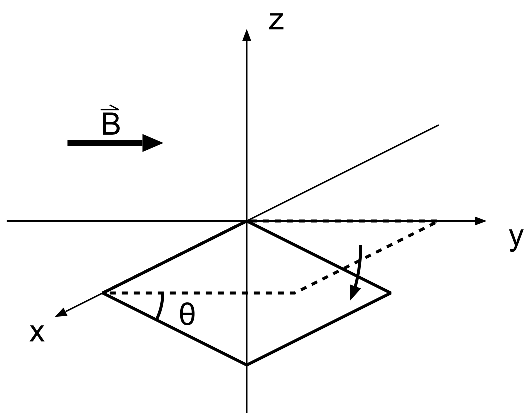
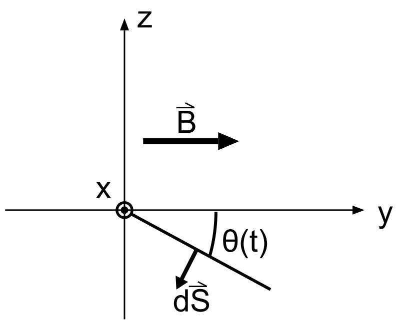
1. Consider a square loop of wire of some finite resistance R with $4\,\mathrm{cm}^2$ surface area that is located in a region of constant magnetic field $\mathbf{B} = 8\hat{y}\,\mathrm{Wb/m}^2$. The loop can rotate around the x-axis as shown in the following figure.





a) If $\theta = 0°$, the area projection onto xz-plane is 0, and therefore, the magnetic flux is 0.
If $\theta = 90°$, the area projection onto xz-plane is $4\,\mathrm{cm}^2$, and $d\mathbf{S} = -dS\hat{y}$, and therefore, the magnetic flux is

$$\Psi = \int_S \mathbf{B}\cdot d\mathbf{S} = \mathbf{B}\cdot\int_S d\mathbf{S} = 8\hat{y}\,\frac{\mathrm{Wb}}{\mathrm{m}^2}\cdot(-4\times10^{-4}\,\mathrm{m}^2\hat{y}) = -32\times10^{-4}\,\mathrm{Wb}.$$

b) For any angle θ, the differential area is $d\mathbf{S} = -dS\left(\cos\theta\hat{z} + \sin\theta\hat{y}\right)$, and therefore, the magnetic flux is

$$\Psi = \int_S \mathbf{B}\cdot d\mathbf{S} = 4\hat{y}\cdot\left(-8\times10^{-4}\left(\cos\theta\hat{z}+\sin\theta\hat{y}\right)\right) = -32\times10^{-4}\sin\theta\,\mathrm{Wb}.$$

c) The induced emf is given by:

$$\mathcal{E} = -\frac{d\Psi}{dt} = 32\times10^{-4}\cos\theta\frac{d\theta}{dt}.$$

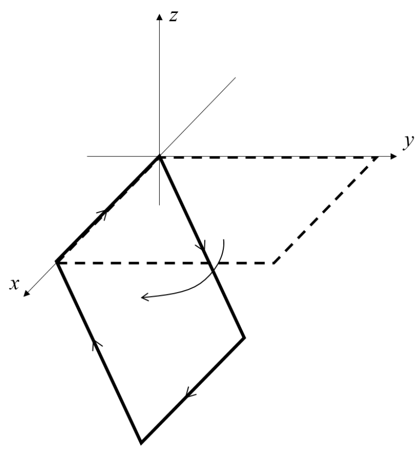
At t = 0.25 s,

$$\theta = \frac{1}{4}\pi$$

Considering $\theta = \frac{\pi}{4}$ rad and $\frac{d\theta}{dt} = \pi$ rad/s, we get

$$\mathcal{E}(\theta = \frac{\pi}{2}) = 32 \times 10^{-4} \left(\cos\frac{\pi}{4}\right)(\pi) = 0.0071\,\text{V}.$$

d) The simple approach to determine the current direction is to remember: current is induced in such a way that the flux associated with the induced current opposes the change of the background flux. At this position, shown in the graph below, the loop rotates towards the xz-plane, which means the area projection onto xz-plane is increasing, which means the flux in the $+\hat{y}$ direction is increasing; as a result, the current is induced in such a way that the induced flux points towards the $-\hat{y}$ direction. Therefore, applying the right-hand rule, the current flow direction is specified in the graph as below.



e) In general you should know that $d\mathbf{S}$ can be assigned arbitrarily without affecting the physical result. Therefore, the answer is that the current flows in the same direction as in part (d). If you would like to think about this problem in terms of vector representations, here $d\mathbf{S}$ changes sign, which means the change of flux also changes sign, and then the induced current also changes "sign". The induced current is used to be in the opposite direction of $d\mathbf{S}$, but now it is in the same direction of $d\mathbf{S}$ (in the sense of right-hand rule, as always).

2.

a) The magnetic field flux through the loop is given by:

$$\Psi = B_0\pi(\frac{1}{2})^2\cos(\theta)$$

and is just the area of the loop multiplied by the strength of the magnetic field and the cosine of the angle made with the yz-plane to account for the changing amount of flux as the loop turns. Because at $\theta = 0$ the direction of the loop is $\hat{x}$ and the direction of the magnetic field is also $\hat{x}$, use cosine because $\theta = 0$ corresponds to a cosine of 1. However, the angle θ is prescribed such that the flux as a function of time is given by:

$$\Psi = B_0\pi(\frac{1}{2})^2\cos(\pi\sin(\pi t))$$

Take a negative time derivative to obtain the instantaneous emf:

$$\varepsilon = -\frac{\partial\Psi}{\partial t} = B_0\frac{\pi^3}{4}\sin(\pi\sin(\pi t))\cos(\pi t)$$

Plugging in $t = 0.3$ and setting the equation equal to the given value for emf at this time gives:

$$10^{-4} = B_0\frac{\pi^3}{4}\sin(\pi\sin(0.3\pi))\cos(0.3\pi)$$

$$B_0 = \frac{4\cdot 10^{-4}}{\pi^3\sin(2.542)\cos(0.943)}$$

$$B_0 = 3.888\times 10^{-5}\text{Wb/m}^2$$

b) A modification of the second equation for part (a) is all that is necessary here:

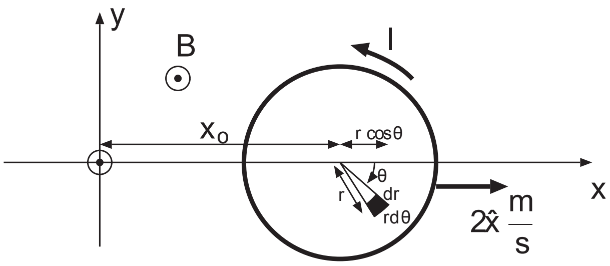
$$\Psi = B_0\cos(\frac{\pi}{2}t)\cdot\pi(\frac{1}{2})^2\cos(\pi\sin(\pi t))$$

c) Take a negative time derivative to obtain the instantaneous emf:

$$\varepsilon = -\frac{\partial\Psi}{\partial t} = \frac{\pi^2 B_0(2\pi\cos(\frac{\pi t}{2})\cos(\pi t)\sin(\pi\sin(\pi t)) + \sin(\frac{\pi t}{2})\cos(\pi\sin(\pi t)))}{8}$$

Plugging in $t = 4$, the resulting emf is just 0 V.

3. The geometry of the problem is shown in the figure below.



a) Looking at the described geometry in cylindrical coordinate, the magnetic flux is given by

$$
\begin{aligned}
\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} &= \int_0^r \int_0^{2\pi} 25 \times 10^{-6} \left(1 - \frac{x_o + r' \cos\theta}{L}\right) r' d\theta dr' \\
&= 25 \times 10^{-6} \left(\int_0^r \int_0^{2\pi} r' d\theta dr' - \int_0^r \int_0^{2\pi} \frac{x_o}{L} r' d\theta dr' - \int_0^r \int_0^{2\pi} \frac{r' \cos\theta}{L} r' d\theta dr' \right) \\
&= 25 \times 10^{-6} \left(\pi r^2 - \frac{x_o}{L} \pi r^2 - \int_0^r r' dr' \int_0^{2\pi} \frac{\cos\theta}{L} d\theta \right) \\
&= 25 \times 10^{-6} \left(\pi r^2 - \frac{x_o}{L} \pi r^2 - 0 \right) \\
&= 25 \times 10^{-6} \pi r^2 \left(1 - \frac{x_o}{L} \right)
\end{aligned}
$$

Thus, the emf $\mathcal{E}$ is

$$
\begin{aligned}
\mathcal{E} &= -\frac{d\Psi}{dt} = 25\pi r^2 \times 10^{-6} \times \frac{1}{L} \times \frac{dx_o}{dt} = 25\pi \times 10^{-6} \times \frac{1}{1000} \times 2 \\
&\approx 157.08\,\text{nV}.
\end{aligned}
$$

b) The magnitude of the loop current with resistance $2\,\Omega$ is

$$
I = \frac{|\mathcal{E}|}{R} = 78.54\,\text{nA}.
$$

4.

a) At time t, the area of the loop= $L(z_0+v_0t)$; for clockwise contour (following ABC), $d\mathbf{S} = dS(-\hat{x})$.

$$\begin{aligned}\Psi &= \int_S \mathbf{B}\cdot d\mathbf{S} \\ &= B_0L(z_0+v_0t) \\ \mathcal{E} &= -\frac{d\Psi}{dt} = -B_0Lv_0\,\text{V}\end{aligned}$$

b) $\mathcal{E} < 0$ so the current flows opposite to the original contour, i.e. along CBA (counterclockwise),

$$\begin{aligned}I_0 &= \frac{\mathcal{E}}{R} = \frac{-B_0v_0L}{R}\,\text{A} \\ |I_0| &= \frac{B_0v_0L}{R}\,\text{A}\end{aligned}$$

c) Lorentz force per unit length

$$\begin{aligned}d\mathbf{F} &= I_0 d\mathbf{l}\times\mathbf{B} \\ &= I_0(dy\hat{y})\times B_0(-\hat{x}) \\ &= I_0B_0dy\hat{z}\end{aligned}$$

Therefore the total $\mathbf{F}$

$$\mathbf{F} = \int d\mathbf{F} = \int_0^L I_0B_0dy\hat{z} = I_0B_0L\hat{z}\,\text{N}$$

For I_0 as above

$$\mathbf{F} = \frac{B_0^2v_0L^2}{R}(-\hat{z})\,\text{N}$$

d) Since the current flows along the contour ABC, I' is in the $+\hat{y}$ direction in the armature. Similar to part (c), at the moment right when the mechanical force is removed:

$$\begin{aligned}\mathbf{F} &= I'L\hat{y}\times B_0(-\hat{x}) \\ &= I'B_0L\hat{z}\,\text{N} \\ \mathbf{a} &= \frac{I'B_0L}{M}\hat{z}\,\text{m/s}^2\end{aligned}$$

Alternatively, we could find the total force by superposing the Lorentz forces on each charge in the armature:

$$\mathbf{F} = NLAq\,(\mathbf{v}\times\mathbf{B})$$

where N is the number density of the charges in the armature, A is its cross-sectional area, and LA is its volume. Since $\mathbf{J} = qN\mathbf{v} = \frac{I'}{A}\hat{y}$, we can arrive at the same result above.

5.

a) Pick an arbitrary angle ϕ, and draw $\hat{s}$. As ϕ increases, the x-component of $\hat{s}$ goes negative and the y-component positive. Therefore, we can write $\hat{s} = \hat{y}\cos\phi - \hat{x}\sin\phi$. As a check, when $\phi = 0$, $\hat{s} = \hat{y}$, which is what our formula evaluates to. When $\phi = \pi/2$, $\hat{s} = -\hat{x}$.

b)

$$\begin{aligned}\Psi = \int_S \mathbf{B} \cdot d\mathbf{s} = B \int_S \hat{x} \cdot \hat{s} ds \\ = B \int_S (-\sin\phi) ds \\ = -B\sin\phi \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} dy' dx' \\ = -Bab\sin\phi.\end{aligned}$$

c) Since $\phi = \omega t$, we can write

$$\Psi(t) = -Bab\sin\omega t,$$

and by definition,

$$\mathcal{E} = -\frac{\partial \Psi}{\partial t} = Bab\omega\cos\omega t.$$

d) The current through the loop is found by

$$I = \mathcal{E}/R = \frac{Bab\omega}{R}\cos\omega t.$$

As ϕ increases slightly from zero, the magnetic flux through the loop decreases (becomes more negative), which will induce a current in the loop. By Lenz' law, the induced current will take on a direction such that it induces a magnetic field to oppose the original, meaning the current is defined as flowing counter-clockwise from the perspective of (A) in the problem set. Continuing the convention of counter-clockwise I, when ϕ surpasses π (one-half rotation), the magnetic flux will change sign as the $\hat{x}$ component of $\hat{s}$ will be positive, pointing in the same direction as $\mathbf{B}$. At this point, the current will be flowing clockwise until $\phi = 2\pi$, when it will cycle back again. This makes sense, as $I(t)$ as defined above will be negative for $t \in [\pi, 2\pi)$, and a negative current flowing counter-clockwise is equivalent to a positive current flowing clockwise. The plot below shows $I(t)$ on the interval $t \in [0, 2\pi)$.