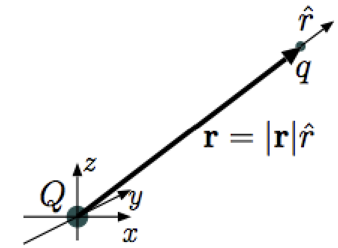
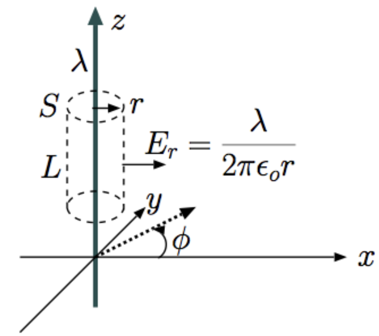
Charge Distributions



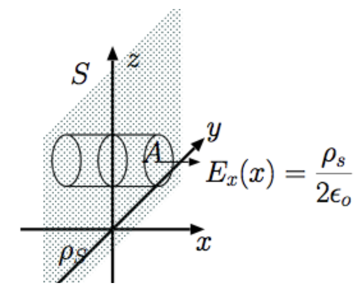
The geometries that we have looked and determined the associated electric field are all examples of “charge distributions”.



We have made assumptions about how these distributions relate to points charges.

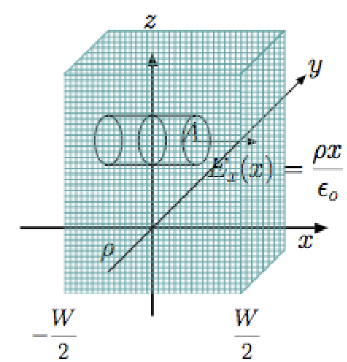
Macroscopic view:

- we are not interested in the fields very close to the charge distribution which is really made up of point charges. We know that the fields near the collection of charges varies quickly, but these variations disappear quickly as you move away from the distribution.

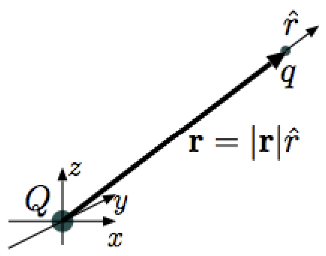


Microscopic view:

- we are interested in the fine details of structure of the fields and so need to identify the exact placement of each charge and its contribution to the total field.

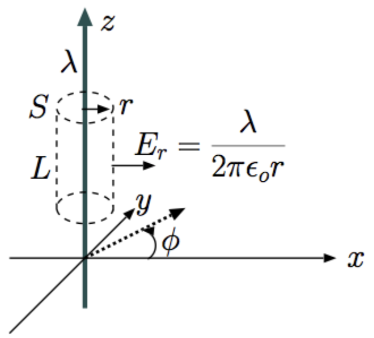


Charge Distributions



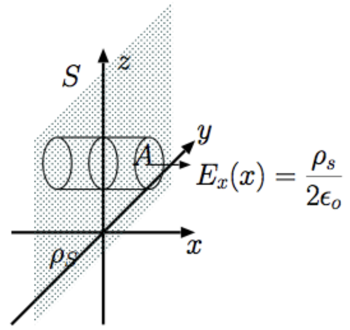
Point Charge at point (x,y,z)

$$Q\,\delta(x, y, z)$$



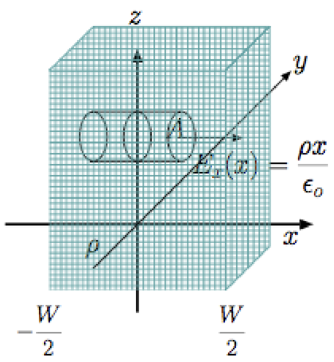
Line Charge along the z-axis

$$\rho_l\,\delta(x, y)$$



Sheet of charge in y-z plane

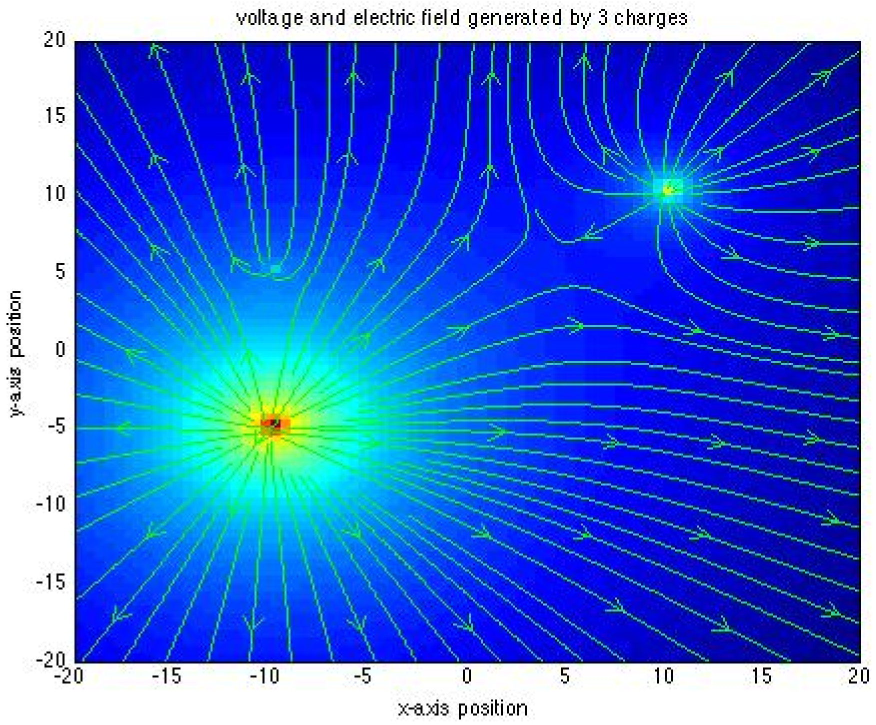
$$\rho_s\,\delta(x)$$

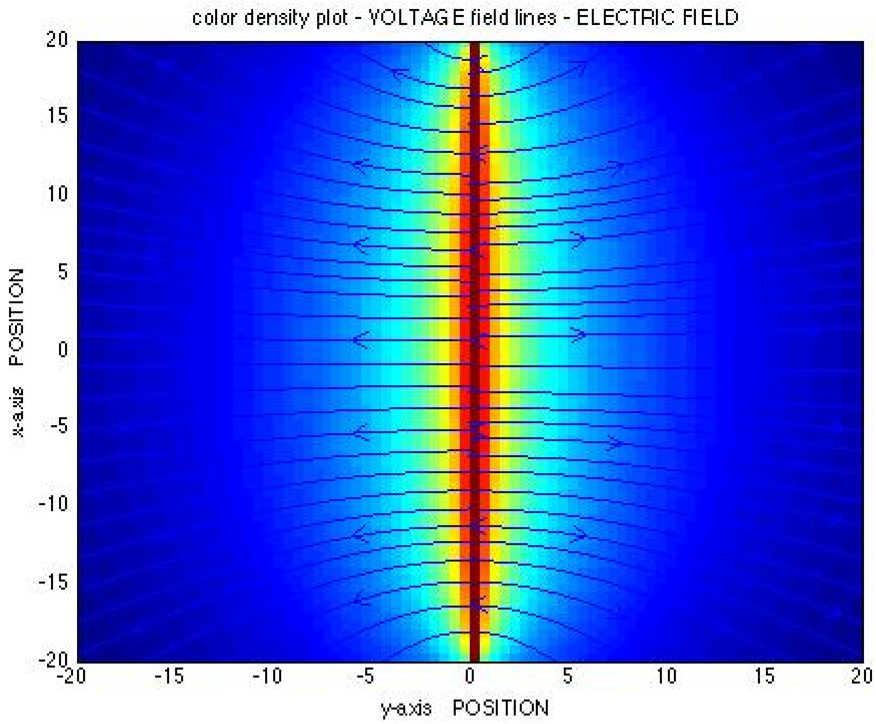


volume of charge

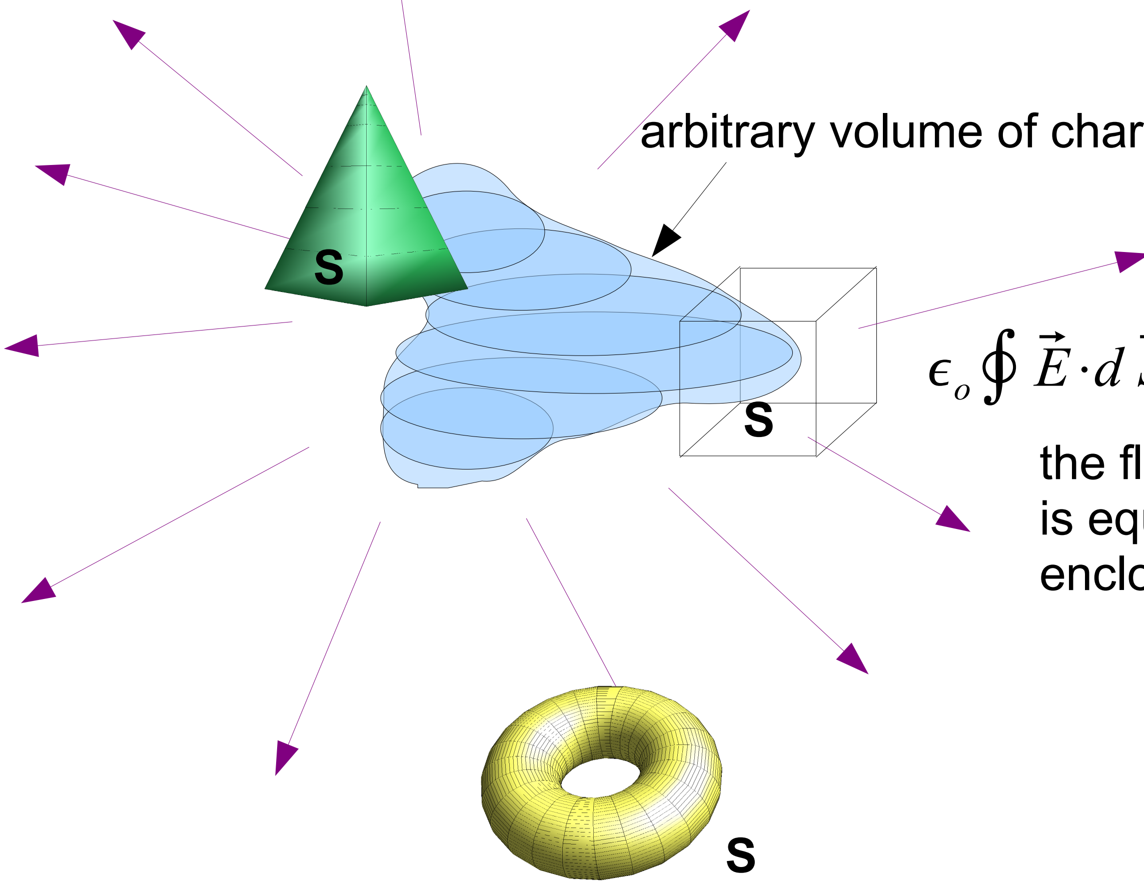
$$\rho_v$$

Charge Distributions





Differential Form of Gauss' law



$$\epsilon_o \oint \vec{E} \cdot d\,\vec{S} = Q$$

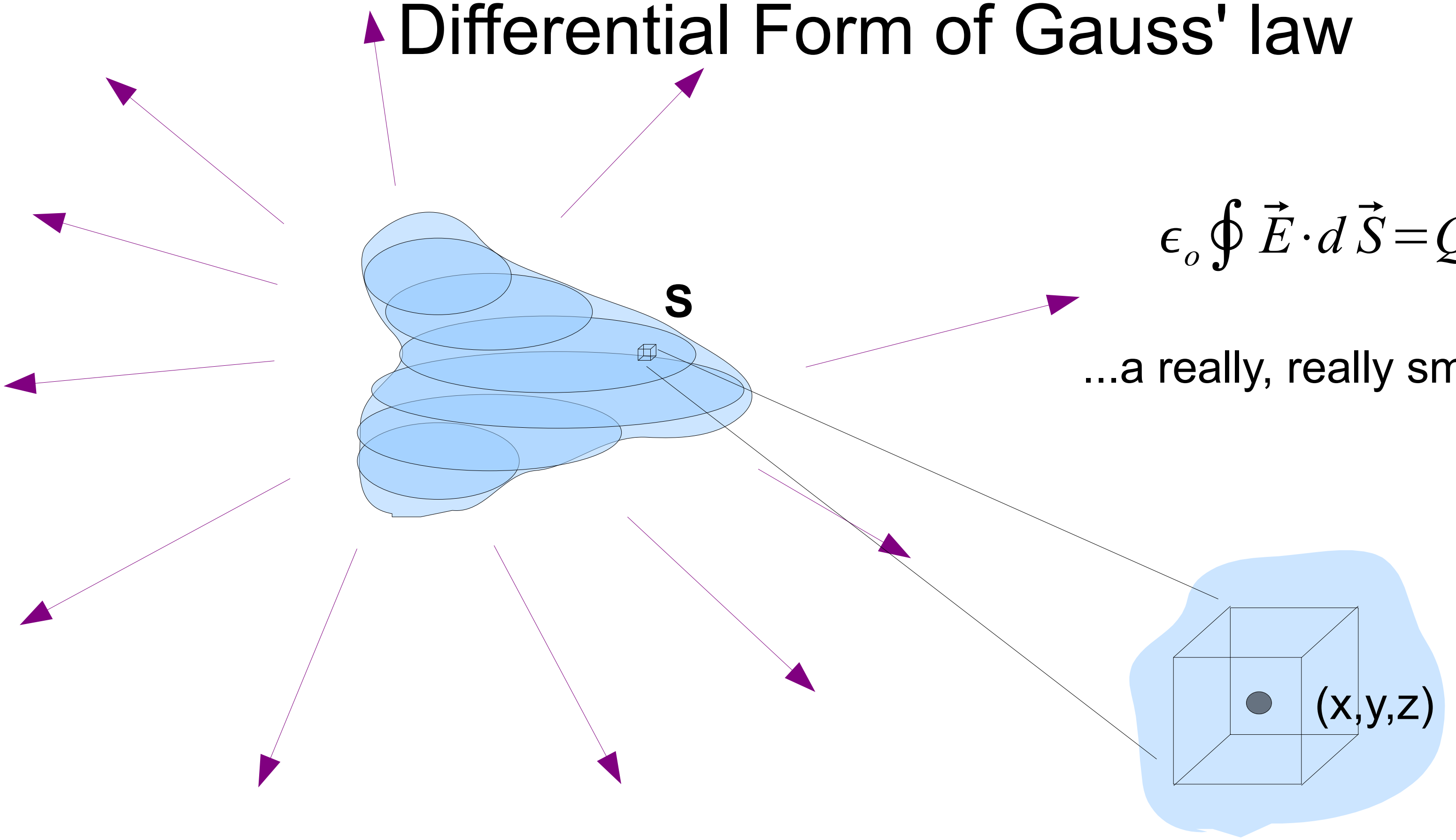
the flux of E out of the closed surface S is equal to the amount of charge enclosed by the surface...

...any surface of any shape, even...

Differential Form of Gauss' law

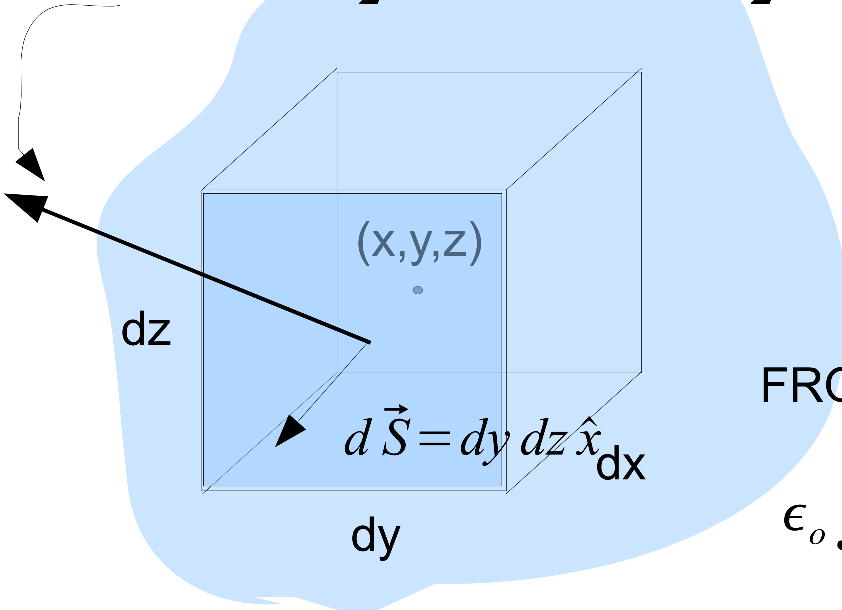
$$\epsilon_o \oint \vec{E} \cdot d\vec{S} = Q$$

...a really, really small one.



Differential Form of Gauss' law

$$\vec{E}=E_x(x+\frac{dx}{2},y,z)\hat{x}+E_y(x+\frac{dx}{2},y,z)\hat{y}+E_z(x+\frac{dx}{2},y,z)\hat{z}$$



$$\epsilon_o\oint\vec{E}\cdot d\vec{S}=Q$$

Let's compute the electric flux flowing out of each of the 6 faces of this surface

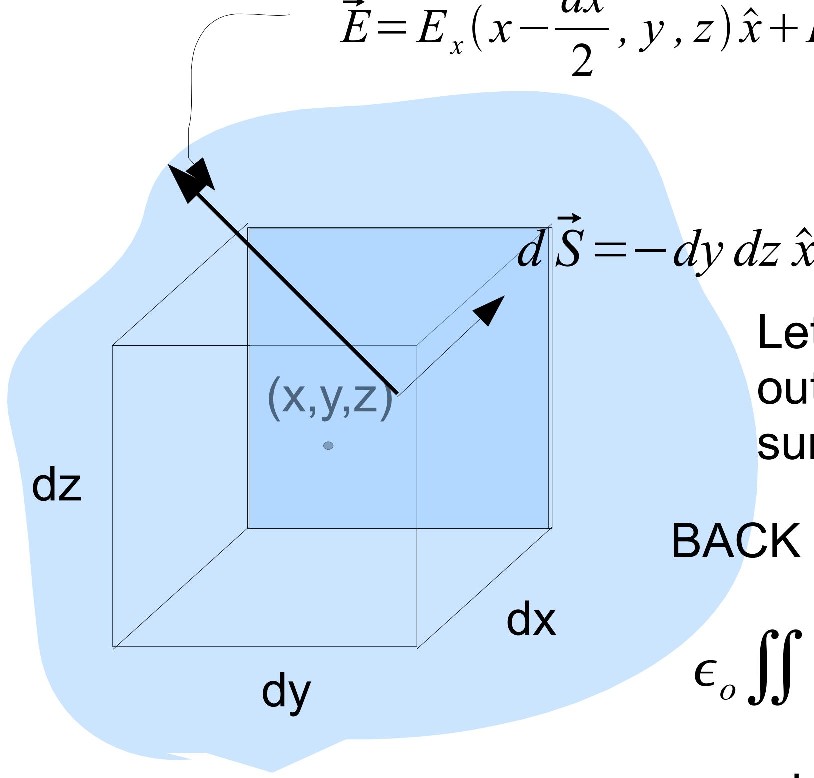
FRONT -

$$\epsilon_o\iint\vec{E}\cdot d\vec{S}=\epsilon_o E_x(x+\frac{dx}{2},y,z)\,dydz$$

what assumption did we make?

Differential Form of Gauss' law

$$\vec{E}=E_x(x-\frac{dx}{2},y,z)\hat{x}+E_y(x-\frac{dx}{2},y,z)\hat{y}+E_z(x-\frac{dx}{2},y,z)\hat{z}$$



$$\epsilon_o\oint\vec{E}\cdot d\vec{S}=Q$$

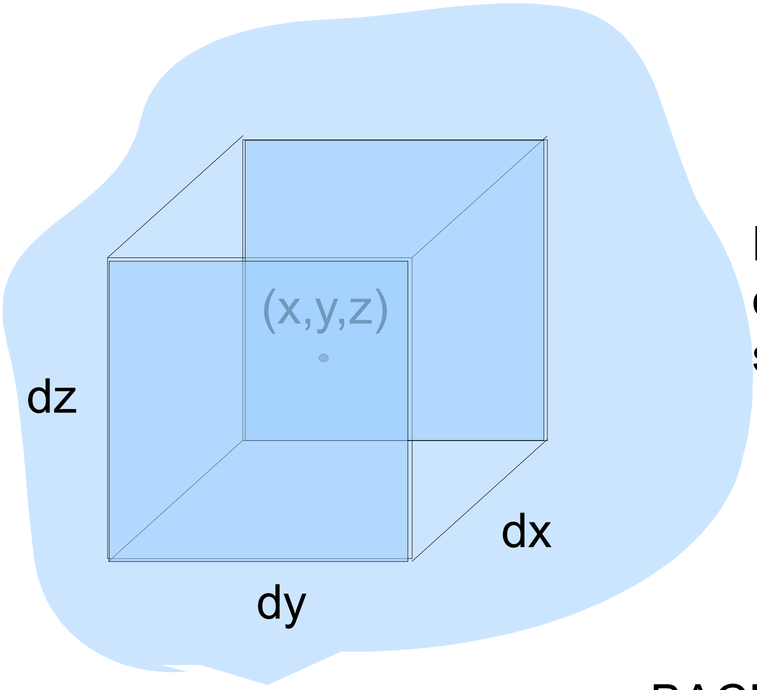
Let's compute the electric flux flowing out of each of the 6 faces of this surface

BACK -

$$\epsilon_o\iint\vec{E}\cdot d\vec{S}=-\epsilon_o E_x(x-\frac{dx}{2},y,z)dydz$$

what assumption did we make?

Differential Form of Gauss' law



$$\epsilon_o \oint \vec{E} \cdot d\vec{S} = Q$$

Let's compute the electric flux flowing out of each of the 6 faces of this surface

FRONT -

$$\epsilon_o \iint \vec{E} \cdot d\vec{S} = \epsilon_o E_x\left(x + \frac{dx}{2}, y, z\right) dy dz$$

BACK -

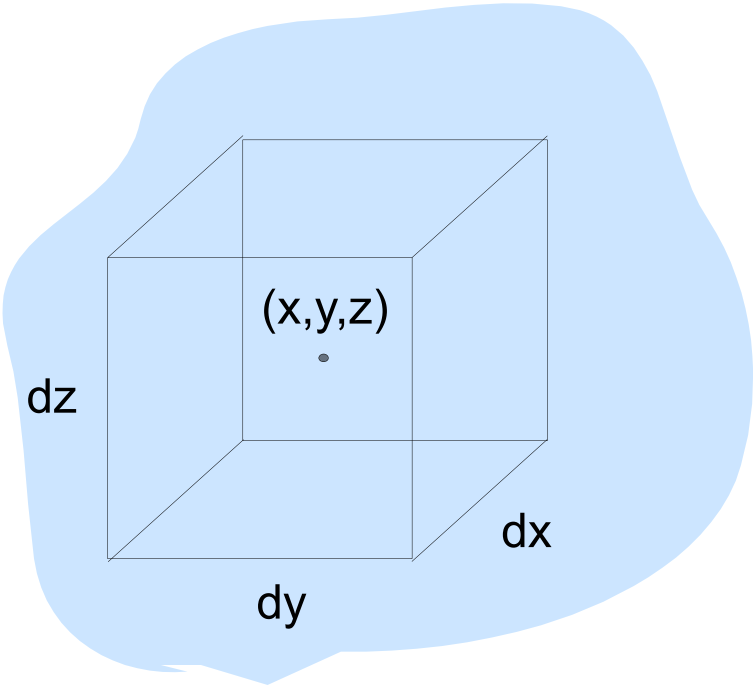
$$\epsilon_o \iint \vec{E} \cdot d\vec{S} = -\epsilon_o E_x\left(x - \frac{dx}{2}, y, z\right) dy dz$$

combining:

electric flux flowing through the volume in the x-direction

$$\epsilon_o \left(E_x\left(x + \frac{dx}{2}, y, z\right) - E_x\left(x - \frac{dx}{2}, y, z\right)\right) dy\, dz$$

Differential Form of Gauss' law



$$\epsilon_o \oint \vec{E} \cdot d\vec{S} = Q$$

Let's compute the electric flux flowing out of each of the 6 faces of this surface

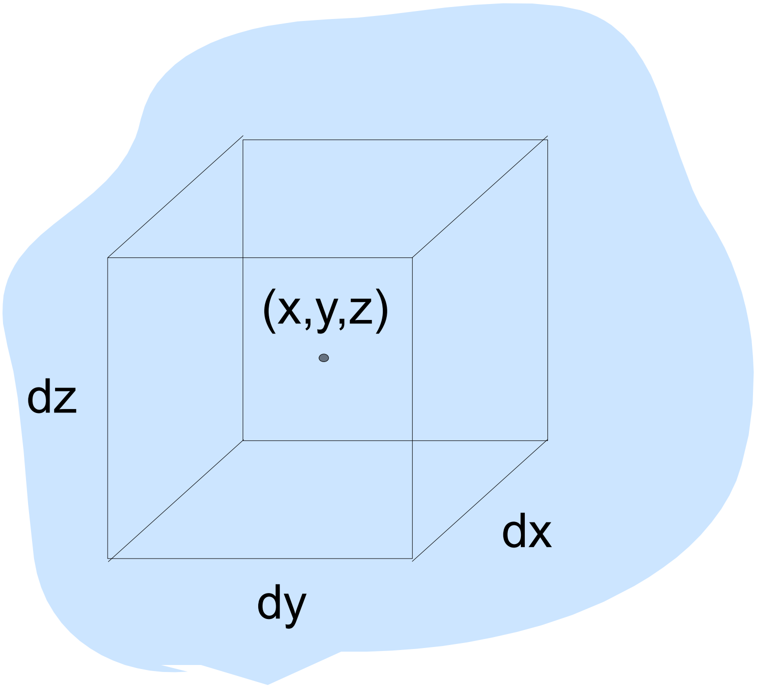
Repeat for the y- and z- directions

$$\epsilon_o \oint \vec{E} \cdot d\vec{S} = \epsilon_o \left(E_x\left(x+\frac{dx}{2}, y, z\right) - E_x\left(x-\frac{dx}{2}, y, z\right)\right) dy\, dz +$$

$$\epsilon_o \left(E_y\left(x, y+\frac{dy}{2}, z\right) - E_y\left(x, y-\frac{dy}{2}, z\right)\right) dx\, dz +$$

$$\epsilon_o \left(E_z\left(x, y, z+\frac{dz}{2}\right) - E_z\left(x, y, z-\frac{dz}{2}\right)\right) dx\, dy$$

Differential Form of Gauss' law



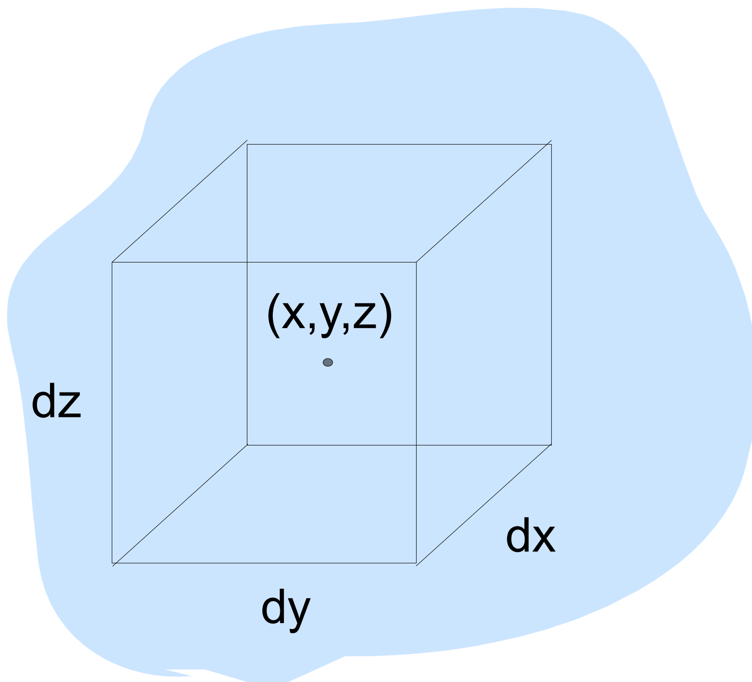
$$\epsilon_o \oint \vec{E} \cdot d\vec{S} = Q$$

Now compute the total charge contained in the volume defined by the small surface

$$Q_{total} = \rho_v \, dx \, dy \, dz$$

again, what assumption did we make?

Differential Form of Gauss' law



$$\epsilon_o \oint \vec{E} \cdot d\vec{S} = Q$$

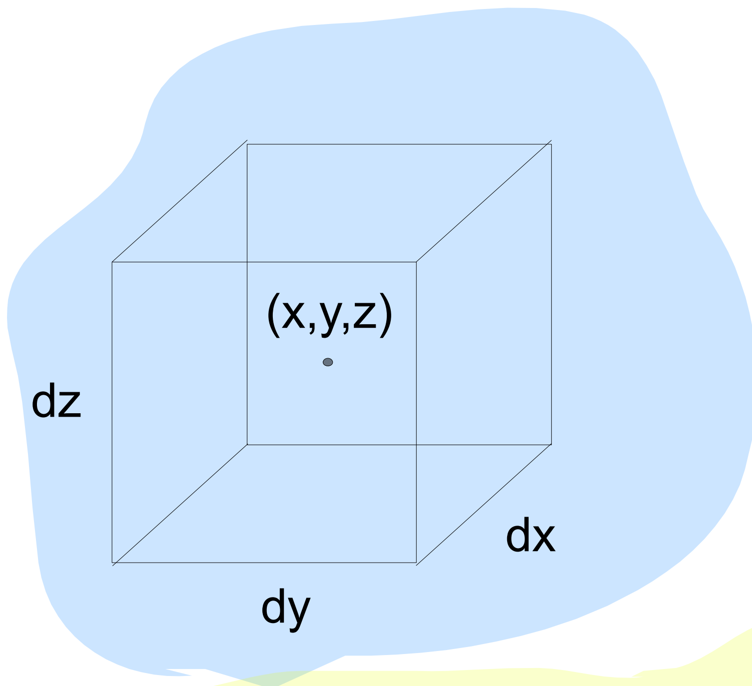
Repeat for the y- and z- directions

$$\epsilon_o \oint \vec{E} \cdot d\vec{S} = \quad \epsilon_o (E_x(x+\frac{dx}{2}, y, z) - E_x(x-\frac{dx}{2}, y, z))\, dy\, dz + \quad = \quad \rho_v\, dx\, dy\, dz$$

$$\epsilon_o (E_x(x, y+\frac{dy}{2}, z) - E_x(x, y-\frac{dy}{2}, z))\, dx\, dz +$$

$$\epsilon_o (E_x(x, y, z+\frac{dz}{2}) - E_x(x, y, z-\frac{dz}{2}))\, dx\, dy$$

Differential Form of Gauss' law



remember - from your early calculus class – the definition of a partial derivative

$$\frac{\partial f(x,y,z)}{\partial x} = \lim \frac{f(x+dx,y,z) - f(x,y,z)}{dx}$$

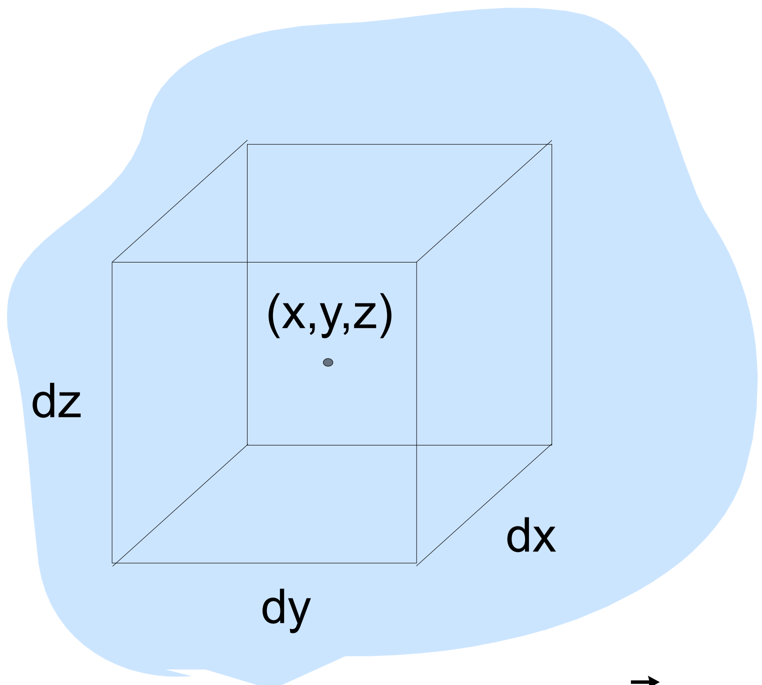
or equivalently,

$$\frac{\partial f(x,y,z)}{\partial x} = \lim \frac{f\left(x+\frac{dx}{2},y,z\right) - f\left(x-\frac{dx}{2},y,z\right)}{dx}$$

$$\epsilon_o \oint \vec{E}\cdot d\vec{S} = \epsilon_o \frac{\left(E_x\left(x+\frac{dx}{2},y,z\right) - E_x\left(x-\frac{dx}{2},y,z\right)\right)}{dx} + \epsilon_o \frac{\left(E_y\left(x,y+\frac{dy}{2},z\right) - E_y\left(x,y-\frac{dy}{2},z\right)\right)}{dy} + \epsilon_o \frac{\left(E_z\left(x,y,z+\frac{dz}{2}\right) - E_z\left(x,y,z-\frac{dz}{2}\right)\right)}{dz} = \rho_v$$

$$\frac{\partial E_x}{\partial x}$$

Differential Form of Gauss' law

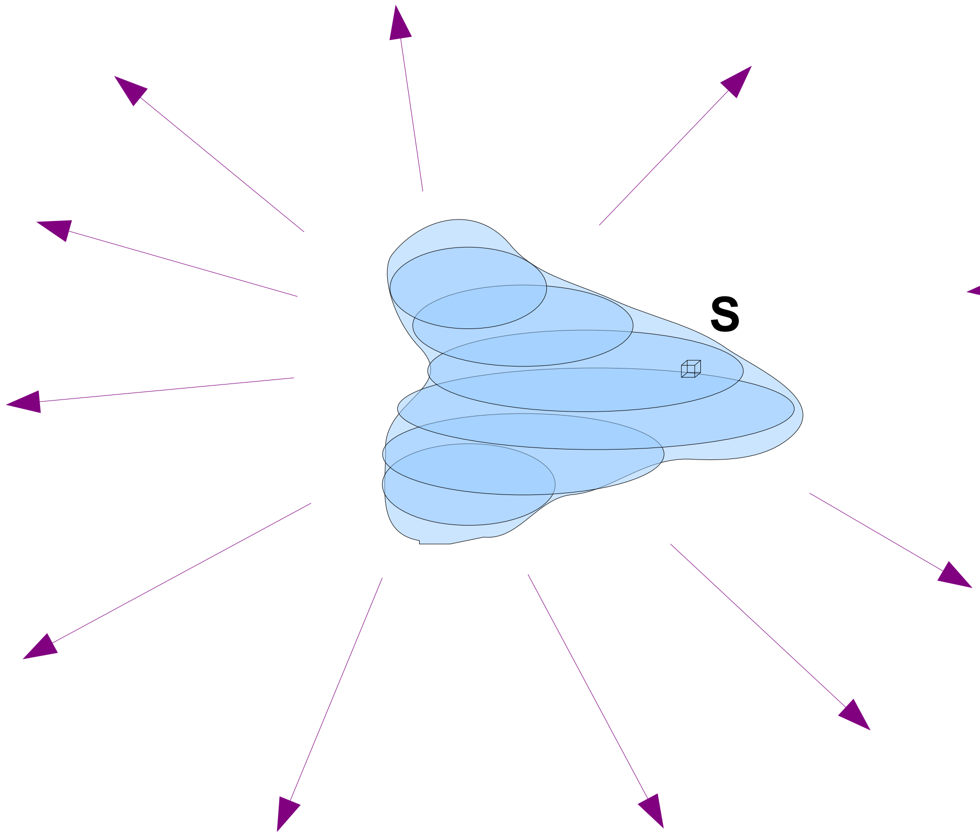


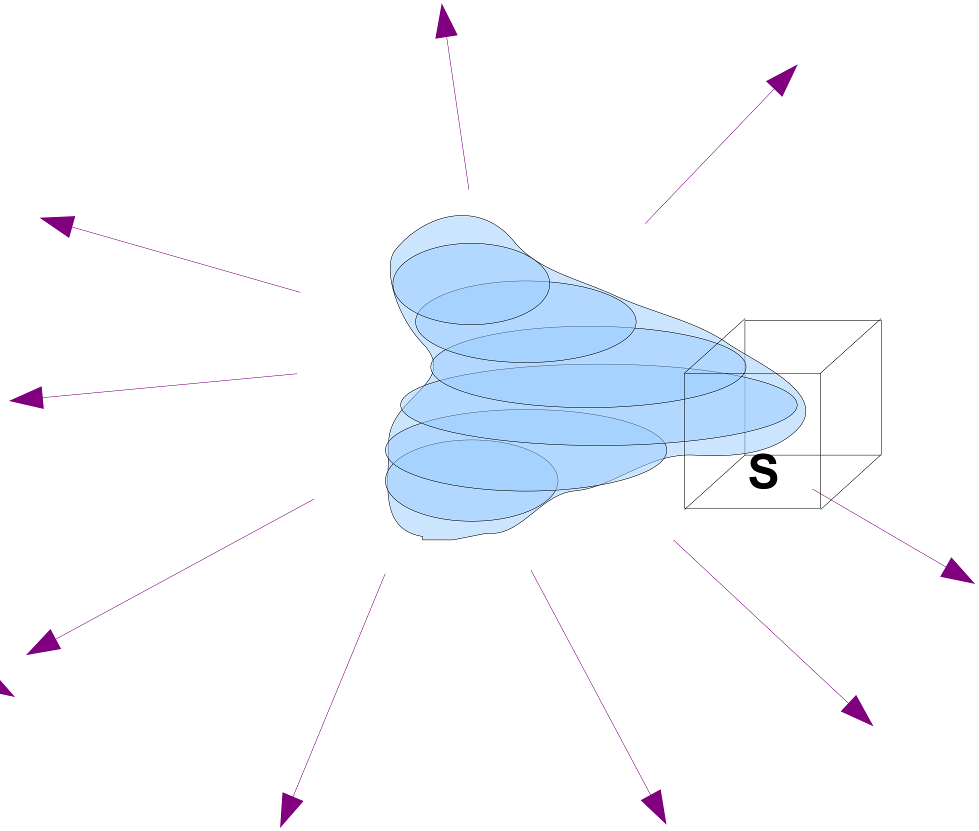
$\nabla \cdot \vec{E}$ ← divergence

$$\epsilon_o \oint \vec{E} \cdot d\vec{S} = \epsilon_o \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = \rho_v$$

over an infinitesimally small volume

Differential Form of Gauss' law





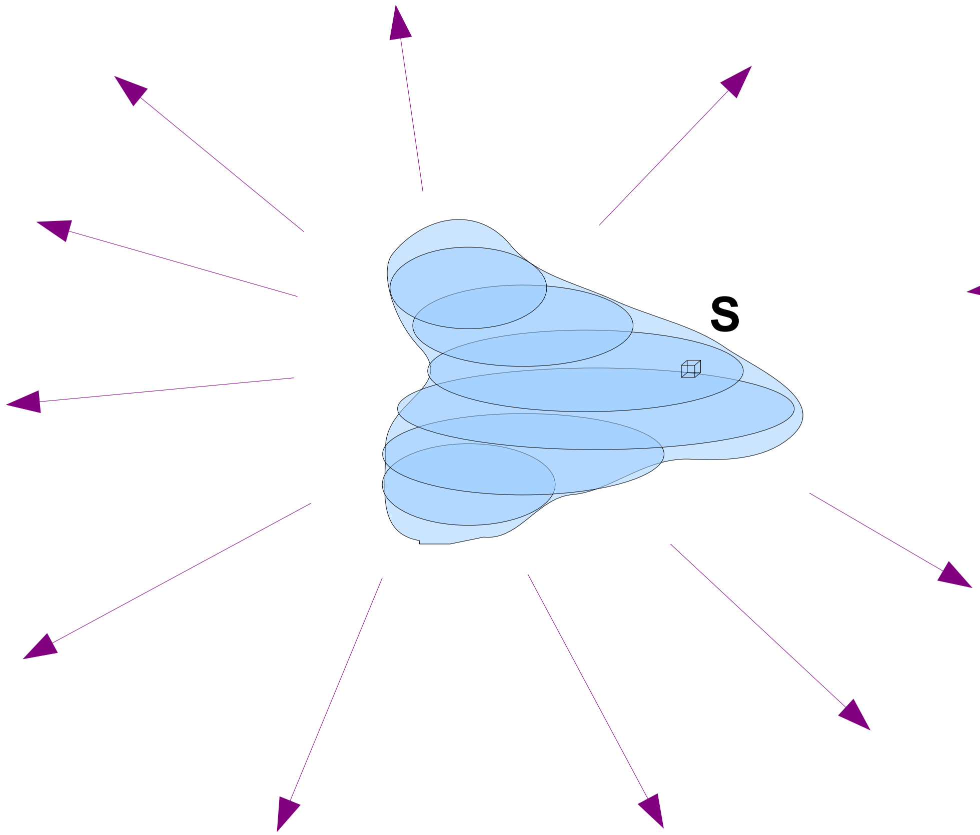
Differential Form of Gauss' Law

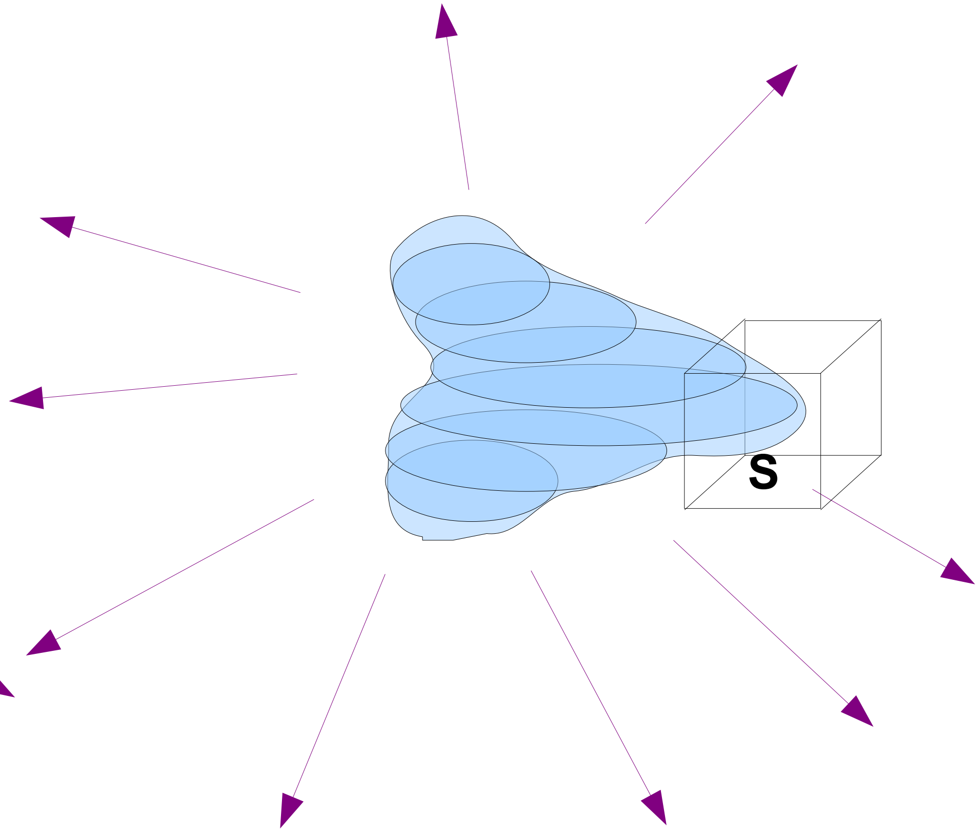
$$\epsilon_o \nabla \cdot \vec{E} = \rho_v$$

Integral Form of Gauss' Law

$$\epsilon_o \oint \vec{E} \cdot d\vec{S} = Q$$

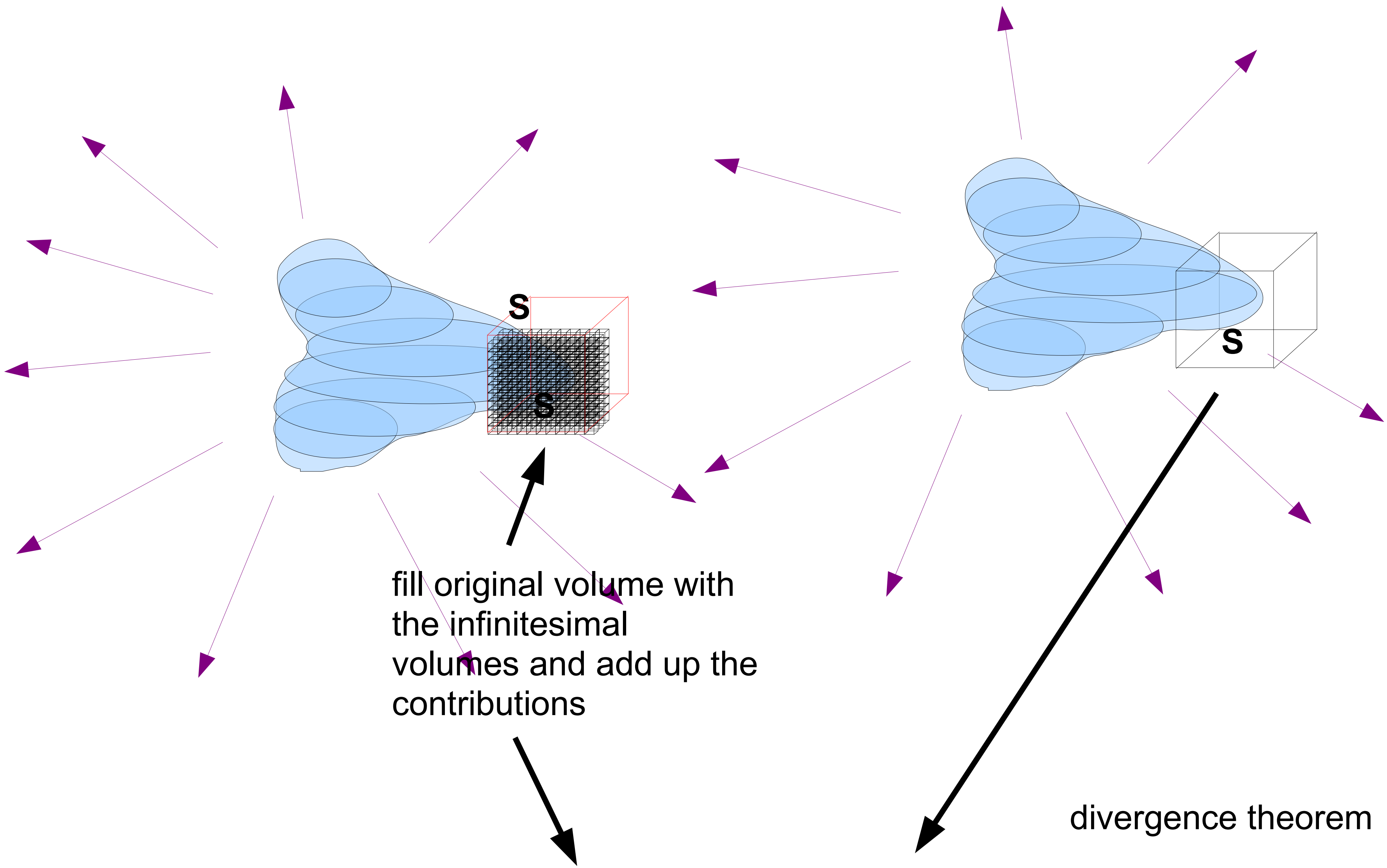
Differential Form of Gauss' law





$$\iiint \epsilon_o \nabla \cdot \vec{E}\, dx\, dy\, dz = \epsilon_o \oint \vec{E} \cdot d\vec{S}$$

Differential Form of Gauss' law



$$\iiint \epsilon_o \nabla \cdot \vec{E}\, dx\, dy\, dz = \epsilon_o \oint \vec{E} \cdot d\vec{S}$$

Maxwell's Equations
static fields

$$\nabla \cdot \vec{D} = \rho$$

Gauss' Law for the electric field

$$\oint \vec{D} \cdot d\vec{S} = Q$$

$$\nabla \cdot \vec{B} = 0$$

Gauss' Law for the magnetic field

$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\nabla \times \vec{E} = \frac{-\partial B}{\partial t}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{S}$$

$$\nabla \times \vec{H} = \frac{\partial D}{\partial t} + \vec{J}$$

$$\oint \vec{H} \cdot d\vec{l} = \frac{\partial}{\partial t} \oint \vec{D} \cdot d\vec{S} + I$$

Maxwell's Equations

static fields

$$\nabla \cdot \vec{D} = \rho \qquad \text{Gauss' Law for the electric field} \qquad \oint \vec{E} \cdot d\vec{S} = Q$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{Gauss' Law for the magnetic field} \qquad \oint \vec{B} \cdot d\vec{S} = 0$$

$$\nabla \times \vec{E} = 0 \qquad \oint \vec{E} \cdot d\vec{l} = 0$$

$$\nabla \times \vec{H} = \vec{J} \qquad \oint \vec{H} \cdot d\vec{l} = I$$

relate sources to fields

constrains the field

Maxwell's Equations

static electric fields

$$\nabla \cdot \vec{D} = \rho$$

Gauss' Law for the electric field

$$\oint \vec{D} \cdot d\vec{S} = Q$$

we know that Gauss' Law tells us that the electric field in any region is related to its source (electrical charges) in a special way. Because all electric fields are generated by charge distributions, if you look at the electric flux through a closed surface it is directly related to the enclosed charge.

$$\nabla \times \vec{E} = 0$$

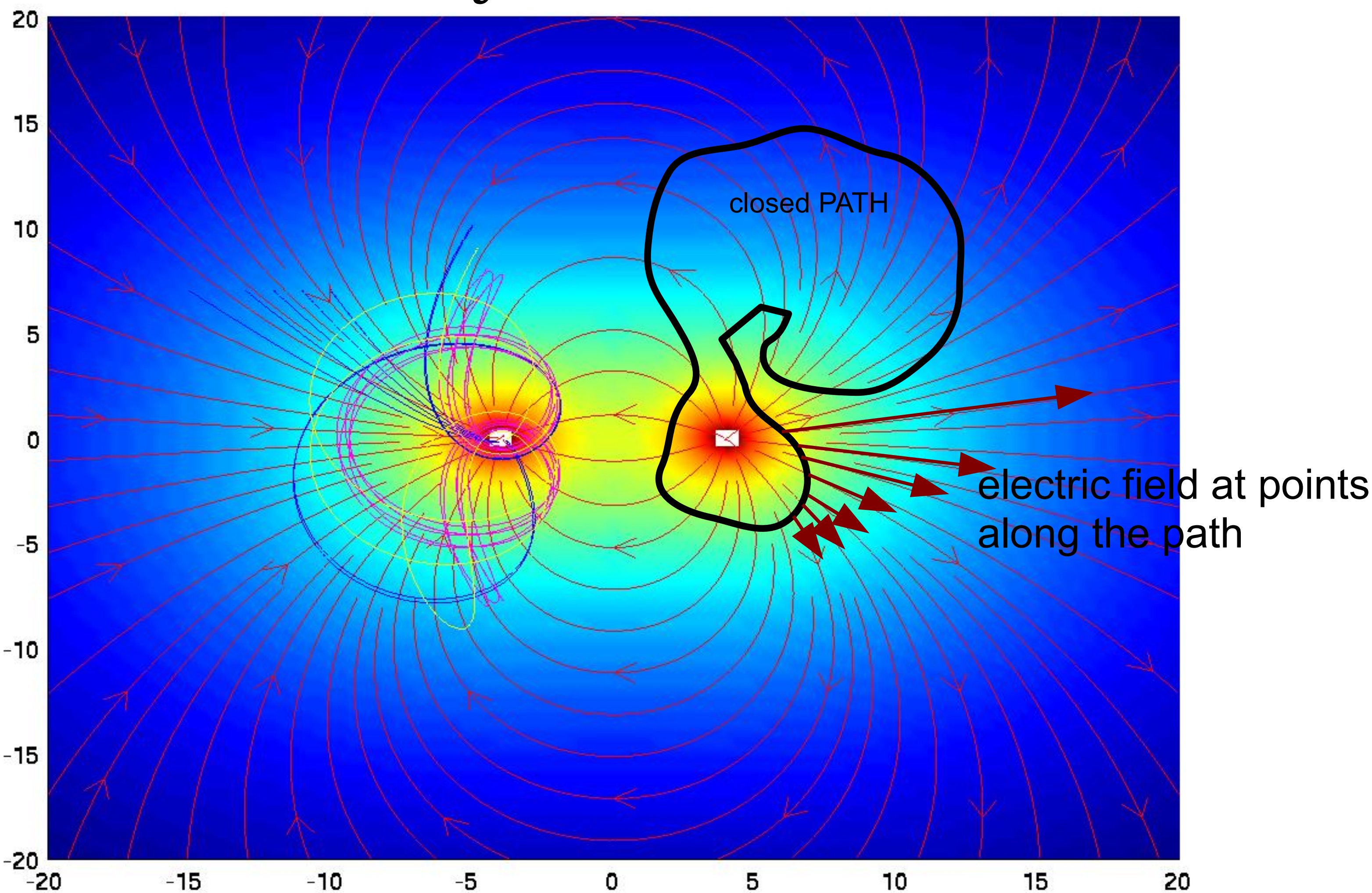
$$\oint \vec{E} \cdot d\vec{l} = 0$$

What do these equations tell us?

Maxwell's Equations

static electric fields

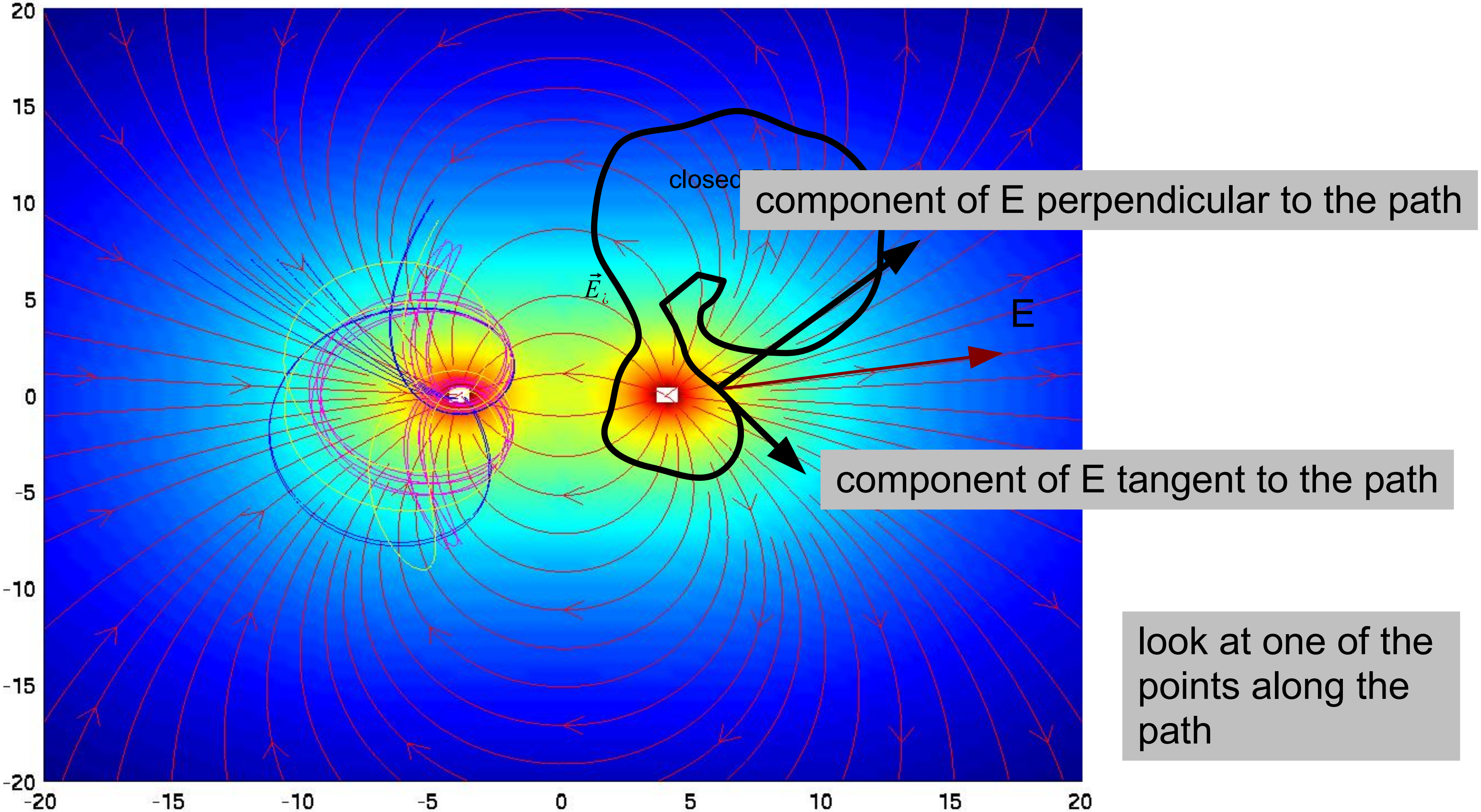
$$\oint \vec{E} \cdot d\,\vec{l} = 0$$



Maxwell's Equations

static electric fields

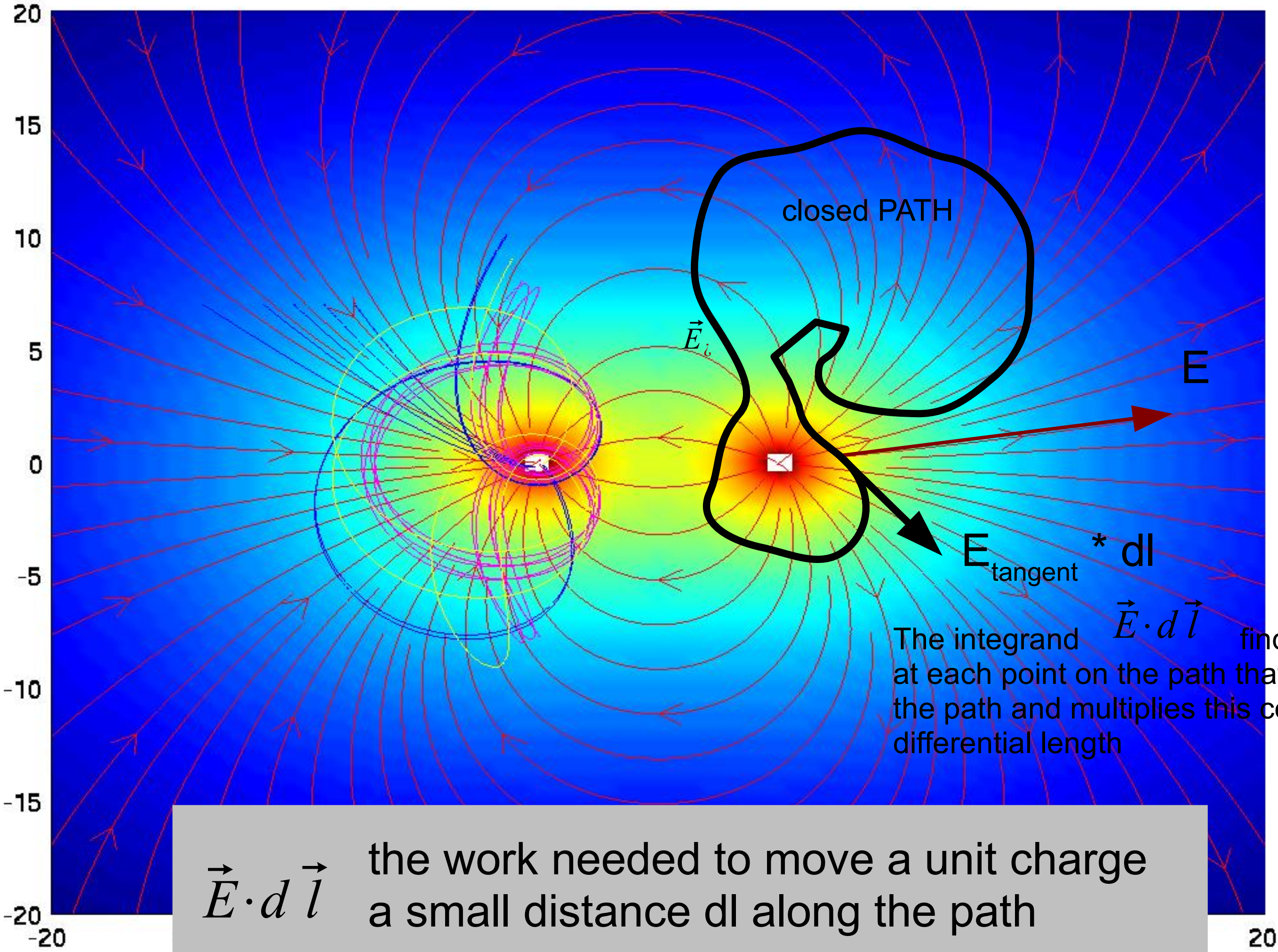
$$\oint \vec{E} \cdot d\vec{l} = 0$$



look at one of the points along the path

Maxwell's Equations

static electric fields

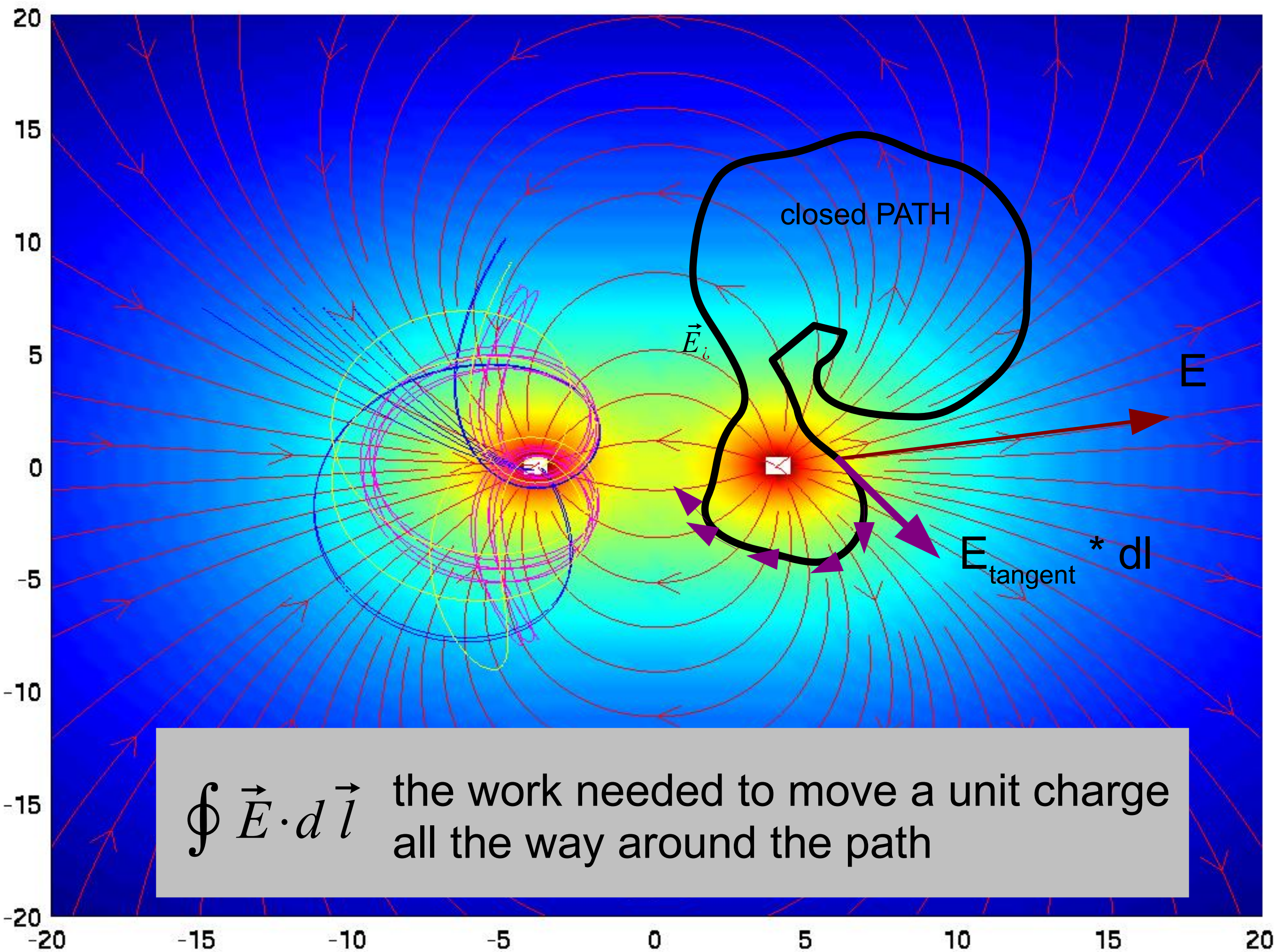


The integrand $\vec{E}\cdot d\vec{l}$ finds the electric field at each point on the path that is TANGENT to the path and multiplies this component by a differential length

$\vec{E}\cdot d\vec{l}$ the work needed to move a unit charge a small distance dl along the path

Maxwell's Equations

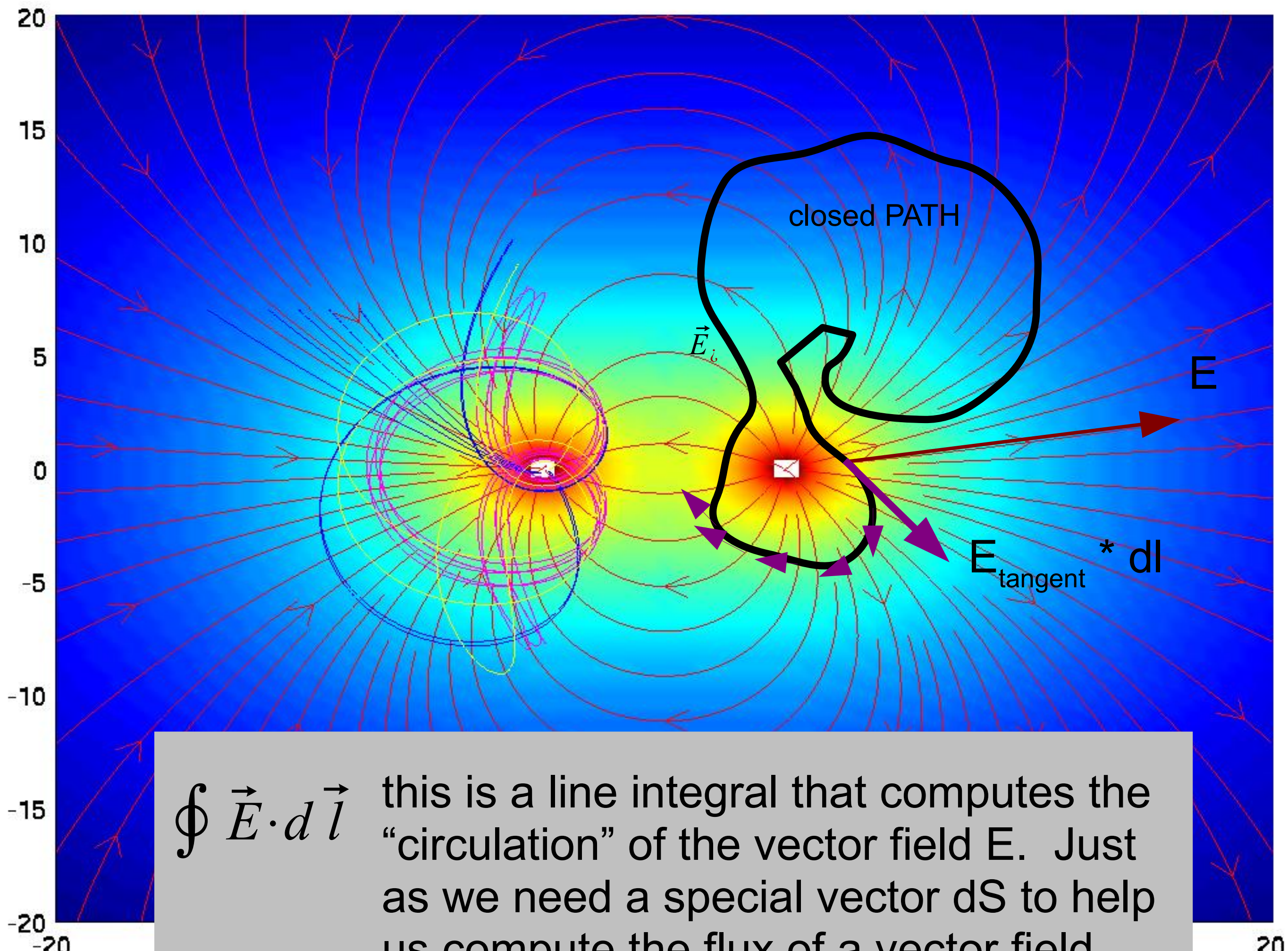
static electric fields



$\oint \vec{E} \cdot d\,\vec{l}$ the work needed to move a unit charge all the way around the path

Maxwell's Equations

static electric fields

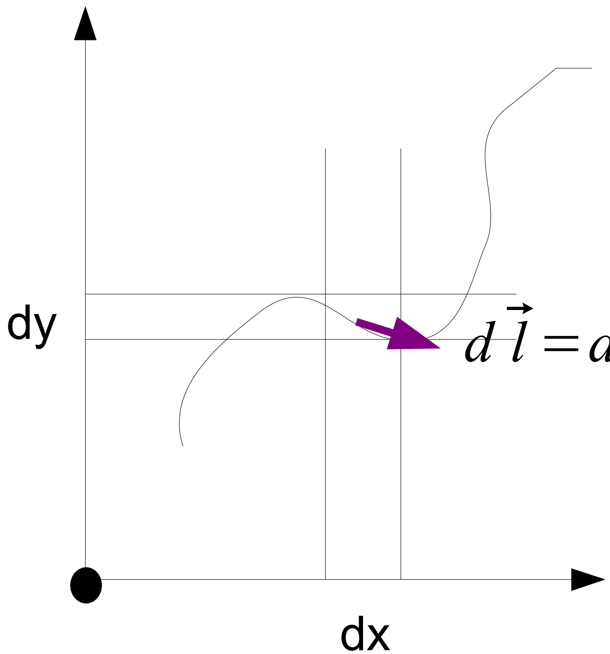


$d\vec{l}$

magnitude specifies an infinitesimal increment of the path

direction specifies a direction TANGENT to the path

$\oint \vec{E}\cdot d\vec{l}$ this is a line integral that computes the “circulation” of the vector field E. Just as we need a special vector dS to help us compute the flux of a vector field through a surface here we need a special vector dl



$$d\vec{l} = dx\,\hat{x} + dy\,\hat{Y}$$

Maxwell's Equations

static electric fields

If $\oint \vec{E}\cdot d\vec{l}$ the work needed to move a unit charge all the way around the path is correct then

$$\oint \vec{E}\cdot d\vec{l} = 0$$

states that the amount of work needed to move a unit charge all the way around the path is 0!?!?!

Maxwell's Equations

static electric fields

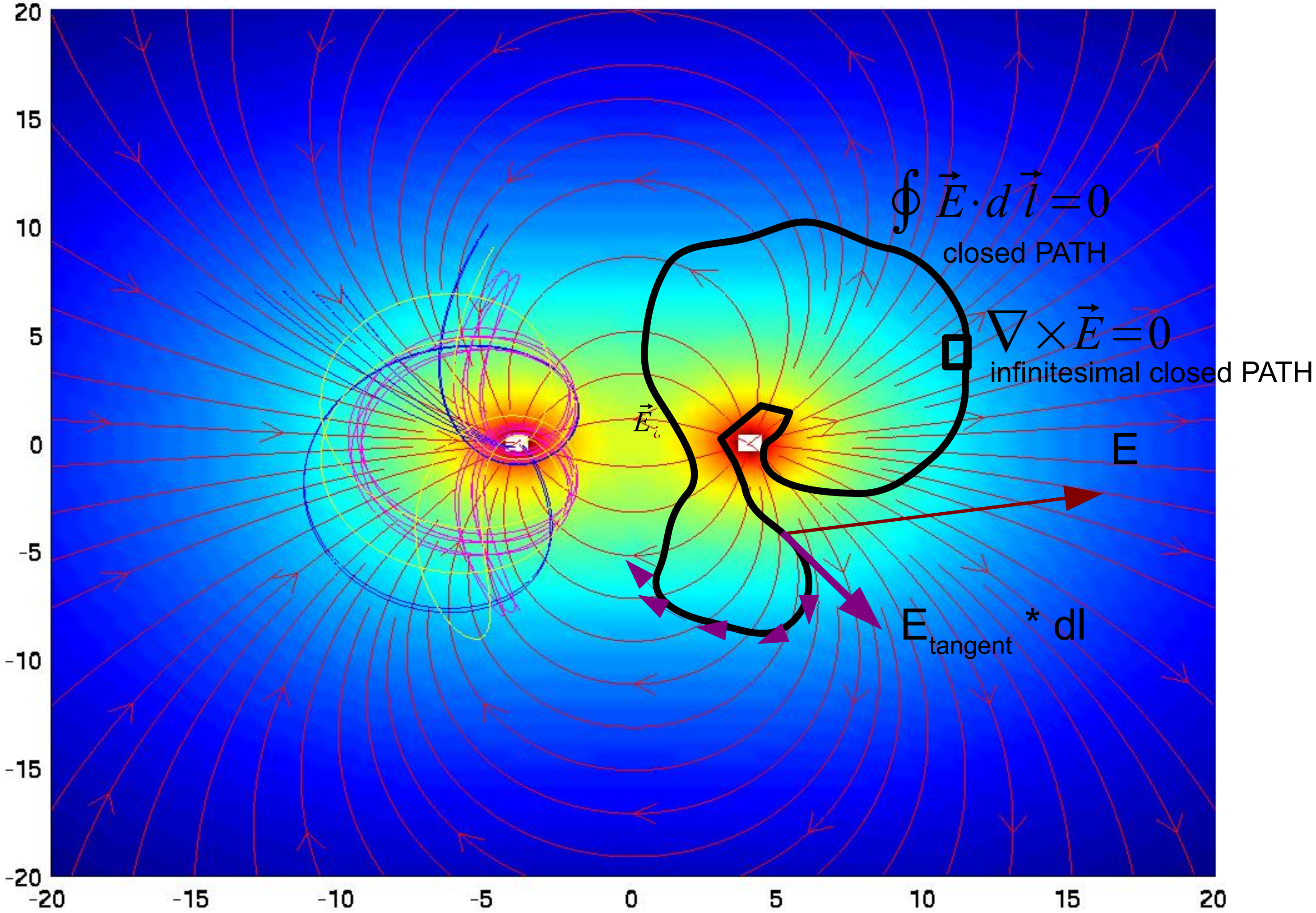
If $\oint \vec{E} \cdot d\vec{l}$ the work needed to move a unit charge all the way around the path is correct then

$$\oint \vec{E} \cdot d\vec{l} = 0$$

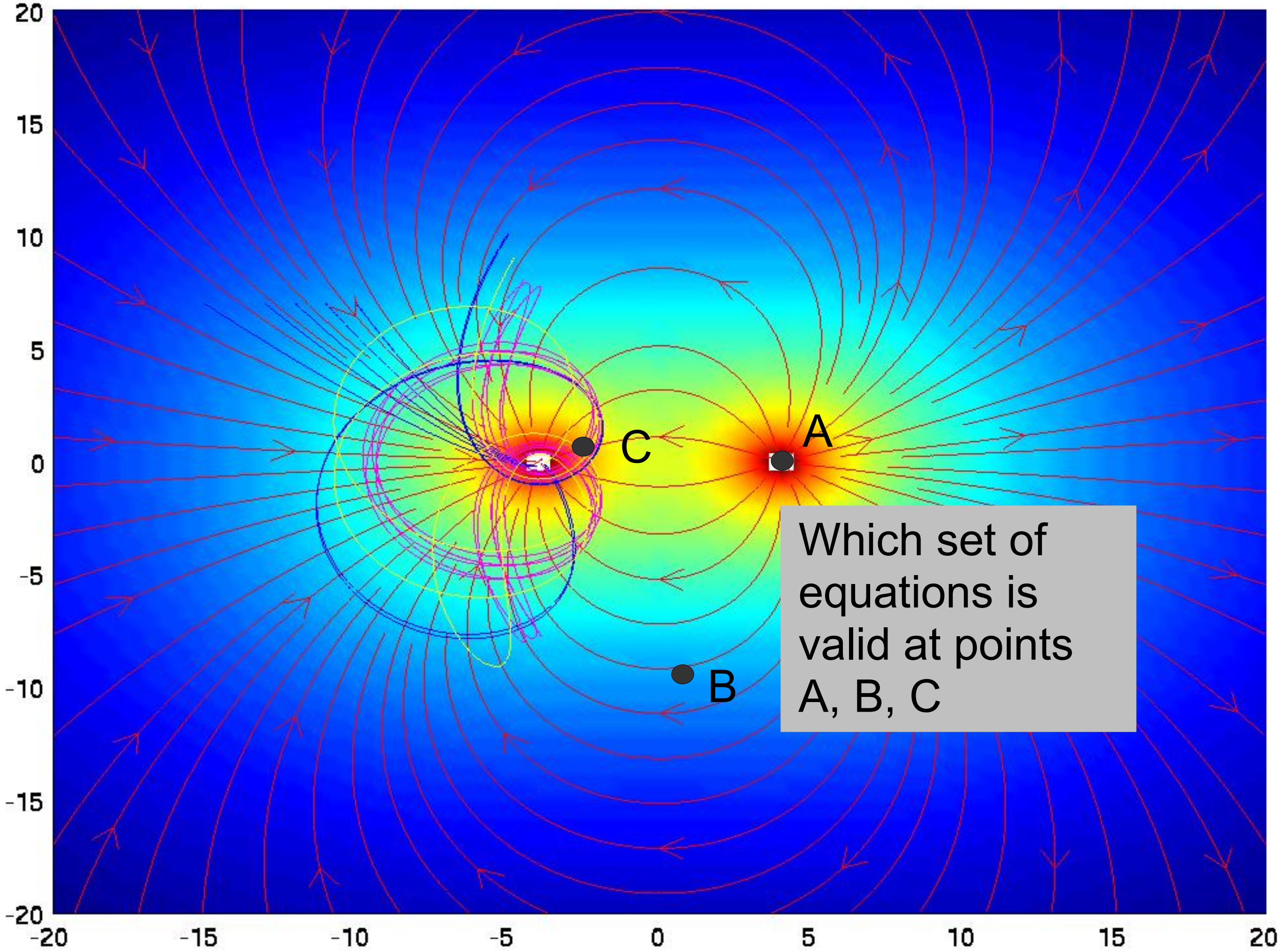
states that the amount of work needed to move a unit charge all the way around the path is 0!?!?!

CONSERVATIVE FIELD

...and just as $\nabla \cdot \vec{D} = \rho$ is equivalent to $\oint \vec{D} \cdot d\vec{S} = Q$ over an infinitely small surface



so $\nabla \times \vec{E} = 0$ is equivalent to $\oint \vec{E} \cdot d\vec{l} = 0$ over an infinitely small path.



$$\nabla \cdot \vec{D} = \rho \quad \nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{D} = 0 \quad \nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{D} = \rho \quad \nabla \times \vec{E} = \vec{J}$$

$$\nabla \cdot \vec{D} = 0 \quad \nabla \times \vec{E} = \vec{J}$$

Example 1: Find the divergence of $\mathbf{D} = \hat{x}5x + \hat{y}12$ C/m^2

Solution: In this case

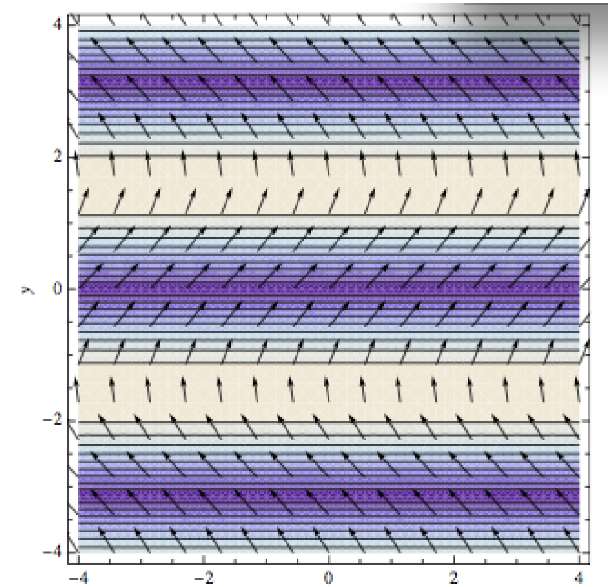
$$D_x = 5x, \quad D_y = 12, \quad \text{and} \quad D_z = 0.$$

Therefore, divergence of $\mathbf{D}$ is

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \\
&= \frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(12) + \frac{\partial}{\partial z}(0) \\
&= 5 + 0 + 0 = 5\,\frac{\mathrm{C}}{\mathrm{m}^3}.
\end{aligned}$$

Note that the divergence of vector $\mathbf{D}$ is a scalar quantity which is the volumetric charge density in space as a consequence of Gauss's law (in differential form).

Example 3: Find the curl of the vector field



$$\mathbf{E} = \hat{x}\cos y + \hat{y}1$$

Solution: The curl is

$$\begin{aligned}
\nabla \times \mathbf{E} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & 1 & 0 \end{vmatrix} \\
&= \hat{x}(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}1) - \hat{y}(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}\cos y) + \hat{z}(\frac{\partial}{\partial x}1 - \frac{\partial}{\partial y}\cos y) \\
&= \hat{x}0 - \hat{y}0 + \hat{z}(0 + \sin y) = \hat{z}\sin y
\end{aligned}$$

which is another vector field.

CURL-free and DIVERGENCE free