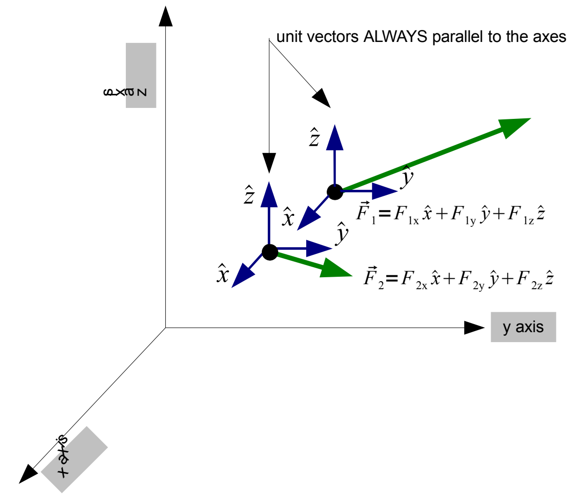
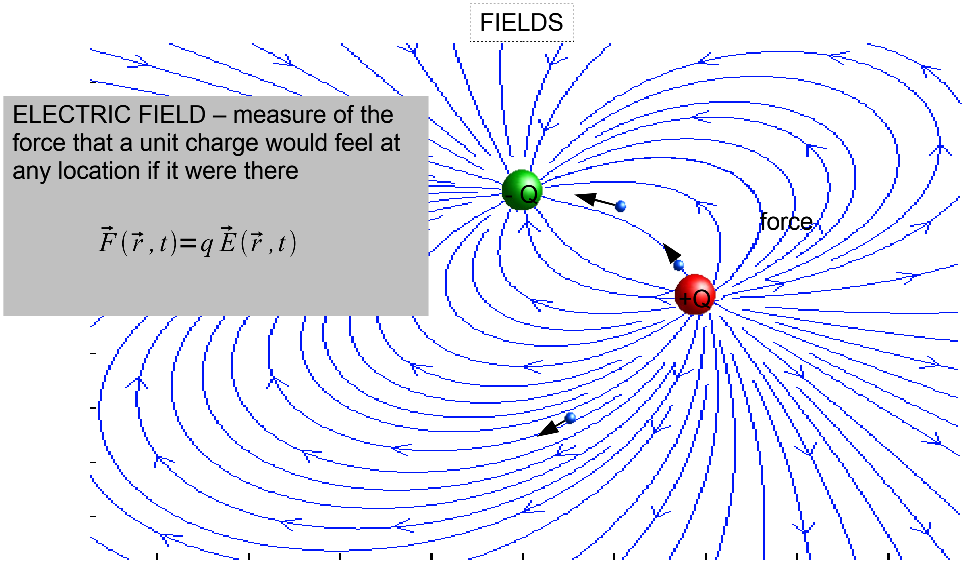
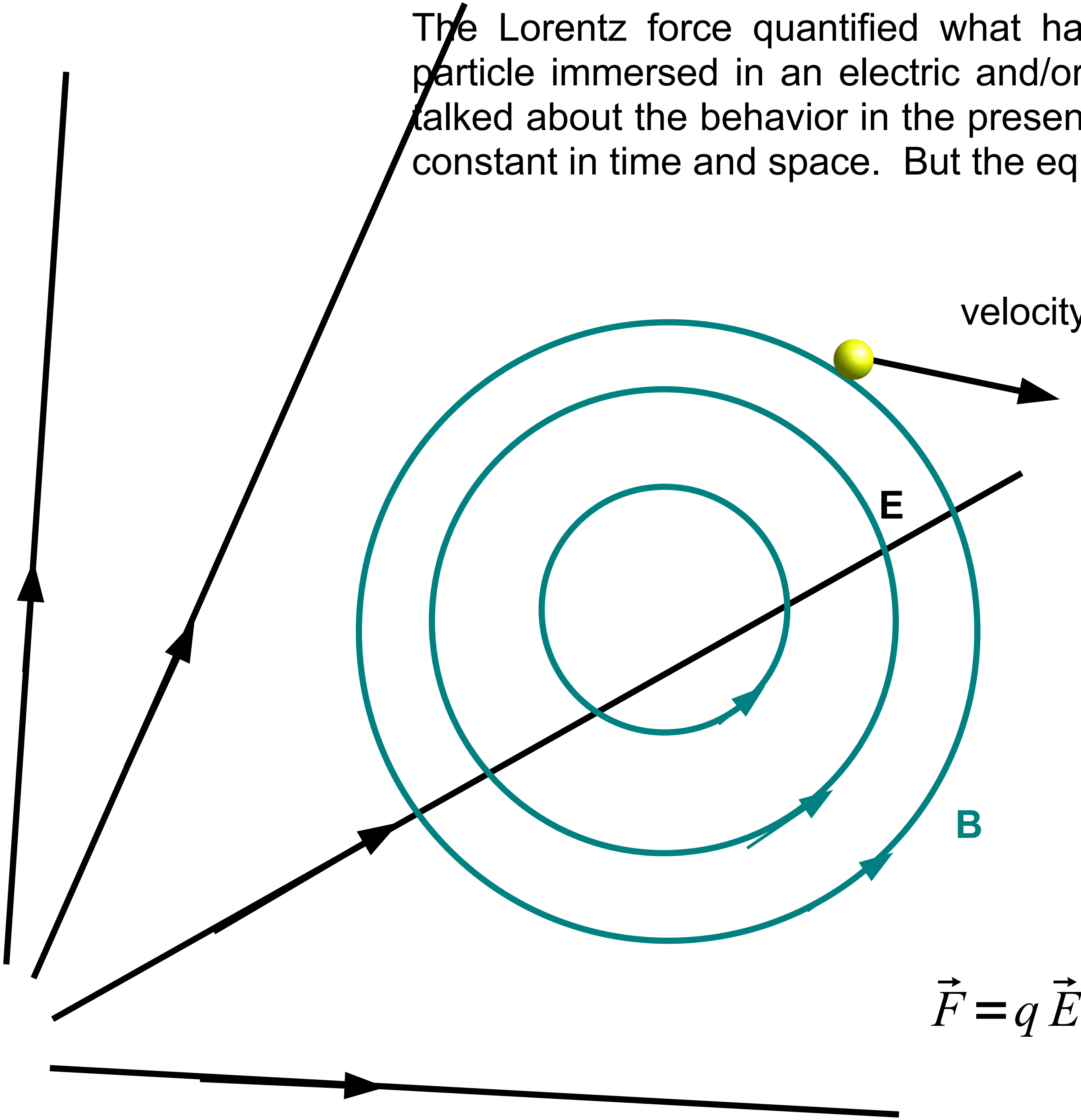
$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

- Summary:
  - How charged particles interact
  - Lorentz force
  - Math Tools:
    - coordinate systems
    - vector notation
    - fields





The Lorentz force quantified what happens to a charged particle immersed in an electric and/or magnetic field. We talked about the behavior in the presence of fields that were constant in time and space. But the equation is general...



$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Now lets address the sources of these fields by looking first at the electric field.

COULOMB's Law -

Charges exert a force on one another according to these observation:

- *magnitude of the force is proportional to the magnitude of both charges*
- *magnitude of the force is inversely proportional to the square of the distance between them*
- *direction of the force lies along the direction of a line connecting the charges. Like charges repel, unlike attract.*

$$\vec{F} = \frac{qQ}{4\pi\epsilon_o r^2}\hat{a}$$

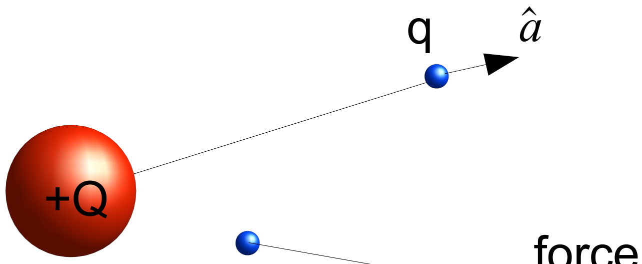
$q-charge$

$Q-charge$

$4\pi\epsilon_o - constant\ of\ proportionality$

$r-distance\ between\ the\ charges$

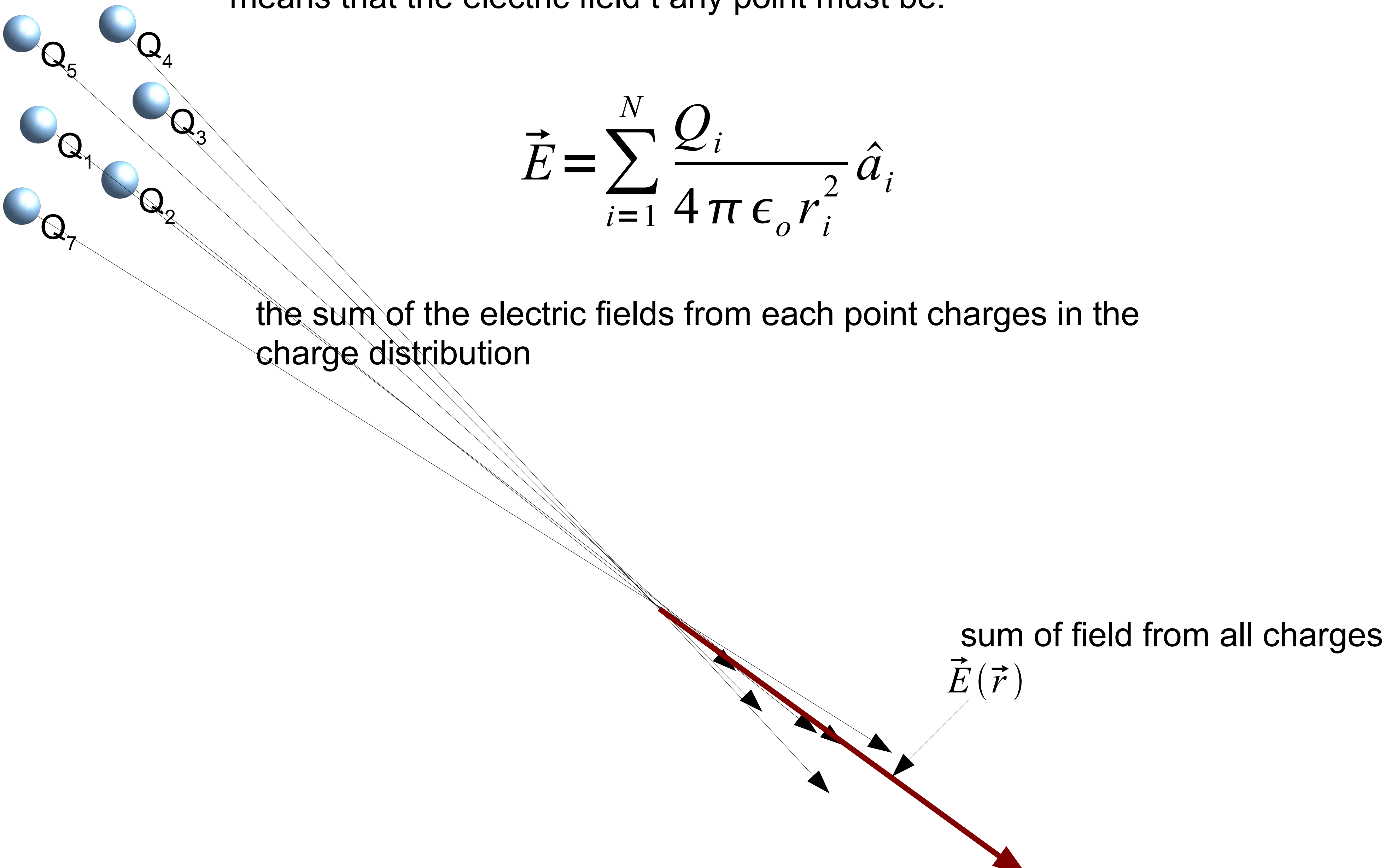
$\hat{a}-unit\ vector\ pointing \in the\ direction\ of\ positive\ field$

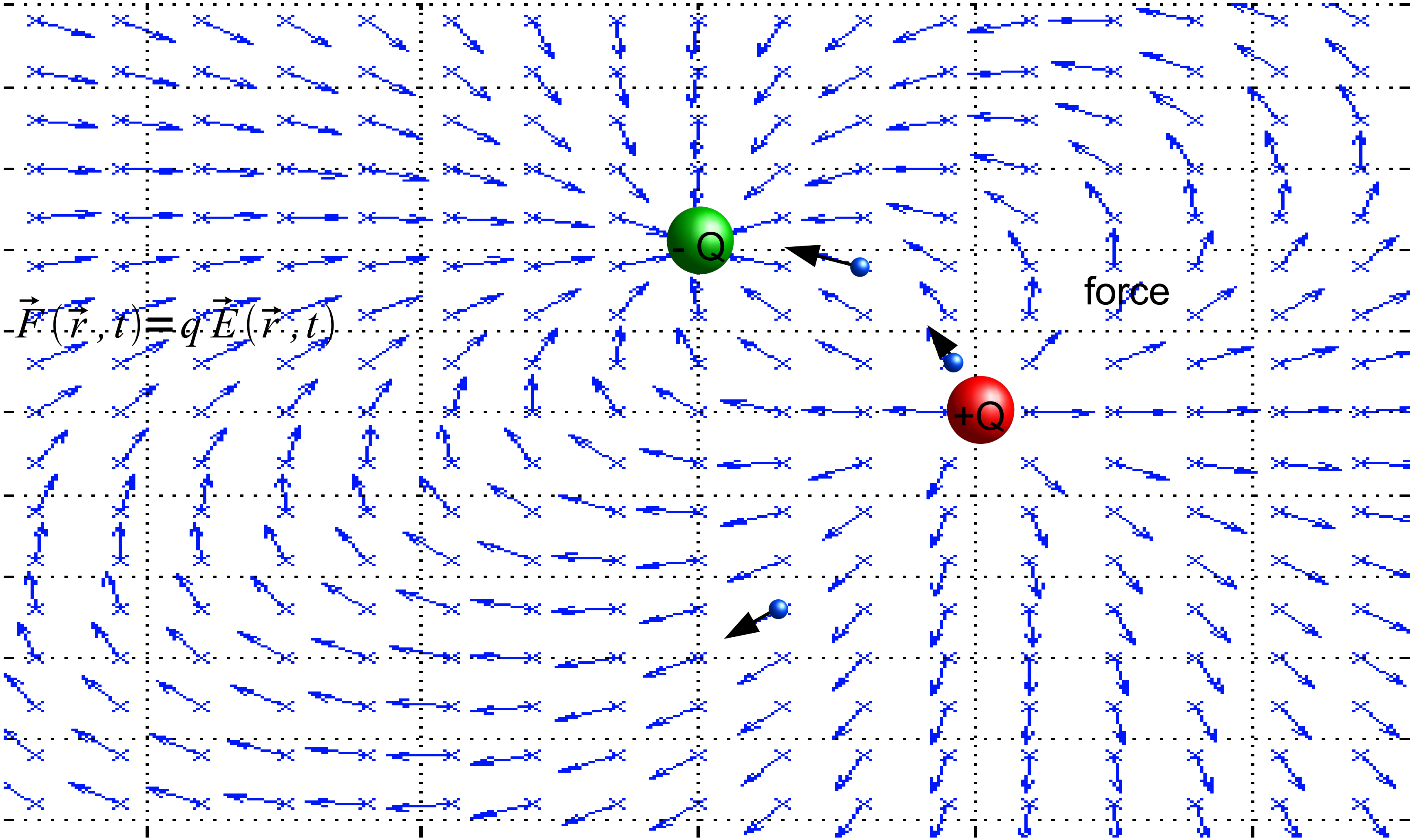


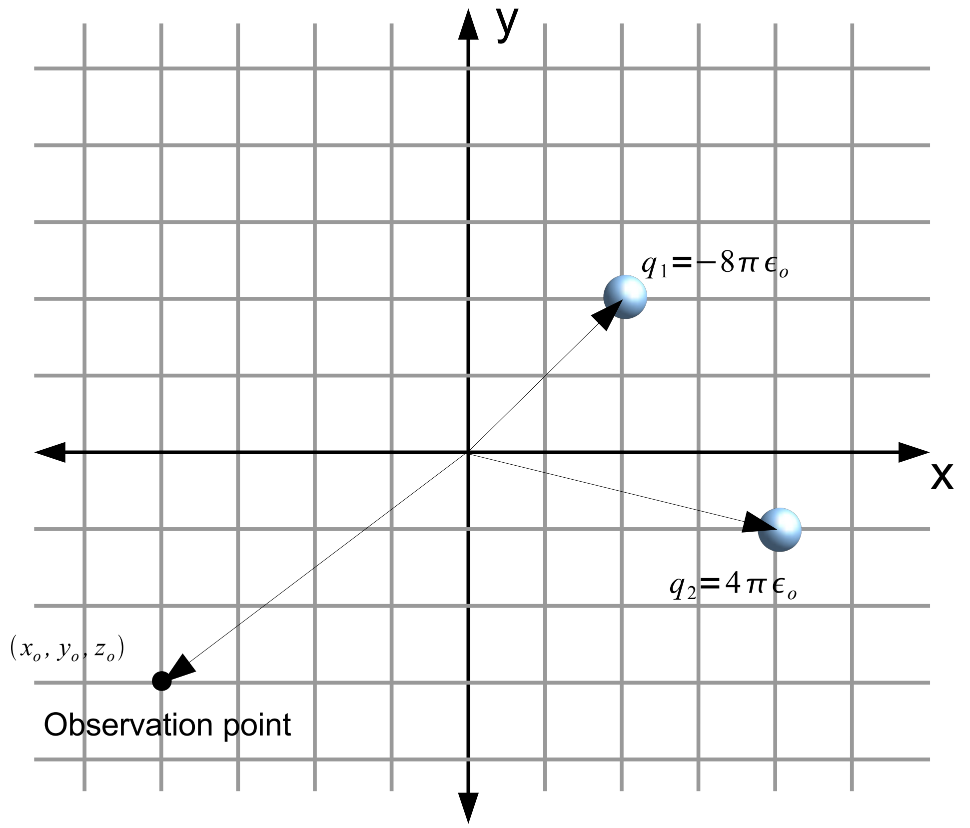
All electric fields are generated by a collection of points sources. Luckily, most physical phenomena are linear which means that the electric field t any point must be:

$$\vec{E} = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon_o r_i^2} \hat{a}_i$$

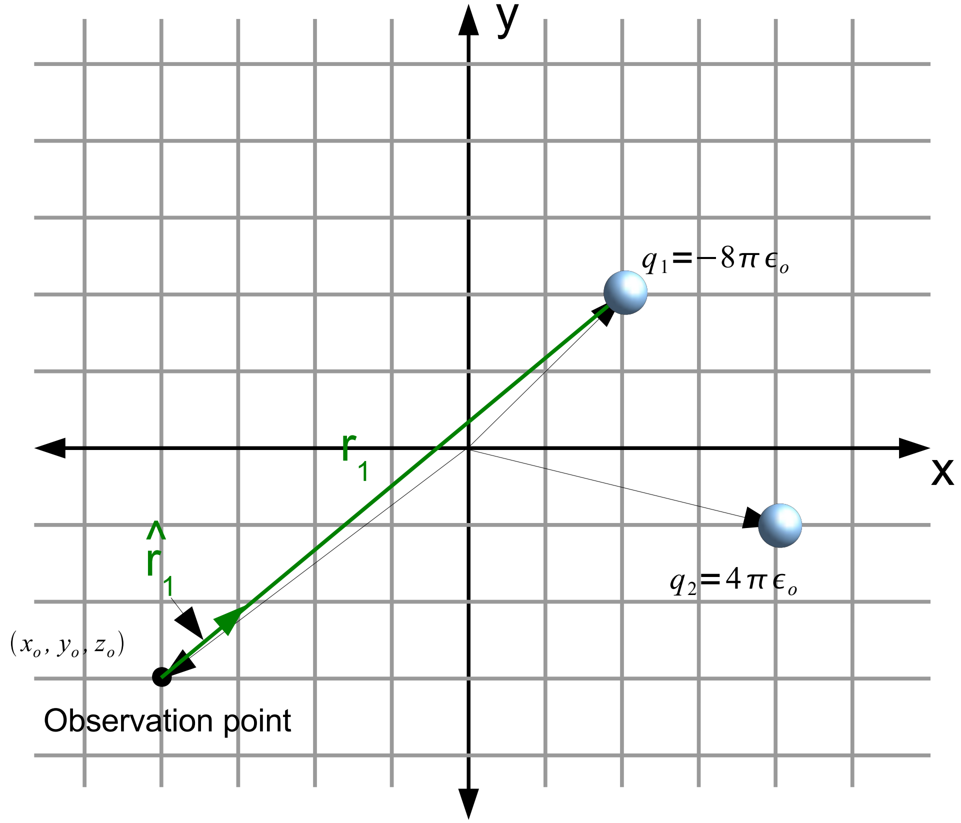
the sum of the electric fields from each point charges in the charge distribution




$\vec{F}(\vec{r},t)=q\vec{E}(\vec{r},t)$
- Q
+Q
force

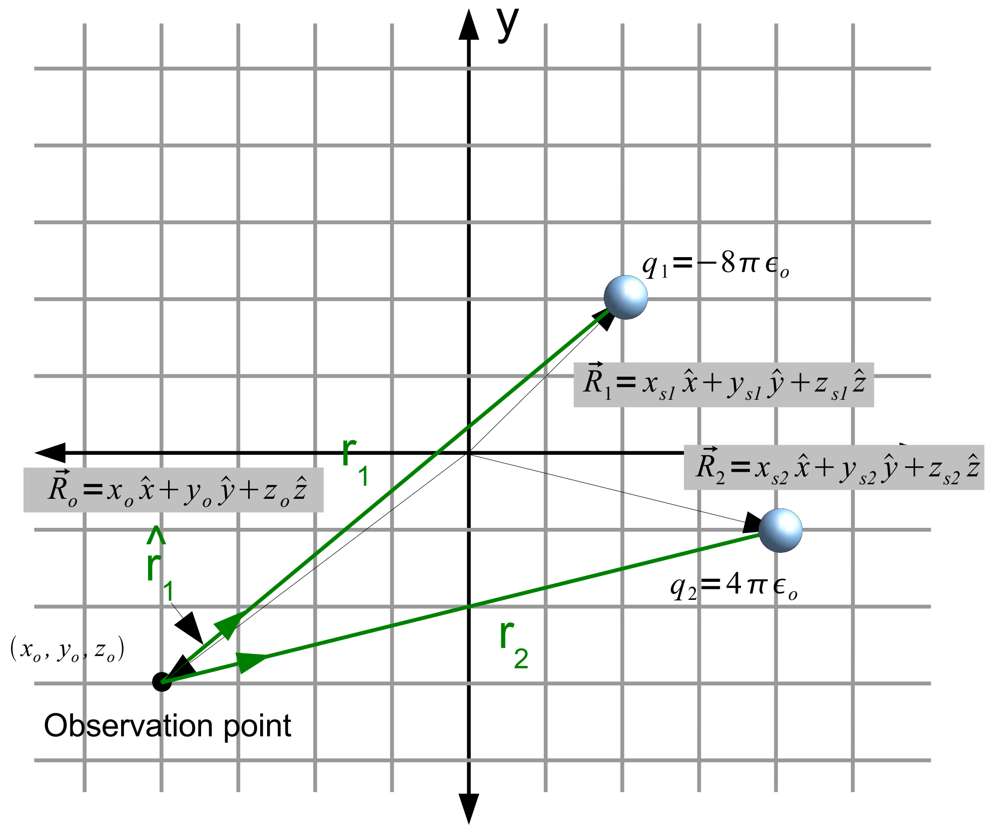


Find the electric field at an arbitrary point (x,y,z) generated by a particle at (2,2,0) with charge $-8\pi\varepsilon_o$ and a particle at (4,-1,0) with charge $4\pi\varepsilon_o$



$$\vec{E}_1 = \frac{-8\pi\epsilon_o}{4\pi\epsilon_o r_1^2}\hat{r}_1$$

$$\vec{E}_2 = \frac{-8\pi\epsilon_o}{4\pi\epsilon_o r_2^2}\hat{r}_2$$

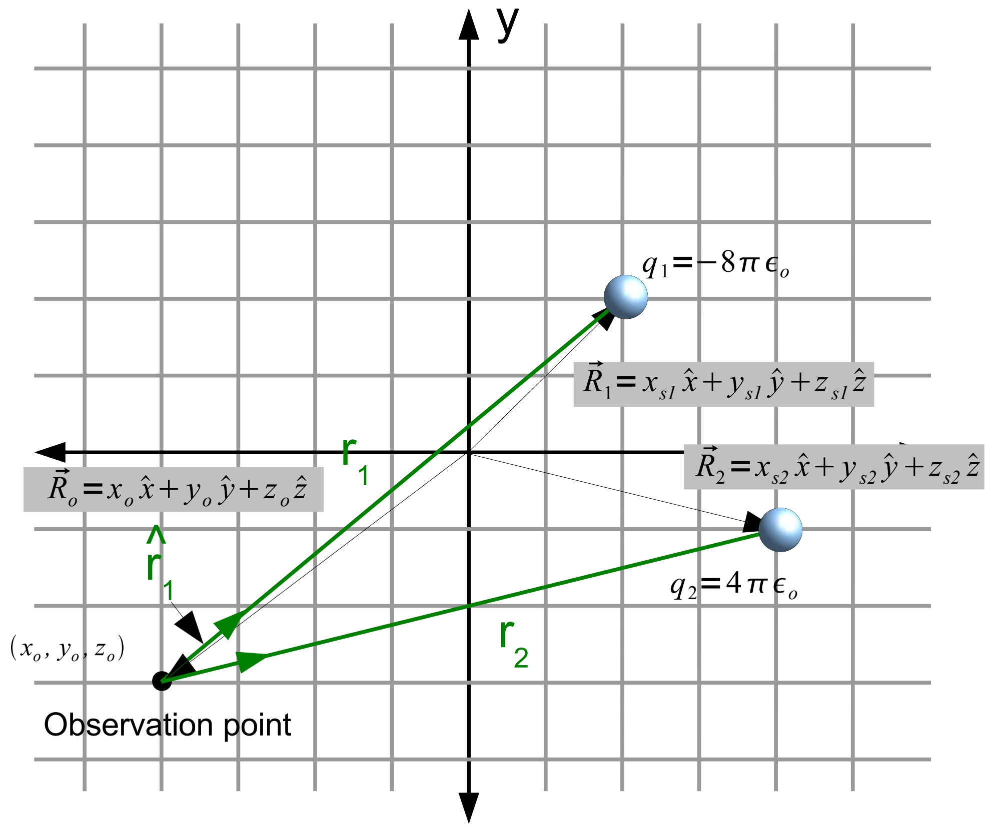


$$\vec{E}_1 = \frac{-8\pi\epsilon_o}{4\pi\epsilon_o r_1^2}\hat{r}_1$$

Use position vectors to find r_1, r_2 and $\hat{r}_1$, $\hat{r}_2$:

$$r_1 = |(x_o - x_{s1})\hat{x} + (y_o - y_{s1})\hat{y} + (z_o - z_{s1})\hat{z}|$$

$$r_2 = |(x_o - x_{s2})\hat{x} + (y_o - y_{s2})\hat{y} + (z_o - z_{s2})\hat{z}|$$

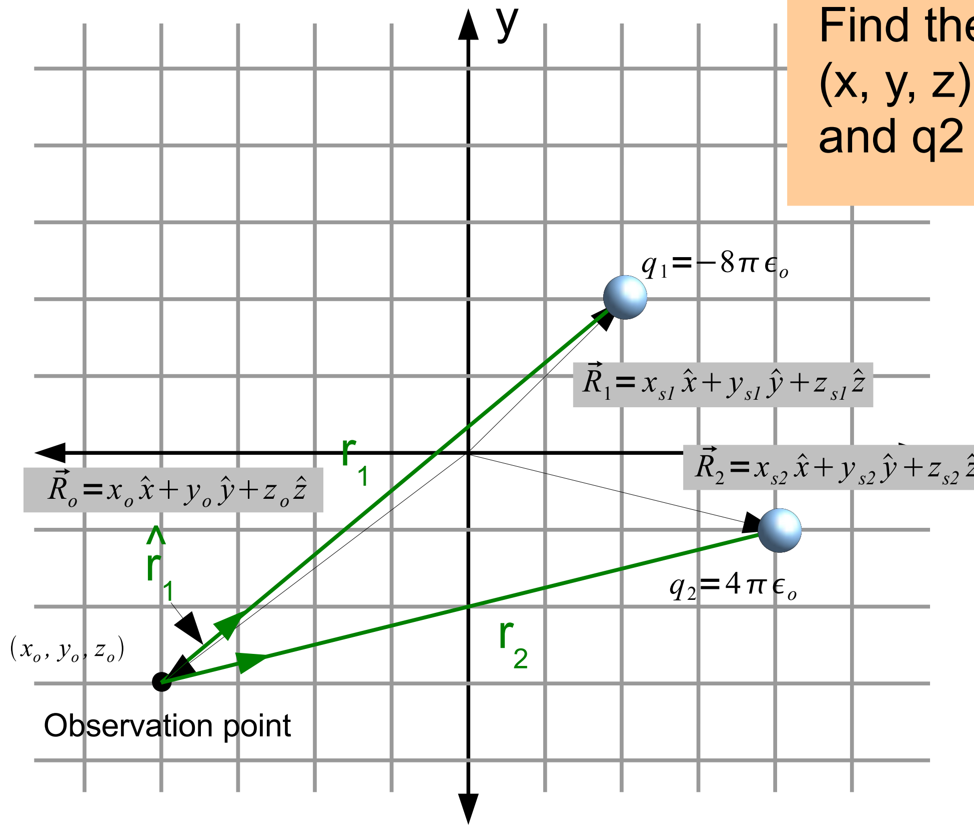


Use position vectors to find r_1, r_2 and $\hat{r}_1$, $\hat{r}_2$:

$$\hat{r}_1 = \frac{(x_o - x_{s1})\hat{x} + (y_o - y_{s1})\hat{y} + (z_o - z_{s1})\hat{z}}{|(x_o - x_{s1})\hat{x} + (y_o - y_{s1})\hat{y} + (z_o - z_{s1})\hat{z}|}$$

$$\hat{r}_2 = \frac{(x_o - x_{s2})\hat{x} + (y_o - y_{s2})\hat{y} + (z_o - z_{s2})\hat{z}}{|(x_o - x_{s2})\hat{x} + (y_o - y_{s2})\hat{y} + (z_o - z_{s2})\hat{z}|}$$

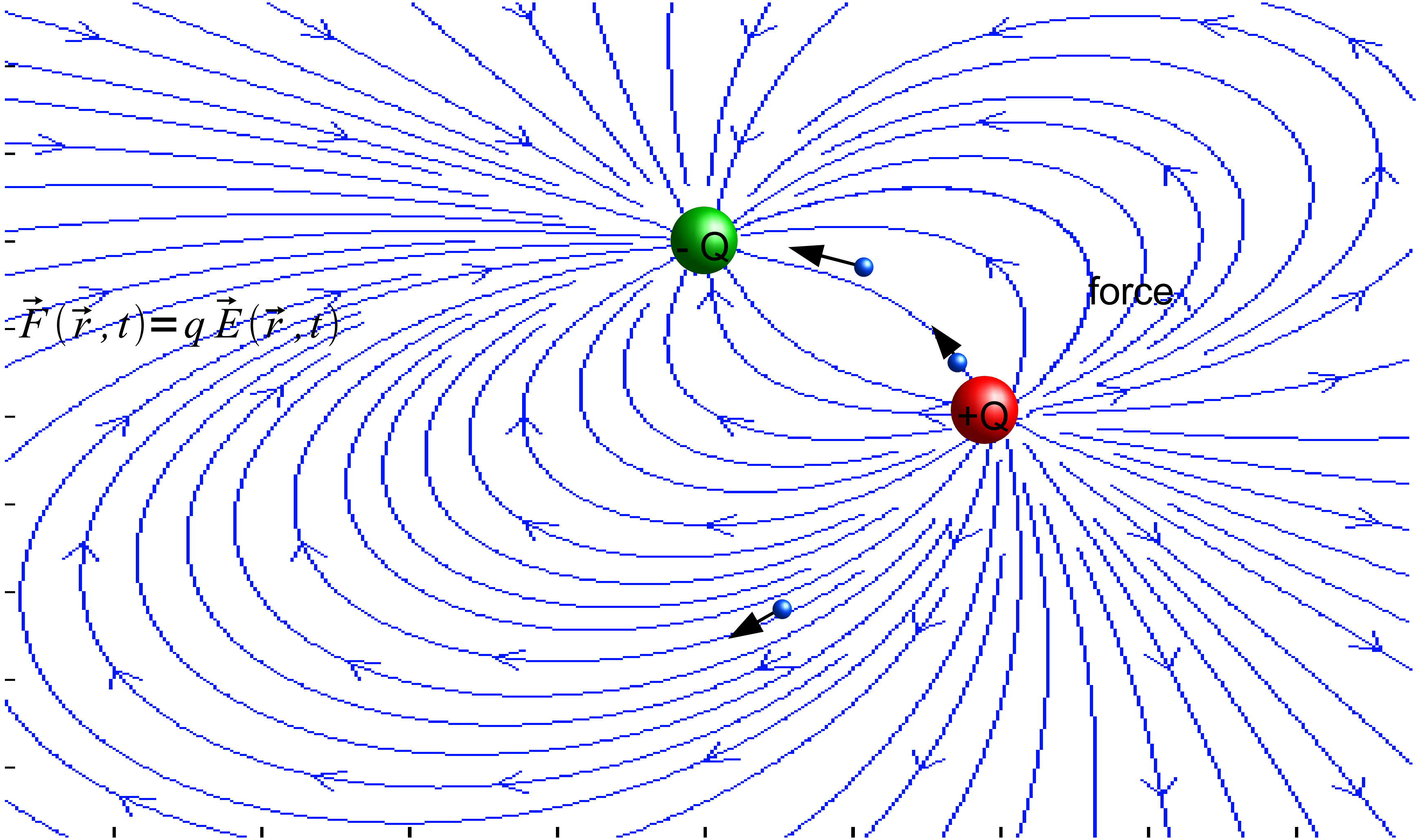
Find the electric field at (x, y, z) if q1 is at (2,2,0) and q2 is at (4,-1,0).



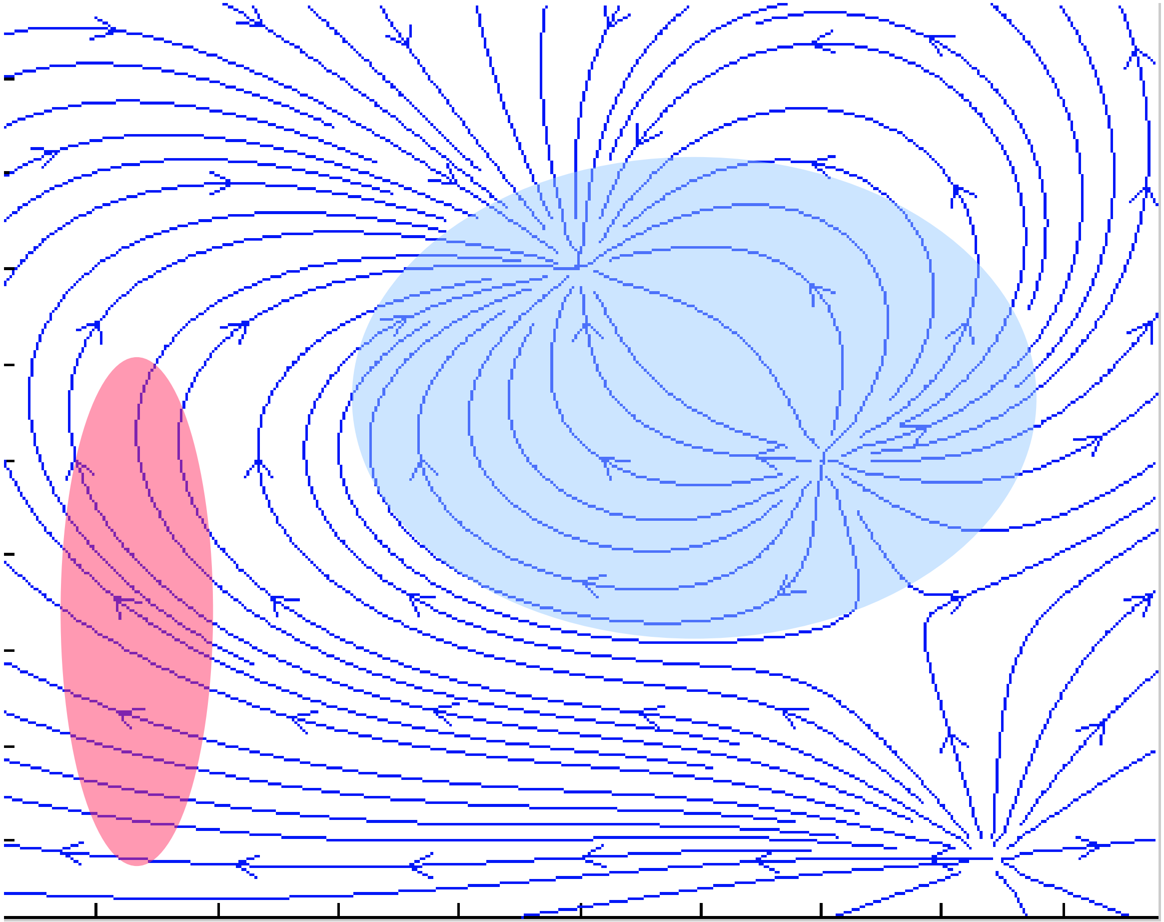
Use position vectors to find r_1, r_2 and $\hat{r}_1$, $\hat{r}_2$:

$$\vec{E}_1 = \frac{-2(x_o - x_{s1})\hat{x} + (y_o - y_{s1})\hat{y} + (z_o - z_{s1})\hat{z}}{|(x_o - x_{s1})\hat{x} + (y_o - y_{s1})\hat{y} + (z_o - z_{s1})\hat{z}|^{3/2}}$$

$$\vec{E}_2 = \frac{(x_o - x_{s2})\hat{x} + (y_o - y_{s2})\hat{y} + (z_o - z_{s2})\hat{z}}{|(x_o - x_{s2})\hat{x} + (y_o - y_{s2})\hat{y} + (z_o - z_{s2})\hat{z}|^{3/2}}$$


- Q
+Q
force
$\vec{F}(\vec{r},t) = q\vec{E}(\vec{r},t)$

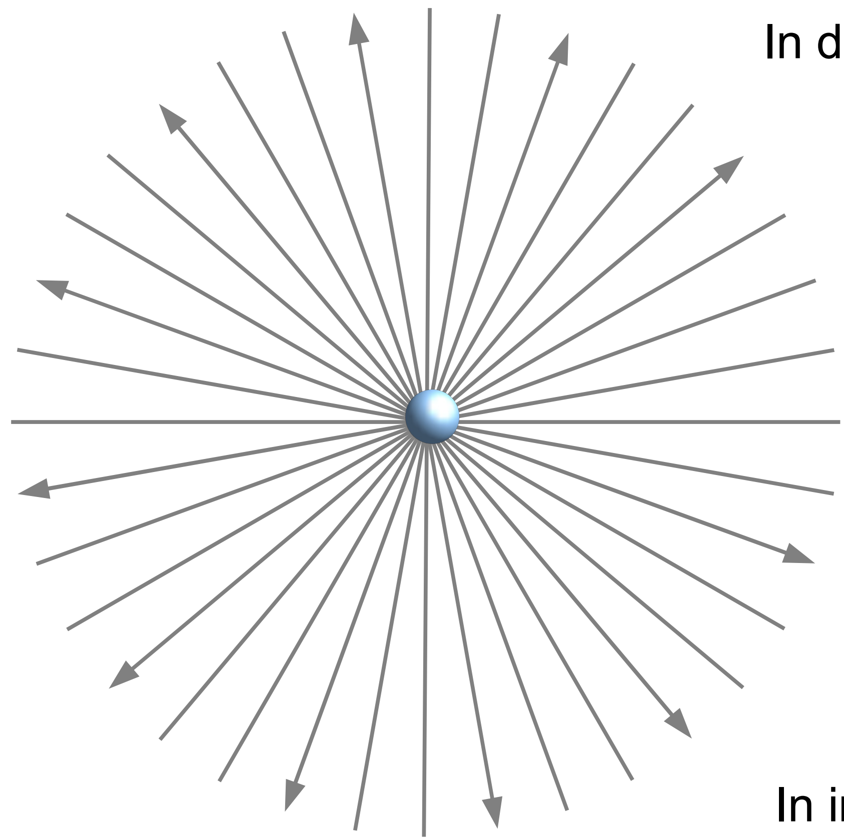
DIVERGENCE



Can you locate the charges?

For the two colored areas – Are there any charges enclosed?

We can guess whether or not charges reside within any given closed surface because we have an intuitive understanding of one of Maxwell's equations – usually written first – Gauss' Law for electric fields.



In differential form

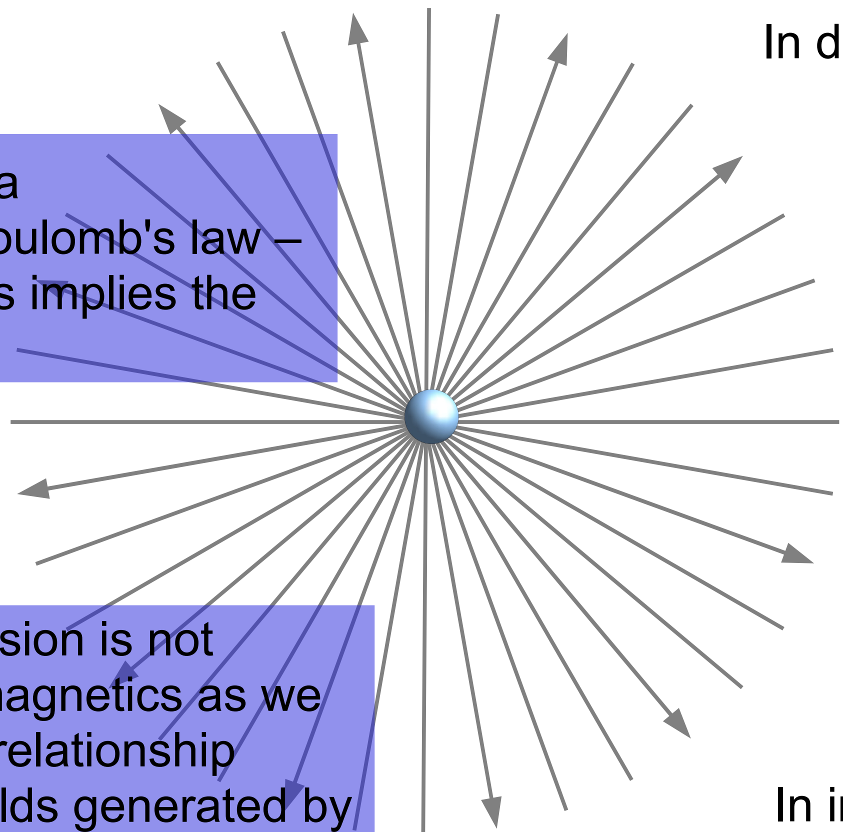
$$\nabla \cdot \vec{E} = \rho / \epsilon_o$$

GAUSS' LAW

In integral form

$$\oint \vec{E} \cdot d\vec{S} = Q / \epsilon_o = \int \frac{\rho}{\epsilon_o} dV$$

Gauss's law is just a generalization of Coulomb's law – knowing one always implies the other.



In differential form

$$\nabla \cdot \vec{E} = \rho / \epsilon_o$$

GAUSS' LAW

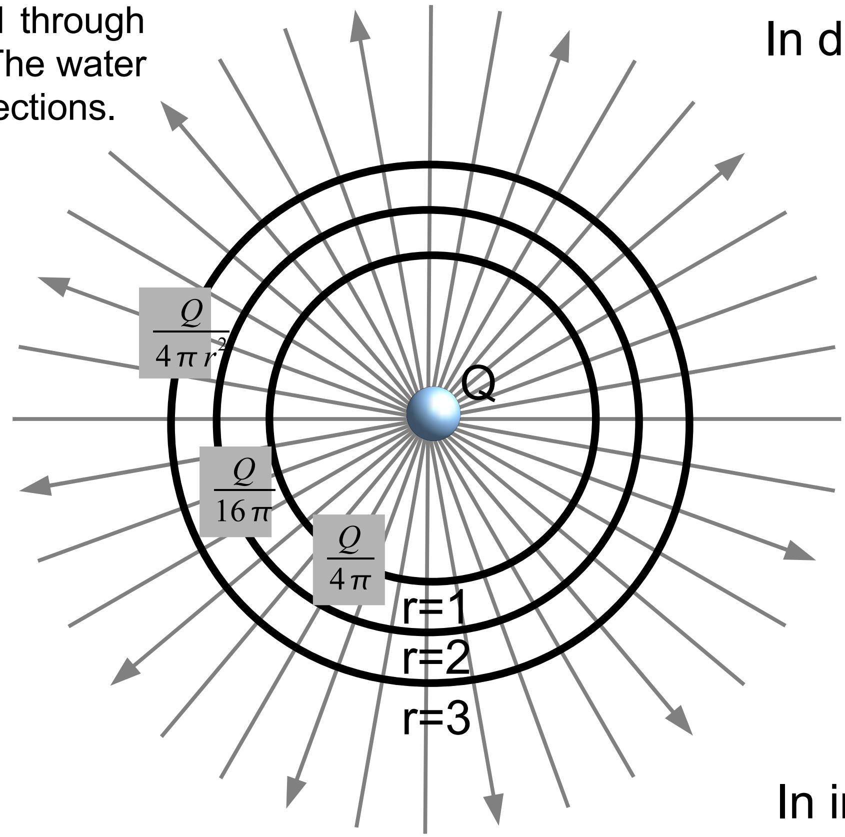
In fact, this expression is not unique to electromagnetics as we will see. General relationship between vector fields generated by sources whose influence drops off as $1/r^2$.

In integral form

$$\oint \vec{E} \cdot d\vec{S} = Q/\epsilon_o = \int \frac{\rho}{\epsilon_o} dV$$

Imagine that the charge Q is a source that emits Q amount of stuff continuously, which spreads out uniformly in all directions.

A 2-D analogy might be a fountain where the water is delivered through a small hole in the ground. The water then flows uniformly in all directions.



In differential form

$$\nabla \cdot \vec{E} = \rho / \epsilon_o$$

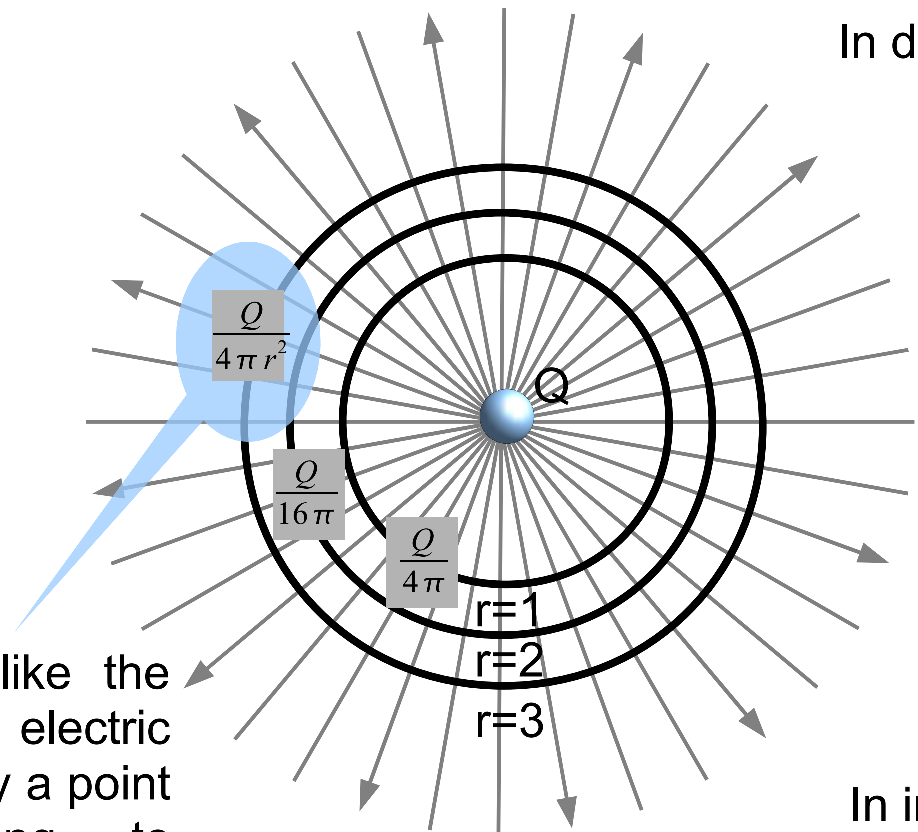
GAUSS' LAW

In integral form

$$\oint \vec{E} \cdot d\vec{S} = Q/\epsilon_o = \int \frac{\rho}{\epsilon_o} dV$$

In differential form

$$\nabla \cdot \vec{E} = \rho / \epsilon_o$$

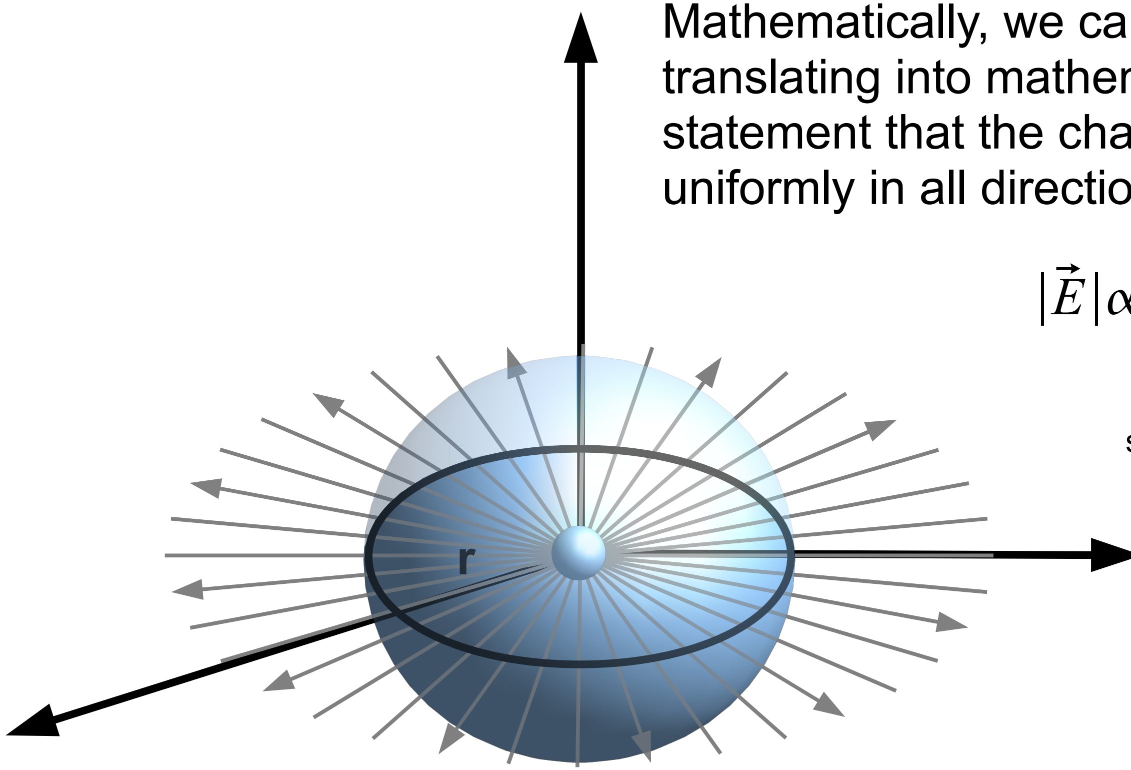


GAUSS' LAW

This looks just like the magnitude of the electric field generated by a point charge according to Coulomb's law.

In integral form

$$\oint \vec{E} \cdot d\vec{S} = Q / \epsilon_o = \int \frac{\rho}{\epsilon_o} dV$$



Mathematically, we can state Gauss' Law by translating into mathematical symbols the statement that the charge Q spreads out uniformly in all directions.

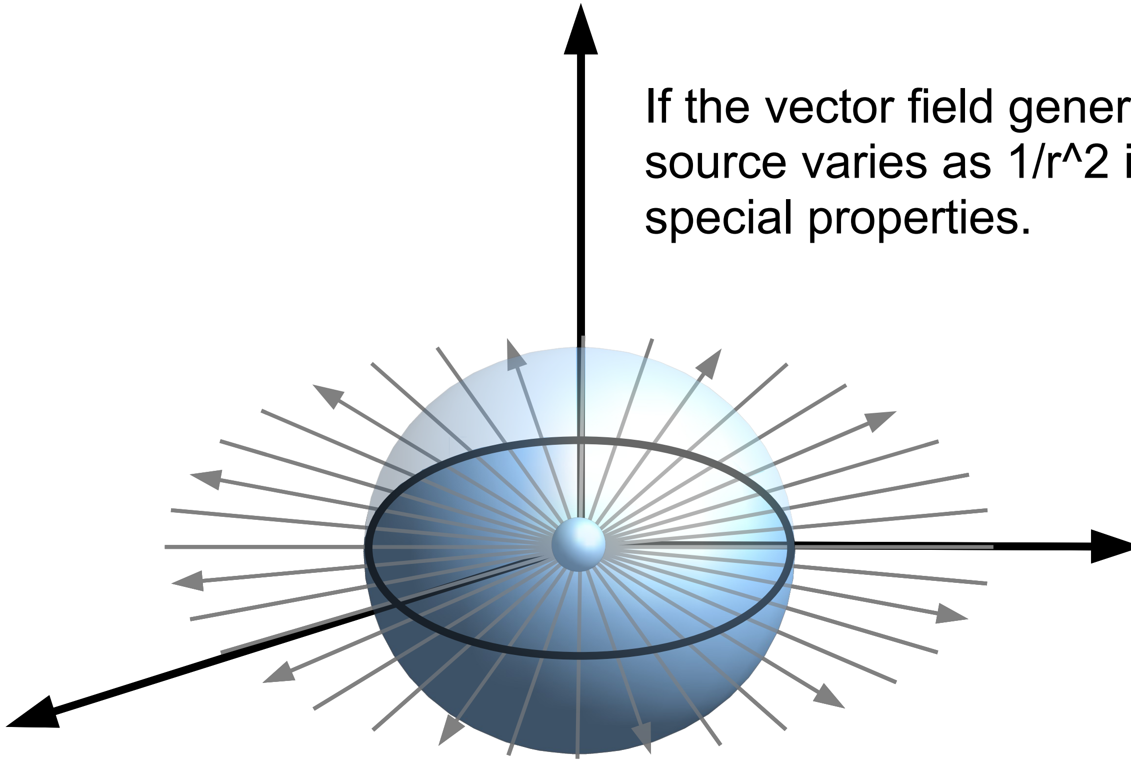
$$|\vec{E}| \alpha \frac{Q}{4\pi r^2}$$

surface area of a sphere.

The surface area of a sphere can be found by integrating over the spherical surface –

$$A_{sphere} = 4\pi r^2 = \int\int dS = \int\int r^2 \sin\theta \, d\phi \, d\theta$$

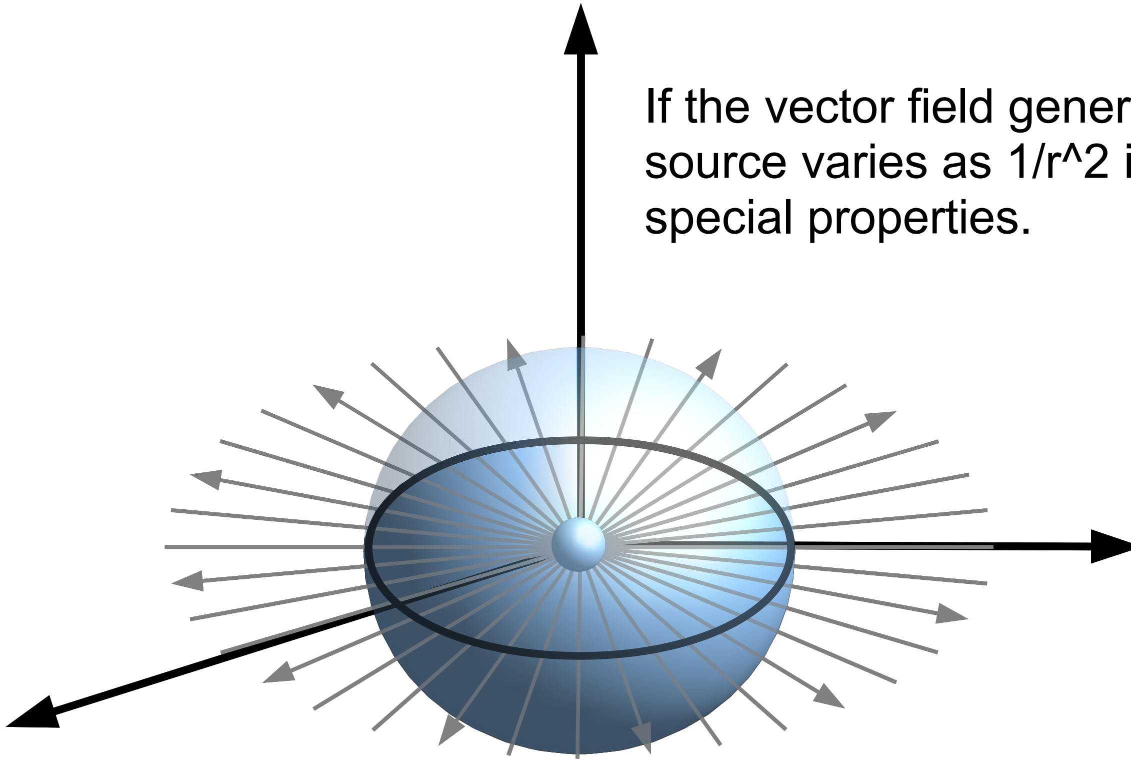
$$|\vec{E}| \alpha \frac{Q}{4\pi r^2} = \frac{Q}{\int\int r^2 \sin\theta \, d\phi \, d\theta}$$



If the vector field generated by a single source varies as 1/r^2 it has some very special properties.

$$|\vec{E}|\int\int r^2 \sin\theta \, d\phi \, d\theta = \frac{Q}{\epsilon_o} = |\vec{E}|\int\int dS$$

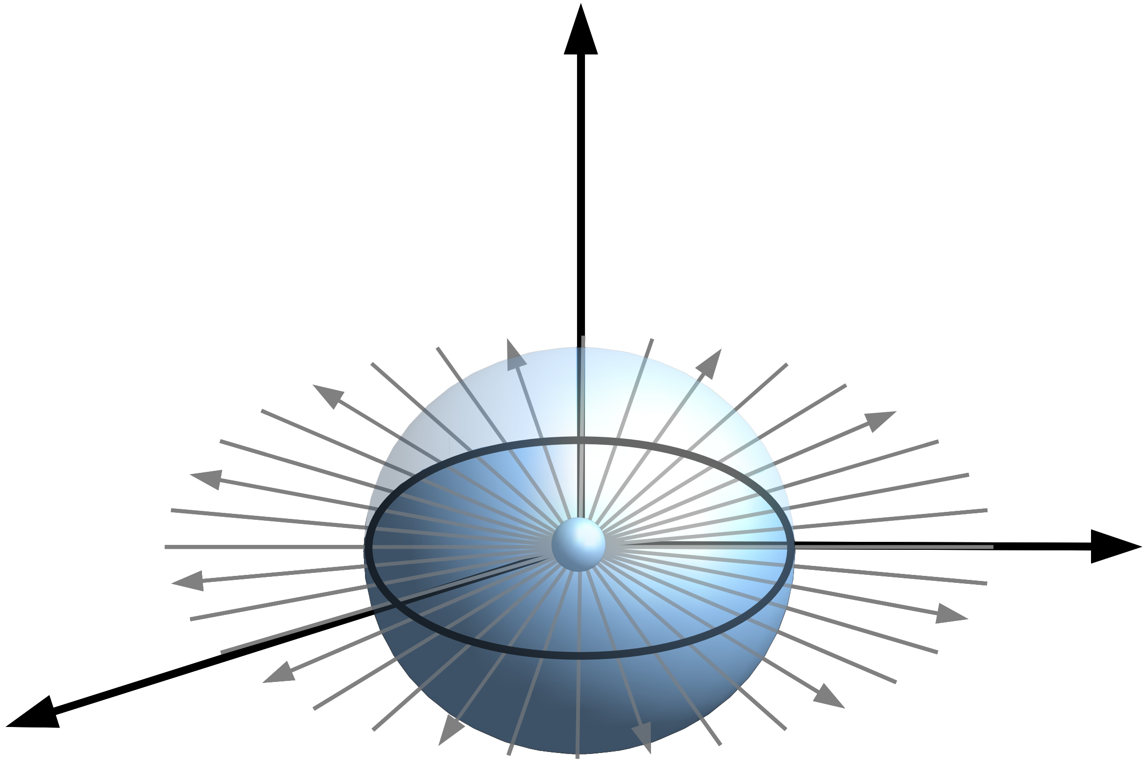
Infinitesimal surface area of a sphere



If the vector field generated by a single source varies as 1/r^2 it has some very special properties.

$$|\vec{E}| \int \int r^2 \sin\theta \, d\phi \, d\theta = \frac{Q}{\epsilon_o} = \int \int |\vec{E}| dS$$

This is a completely general result, true for any shaped surface and any charge distribution. By drawing any CLOSED surface around a charge distribution and finding the amount of the electric field flowing OUT of the surface as compared to the amount of the electric field flowing IN to the surface, the amount of charge can be deduced.

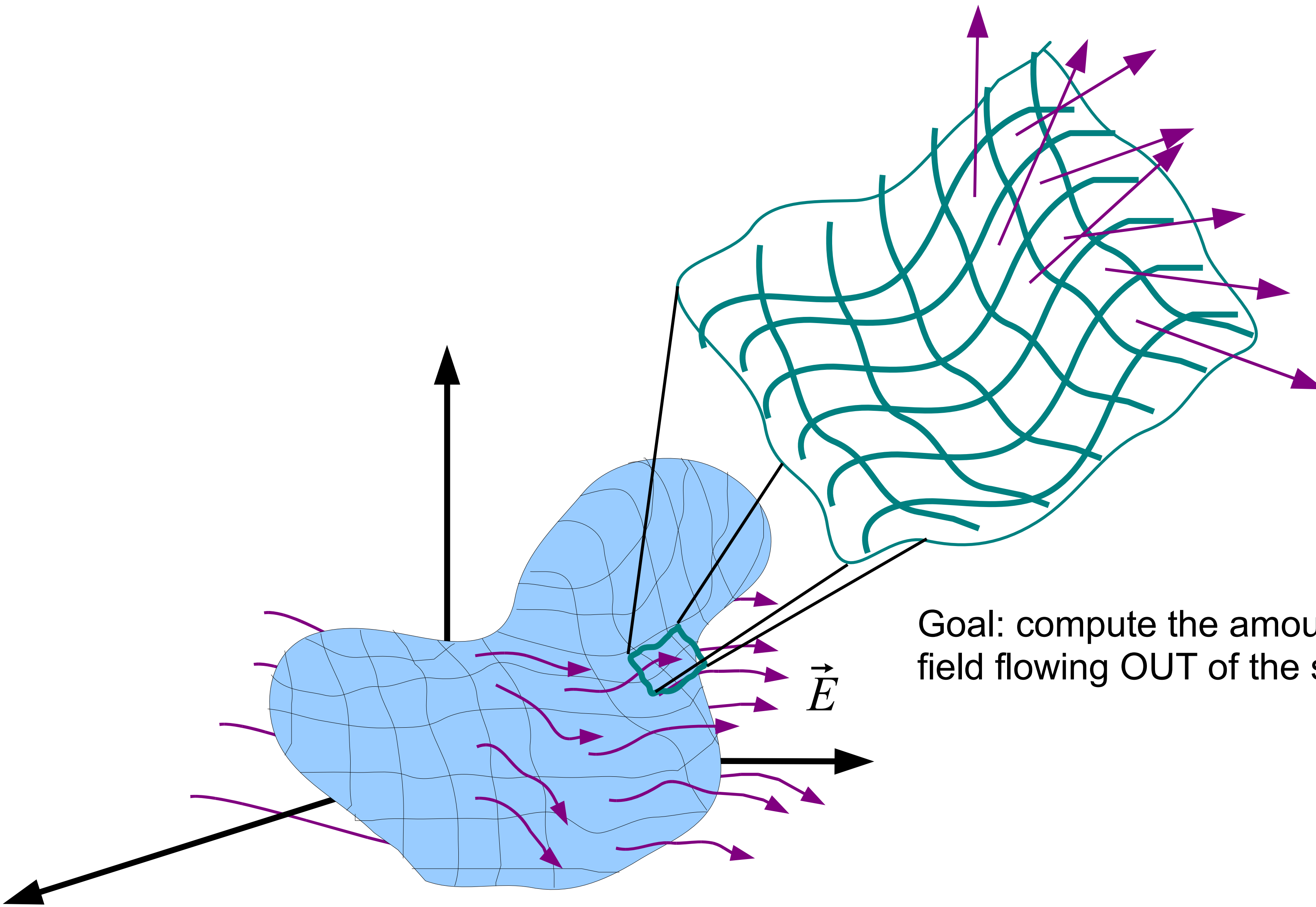


$$\int\int |\vec{E}|\, r^2 \sin\theta \, d\phi \, d\theta \rightarrow \oint |\vec{E}|\, dS$$

Generalize this formula

By looking at the electric field at the surface of a closed figure you can deduce information about the charge distribution producing the field.

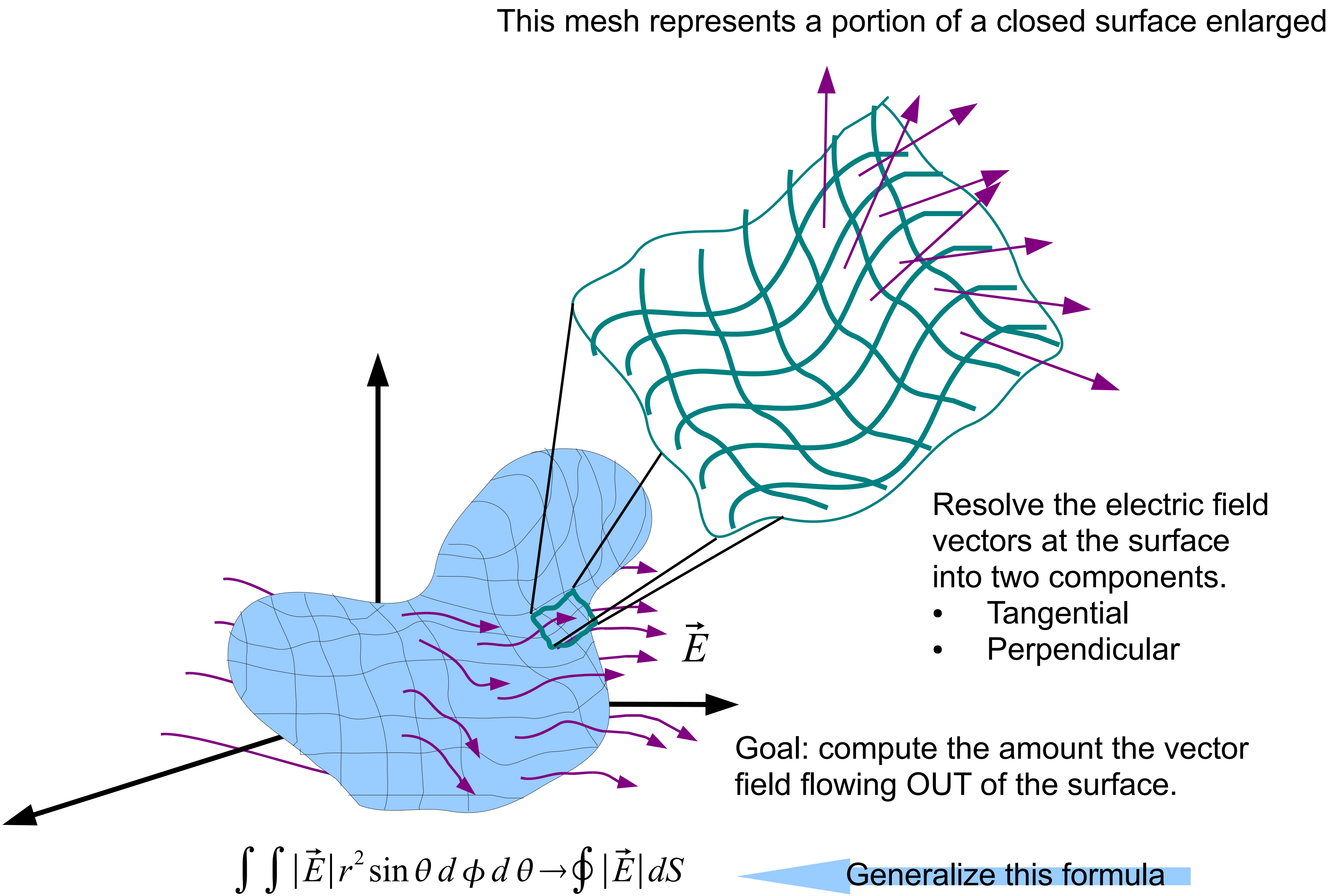
This mesh represents a portion of a closed surface enlarged

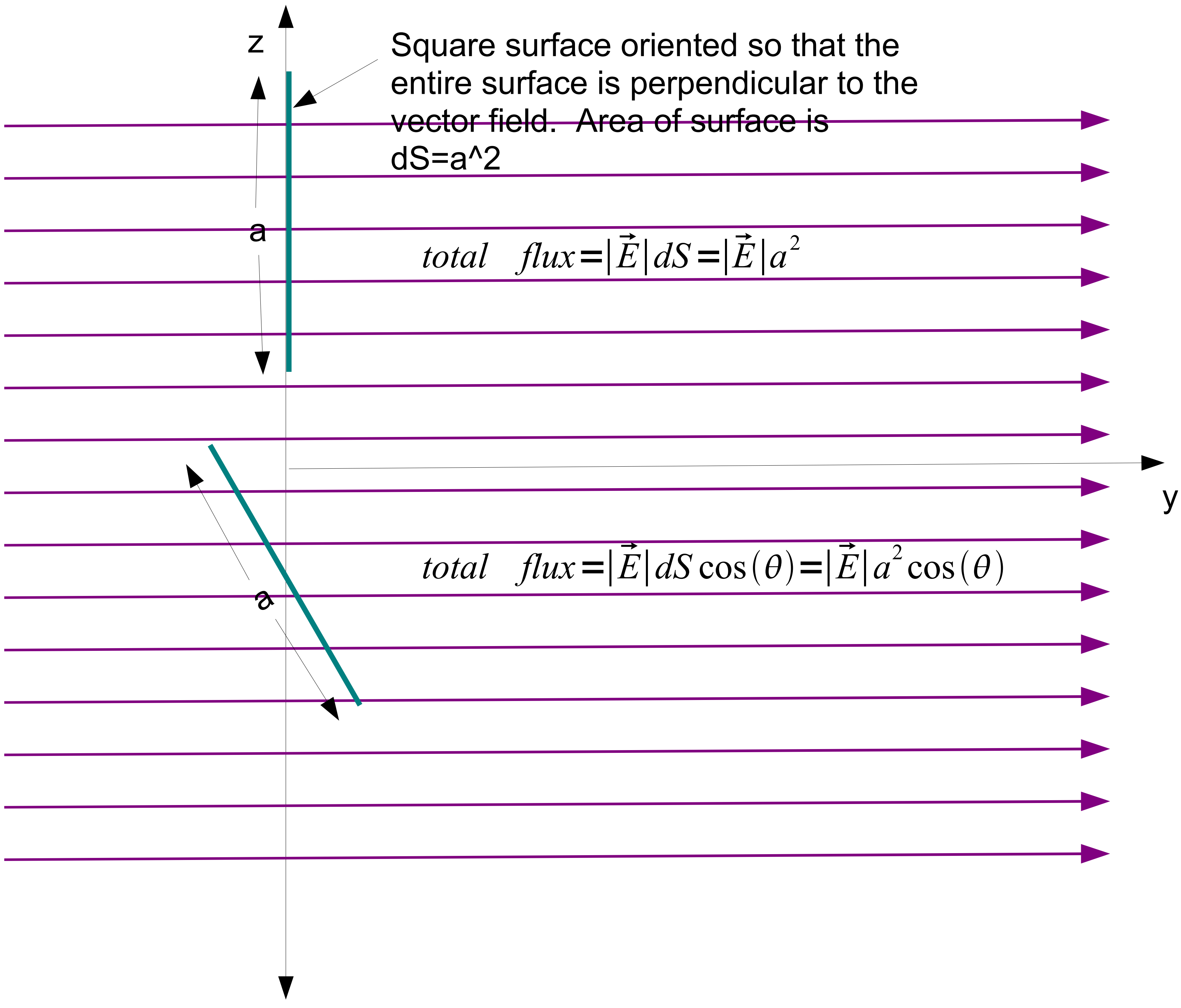


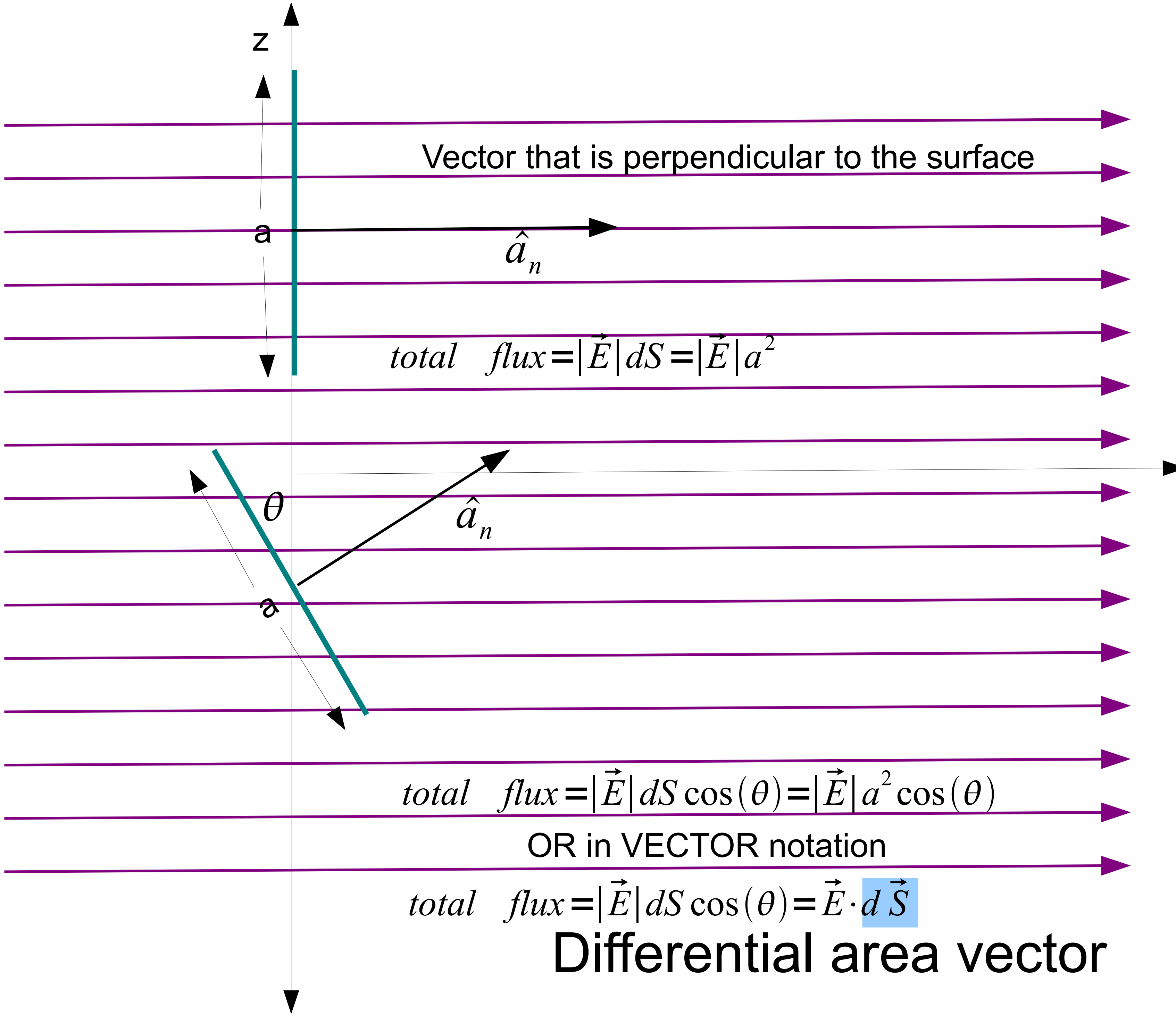
Goal: compute the amount the vector field flowing OUT of the surface.

$$\int\int |\vec{E}| r^2 \sin\theta \, d\phi \, d\theta \rightarrow \oint |\vec{E}| dS$$

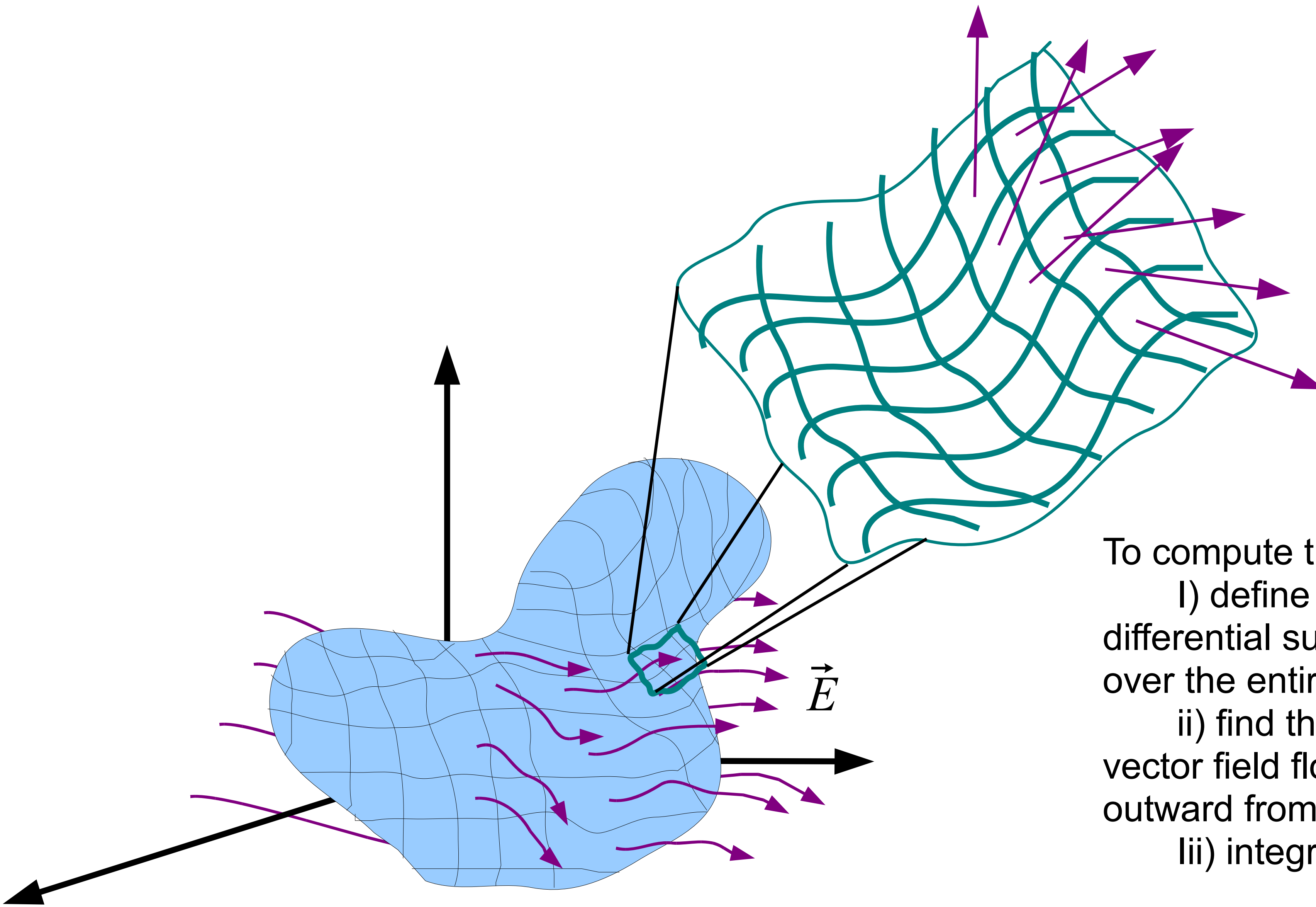
Generalize this formula


This mesh represents a portion of a closed surface enlarged
$\vec{E}$
Resolve the electric field vectors at the surface into two components.
• Tangential
• Perpendicular
Goal: compute the amount the vector field flowing OUT of the surface.
$\int\int|\vec{E}|r^2\sin\theta\, d\phi\, d\theta \rightarrow \oint|\vec{E}|dS$
Generalize this formula


z
Square surface oriented so that the entire surface is perpendicular to the vector field. Area of surface is dS=a^2
a
$total \quad flux = |\vec{E}| dS = |\vec{E}| a^2$
y
a
$total \quad flux = |\vec{E}| dS \cos(\theta) = |\vec{E}| a^2 \cos(\theta)$


z
y
a
a
θ
Vector that is perpendicular to the surface
$\hat{a}_n$
$\hat{a}_n$
$total \quad flux = |\vec{E}|\, dS = |\vec{E}|\, a^2$
$total \quad flux = |\vec{E}|\, dS \cos(\theta) = |\vec{E}|\, a^2 \cos(\theta)$
OR in VECTOR notation
$total \quad flux = |\vec{E}|\, dS \cos(\theta) = \vec{E} \cdot d\vec{S}$
Differential area vector

This mesh represents a portion of a closed surface enlarged



To compute the flux –

I) define the differential surface vector over the entire surface.

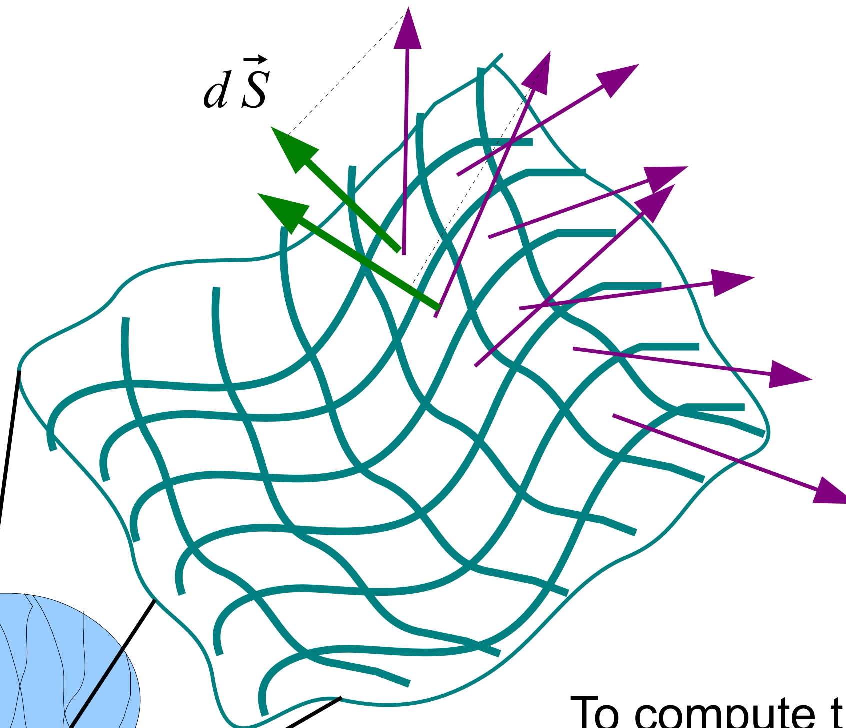
ii) find the amount of vector field flowing outward from the surace

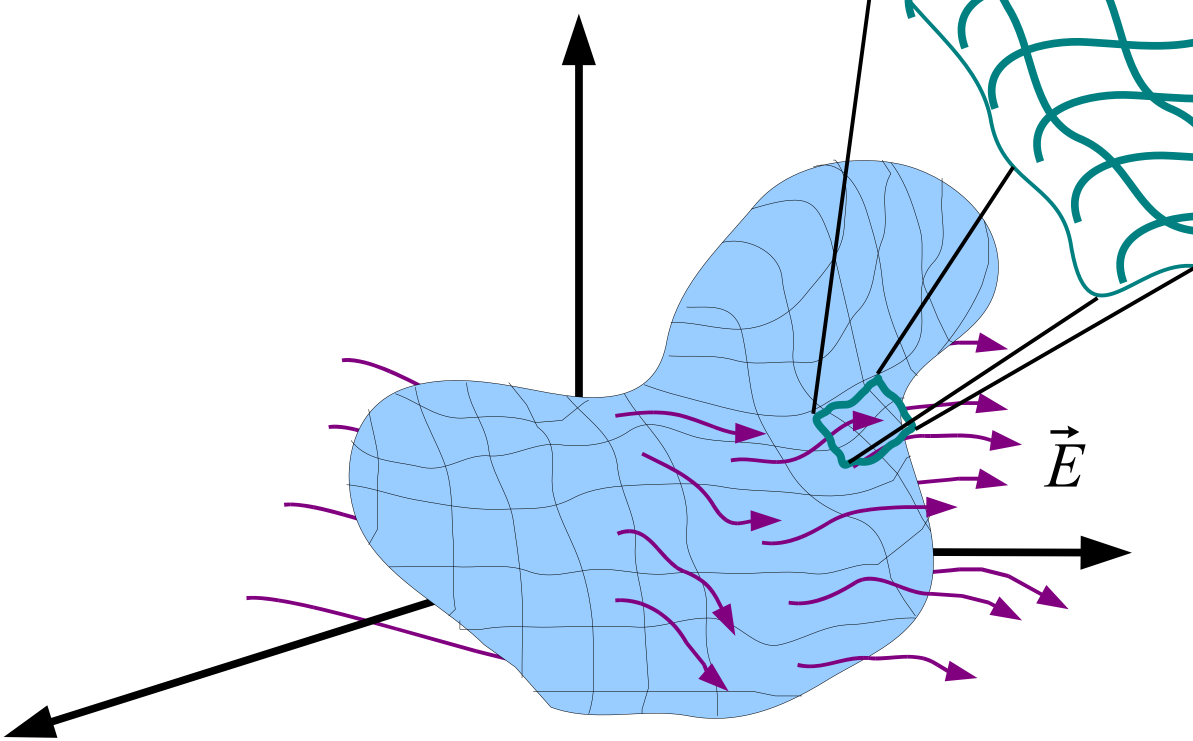
Iii) integrate

$$\int\int |\vec{E}| r^2 \sin\theta \, d\phi \, d\theta \rightarrow \oint |\vec{E}| dS$$

Generalize this formula

This mesh represents a portion of a closed surface enlarged

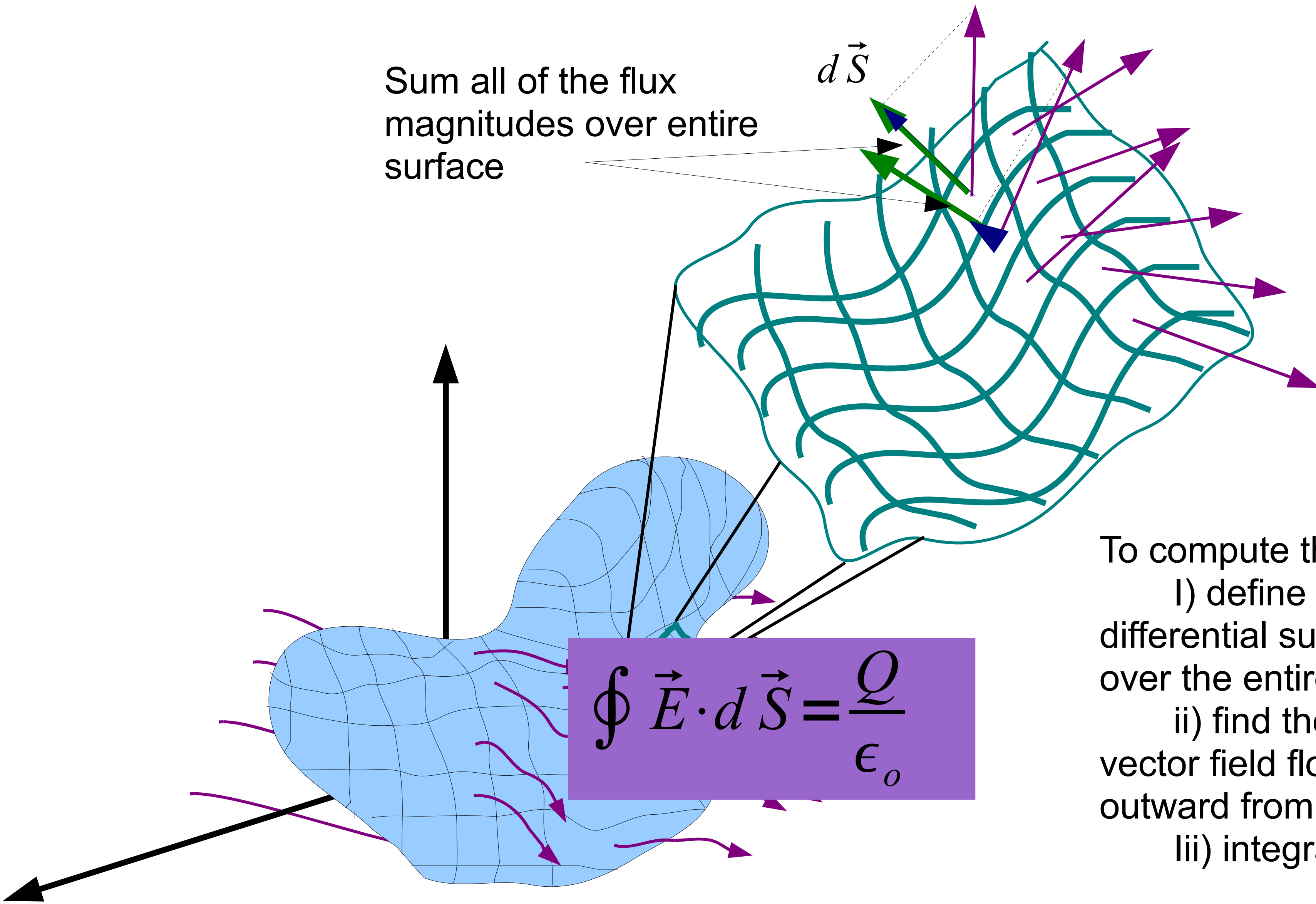




To compute the flux –
I) define the differential surface vector over the entire surface.
ii) find the amount of vector field flowing outward from the surace
Iii) integrate

Generalize this formula

This mesh represents a portion of a closed surface enlarged



$$\oint \vec{E}\cdot d\vec{S} = \frac{Q}{\epsilon_o}$$

To compute the flux –

I) define the differential surface vector over the entire surface.

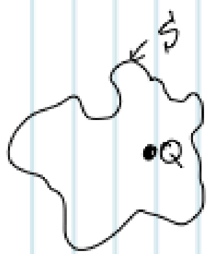
ii) find the amount of vector field flowing outward from the surace

Iii) integrate

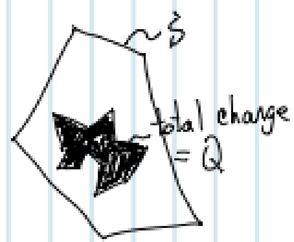
THIS is Exactly Gauss' LAW

$$\oint_{S} \underline{E} \cdot d\underline{s} = \frac{Q}{\varepsilon_0}$$

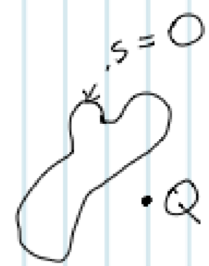
Except that Q can be any arbitrary charge distribution, and S can be any surface



$\oint \underline{E} \cdot d\underline{s} = Q$



$\oint \underline{E} \cdot d\underline{s} = Q$



$\oint \underline{E} \cdot d\underline{s} = ?$

So $\oint \underline{E} \cdot d\underline{l} =$

encloses charge –

$= Q$ ← total charge enclosed.

? why does this integral only depend on charges INSIDE surface

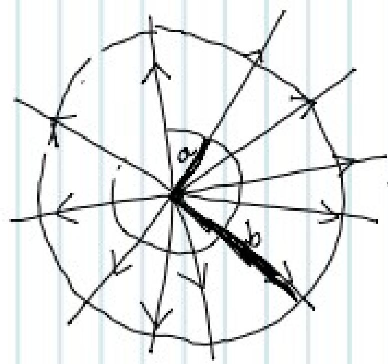
does not enclose charge –

$= 0$

? WHY?

the answer is – the inverse-square law!

Look at point charge:



$$\underline{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

– on the surface of a sphere of radius a

$$\underline{E} = \frac{Q}{4\pi\varepsilon_0 a^2}\hat{r}$$

and

$$\oint_S \frac{Q}{4\pi\varepsilon_0 a^2}\hat{r}\cdot a^2 \sin\theta \, d\theta \, d\phi \, \hat{r} = \frac{Q}{4\pi\varepsilon_0 a^2}\cdot \underbrace{4\pi a^2}_{\text{surface area of sphere.}}$$

$$= \frac{Q}{4\pi\varepsilon_0}\underbrace{\int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\theta \, d\phi}_{\text{integrates to } 4\pi} = \boxed{\frac{Q}{\varepsilon_0}}$$

– on the surface of a sphere of radius b

$$\underline{E} = \frac{Q}{4\pi\varepsilon_0 b^2}\hat{r} \Rightarrow \oint_{\text{sphere } r=b} \underline{E}\cdot d\underline{s} = \boxed{\frac{Q}{\varepsilon_0}}$$

Both integrals have the same value—

$$\oint_{\substack{\text{sphere of}\\ \text{radius } a}} \bar{E} \cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi} \frac{Q}{4\pi\varepsilon_0 a^2}\, a^2 \sin\theta \, d\theta \, d\phi = \frac{Q}{\varepsilon_0}$$

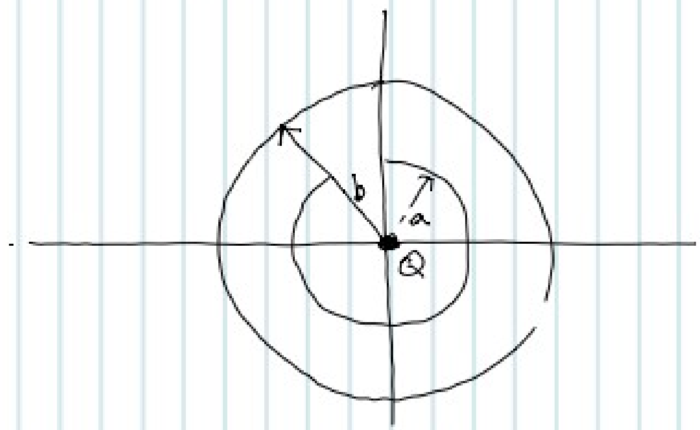
For a POINT CHARGE

$$\oint_{\substack{\text{sphere of}\\ \text{radius } b}} \bar{E} \cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi} \frac{Q}{4\pi\varepsilon_0 b^2}\, b^2 \sin\theta \, d\theta \, d\phi = \frac{Q}{\varepsilon_0}$$

which we expect because in general

$$\oint \bar{E} \cdot d\bar{s} = \frac{Q}{\varepsilon_0}$$

Now consider this surface —



Now $\oint \bar{E} \cdot d\bar{s} = \oint_{\substack{\text{sphere} \\ r=a}} \bar{E} \cdot d\bar{s} + \oint_{\substack{\text{sphere} \\ r=a}} \bar{E} \cdot d\bar{s}$

$$\oint_{\substack{\text{sphere} \\ r=a}} \bar{E} \cdot d\bar{s} = -\frac{Q}{\varepsilon_0} \quad \text{AND} \quad \oint_{\substack{\text{sphere} \\ r=b}} \bar{E} \cdot d\bar{s} = \frac{Q}{\varepsilon_0}$$

why the minus sign? Before it was a +

$$\oint_{\substack{\text{shell} \\ r_i=a \\ r_o=b}} \bar{E} \cdot d\bar{s} = 0$$

DEFINE:

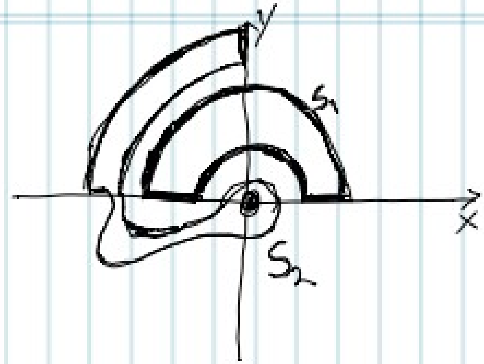
Flux — associate with electric field a measure of the "stuff" flowing at each point —

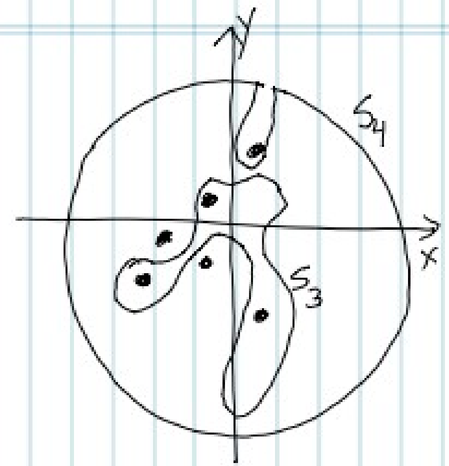
$\Rightarrow \varepsilon_0 \oint \bar{E} \cdot d\bar{s} =$ total Electric flux

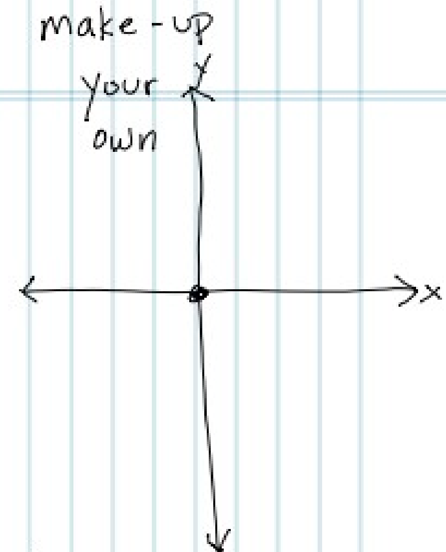
WHAT DOES this equation tell us?

If we have a vector field generated by sources then - if we look at any closed surface in space if the net amount of flux INTO the surface is NOT zero => surfaces contain sources!

If there are No sources inside the closed surface => the flux going into a surface must equal that leaving the surface.





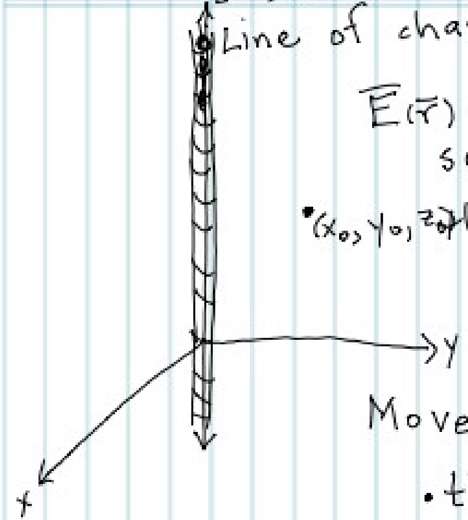


You can convince yourself that

$$\varepsilon_0 \oint_S \bar{E} \cdot d\bar{s} = Q$$

for any charge distribution and any surface if you remember that ANY charge distribution can be reduced to a sum of point charges ___

INFINITE Line of charge



$\bar{E}(\bar{r})$ there is alot of symmetry associate with this charge configuration

microscopic ↕ macro-scopic

Move up and down - in the z-direction

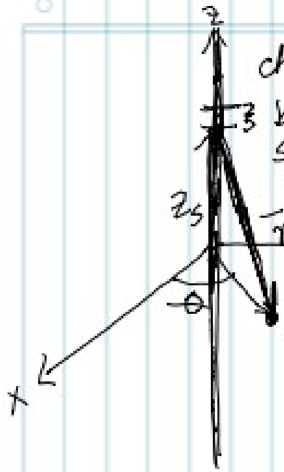
- the line of charge looks the same no matter what your z_0

$\Rightarrow \bar{E}$ NOT a function of z.

Move around the line in the cyl. or spherical ϕ direction.

- again the line looks the same from all angles

$\Rightarrow \bar{E}$ NOT a function of r (cyl. r)



charge density

break line up into small pieces. Assume each piece is a point source.

$(x_0, y_0, 0)$ ⇐ find electric field here

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

⇓

translate to this geometry using

— POSITION VECTORS —

$\bar{r}_s$ = piece of line position vector = $x_0 \cos\phi \, \hat{x} + y_0 \sin\phi \, \hat{y}$

$\bar{r}_0$ = observation point position vector = $z_s z$

vector pointing from charge to observation point

$$\bar{r}_0 - \bar{r}_s = x_0 \cos\phi \, \hat{x} + y_0 \sin\phi \, \hat{y} - z_s \hat{z}$$

$$\Rightarrow |\bar{r}_0 - \bar{r}_s| = \sqrt{x_0^2 + y_0^2 + (-z_s)^2} = \underbrace{\sqrt{r^2 + z^2}}_{\text{standard notation}}$$

$$d\vec{E}(x_0, y_0, 0) = \frac{\lambda\, dz}{4\pi\varepsilon_0 (r^2+z^2)} \frac{x_0 \hat{x} + y_0 \hat{y} - z\hat{z}}{\sqrt{r^2+z^2}}$$

contribution of 1 small piece located at z

To find the total field:

$$\vec{E}(x_0, y_0, 0) = \int_{-\infty}^{\infty} \frac{\cos\phi\, \lambda x_0\, dz}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} \hat{x} + \int_{-\infty}^{\infty} \frac{\sin\phi\, \lambda y_0\, dz}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} \hat{y}$$

$$+ \int_{-\infty}^{\infty} \frac{\lambda z\, dz}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}$$

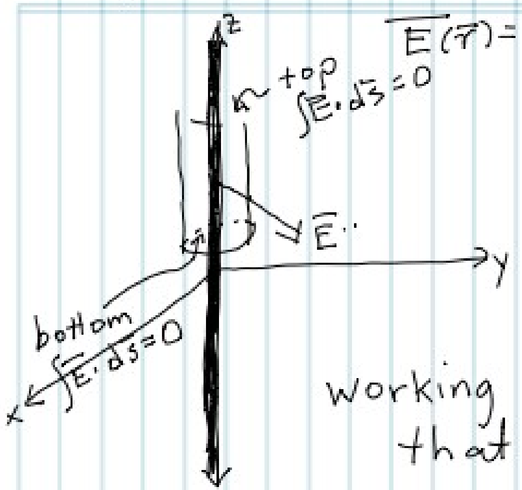
→ EXPECT this to be ZERO

WHY ??

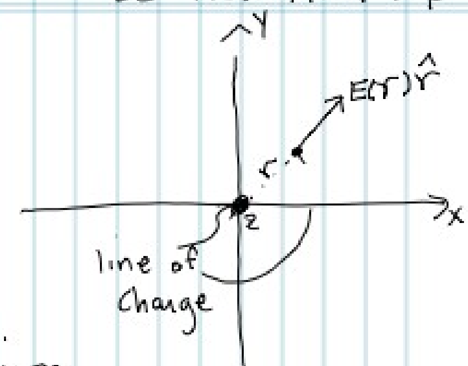
and can rewrite remaining 2 integrals

$$= \int_{-\infty}^{\infty} \frac{\lambda r \hat{r}\, dz}{(r^2+z^2)^{3/2}(4\pi\varepsilon_0)} = \boxed{\frac{\lambda}{2\pi\varepsilon_0 r} \hat{r}}$$

Find $\overline{E}(\overline{r})$ USING Gauss' Law



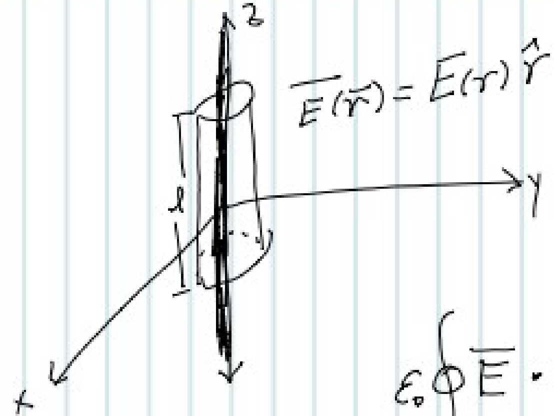
2D view from top



working with this assumption that $\overline{E}(\overline{r}) = E(r)\,\hat{r}$ we can compute the total electric flux through a surface

$$\oint \overline{E}\cdot d\overline{s} = \oint E(r)\,\hat{r}\cdot d\overline{s}$$

Pick a highly symmetric surface like a cylinder centered on the line of charge.



$\overline{E}(\overline{r}) = E(r)\,\hat{r}$

$$\oint_{cylinder} \overline{E}\cdot d\overline{s} = \underbrace{\int_{top} \overline{E}\cdot d\overline{s}}_{0} + \underbrace{\int_{bottom} \overline{E}\cdot d\overline{s}}_{0} + \int_{side} \overline{E}\cdot d\overline{s}$$

WHY?

WHY?

$$\varepsilon_0 \oint \overline{E}\cdot d\overline{s} = \varepsilon_0 \int_{\substack{side\ of\\ cylinder}} E(r)\,\hat{r}\cdot d\overline{s} = \varepsilon_0 E(r)\cdot(\text{area of side})$$

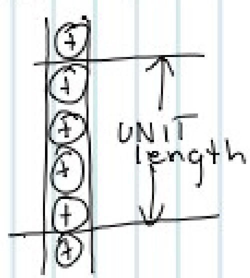
$$= \varepsilon_0 E(r)\, 2\pi r \cdot l$$

makes valid for all r

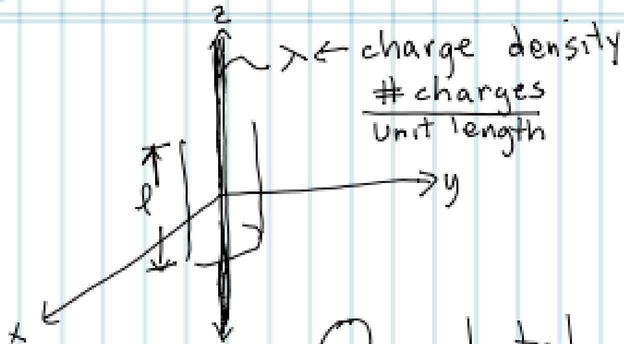
Now $\varepsilon_0 E(r)\, 2\pi r l$ = (total charge enclosed by cylinder)

Compute total charge

Expand z-axis



a continuos density is usually specified by a number designating how changes are in a UNIT - in this case length.



Q = total charge enclosed in cylinder

$$= \lambda \, \frac{\text{charge}}{\cancel{\text{unit length}}} \cdot \ell \, \cancel{\text{unit length}} = \lambda \ell$$

Equating $\varepsilon_0 \oint \bar{E}(\bar{r}) \cdot d\bar{s} = Q$

$$\varepsilon_0 E(r) \, 2\pi r \cancel{\ell} = \lambda \cancel{\ell}$$

$$E(r) = \frac{\lambda}{2\pi \varepsilon_0 r} \quad \text{or} \quad \boxed{\bar{E}(\bar{r}) = \frac{\lambda}{2\pi \varepsilon_0 r} \hat{r}}$$