frac{\partial V}{\partial y}$$

$$E_z = \frac{\partial V}{\partial z}$$

$$\oint \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = 0$$

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$$\nabla \times \vec{E} = 0 \quad \text{E is a conservative field} \quad \oint \vec{E} \cdot d\vec{l} = 0$$

write out components for both integral and differential form

$$E_x = \frac{\partial V}{\partial x} \quad E_y = \frac{\partial V}{\partial y} \quad E_z = \frac{\partial V}{\partial z}$$

Looking at both the differential form and the integral form we see that if V has this special relationship to E then this equation is automatically satisfied!

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= \frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial y \partial z} = 0 \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= \frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial x \partial z} = 0 \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= \frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial x \partial y} = 0 \end{aligned}$$

$$\oint E_x dx + E_y dy + E_z dz =$$

$$\oint \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = 0$$

remembering that - $dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$

$$\oint dV = V(x, y, z) \Big|_{\text{pt. a}}^{\text{pt. a}} = 0$$

Define V to be the electrostatic voltage and can be used as a proxy for the electric field.
The voltage V and electric field are related by:

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)$$

the minus sign is convention
chosen so that the electric field
always points FROM a HIGHER
potential to a LOWER potential



gradient

Example 1: Given an electrostatic potential

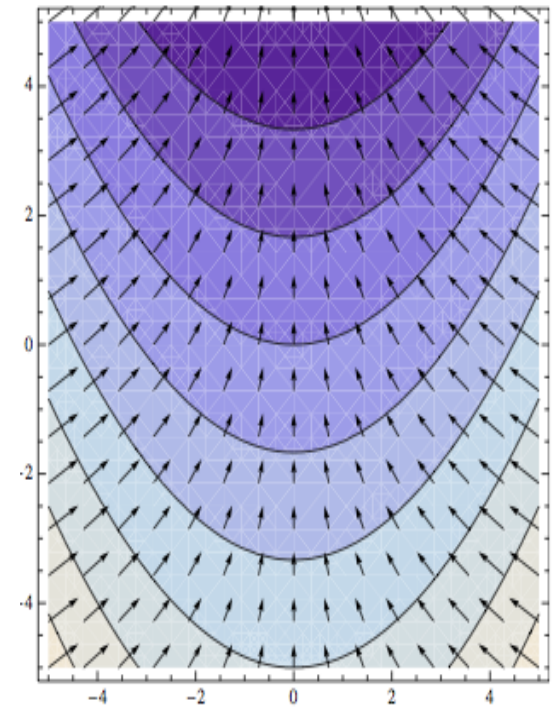
$$V(x, y, z) = x^2 - 6y \text{ V}$$

in a certain region of space, determine the corresponding electrostatic field $\mathbf{E} = -\nabla V$ in the same region.

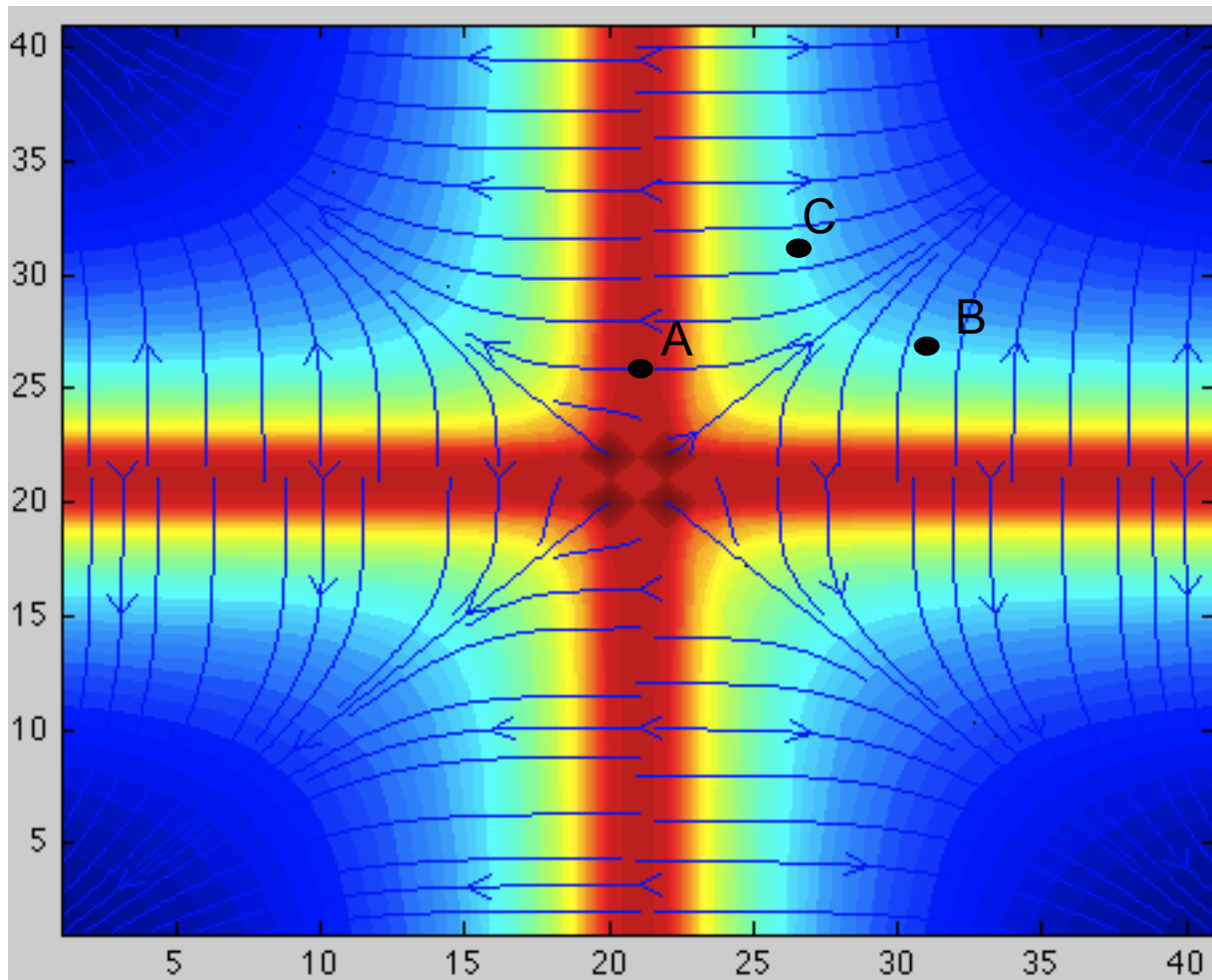
Solution: The electrostatic field is

$$\mathbf{E} = -\nabla(x^2 - 6y) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(x^2 - 6y) = (-2x, 6, 0) = -\hat{x} 2x + \hat{y} 6 \text{ V/m}.$$

Note that this field is directed from regions of high potential to low potential. Also note that electric field vectors are perpendicular everywhere to “equipotential” contours.



Suppose we have a description of the voltage $V(x,y,z)$ and we wish to find the electric field



this inverse gradient operation must be done with some care because taking the derivative of the voltage implies that there are many possible electric fields that would produce the same V .

$$\vec{E} = x \hat{x} + y \hat{y} \rightarrow V(x, y, z) = -\hat{x} + -\hat{y}$$

but

$$\vec{E} = (x+z) \hat{x} + y \hat{y} \rightarrow V(x, y, z) = -\hat{x} + -\hat{y}$$

$$-\int_A^B \vec{E} \cdot d\vec{l} = -(V(B) - V(A)) = V(A) - V(B)$$

voltage between points A and B

Example 3: Given that $V_o = V(0, 0, 0) = 0$ and

$$\mathbf{E} = 2x\hat{x} + 3z\hat{y} + 3(y+1)\hat{z} \frac{\text{V}}{\text{m}},$$

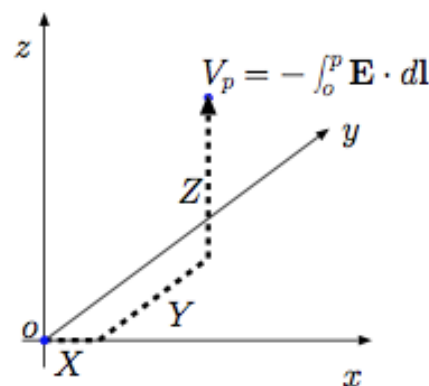
determine the electrostatic potential $V_p = V(X, Y, Z)$ at point $p = (X, Y, Z)$ in volts.

Solution: Assuming that the field is curl-free (it is), so that any integration path can be used, we find that

$$\begin{aligned} V_p &= \int_p^o \mathbf{E} \cdot d\mathbf{l} = - \int_o^p \mathbf{E} \cdot d\mathbf{l} = - \int_o^p (2x dx + 3z dy + 3(y+1) dz) \\ &= - \int_0^X 2x dx|_{y,z=0} - \int_0^Y 3z dy|_{x=X,z=0} - \int_0^Z 3(y+1) dz|_{x=X,y=Y} \\ &= -X^2 - 0 - 3(Y+1)Z. \end{aligned}$$

This implies

$$V(x, y, z) = -x^2 - 3(y+1)z \text{ V}.$$



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This relationship between E and V (for static fields) allows manipulating a scalar function rather than a vector function.

