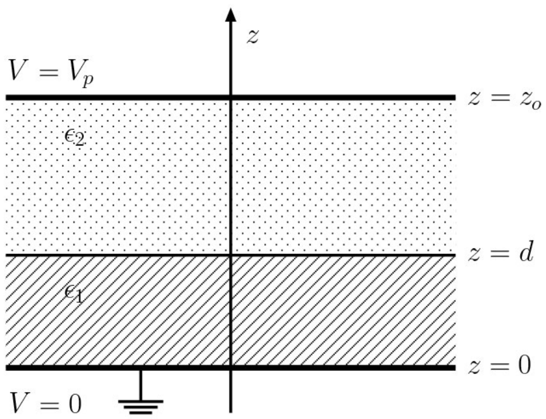
1.

a) Considering the medium in each slab to be homogeneous, we can refer to Laplace's equation, $\nabla^2 V = 0$. Hence, the potential in each dielectric slab will be written

$$V(z) = \begin{cases} A_1 z + B_1 & , \quad 0 < z < d \\ A_2 z + B_2 & , \quad d < z < z_0 \end{cases}$$

from which we find the electric field $\mathbf{E}$ as

$$\mathbf{E}(z) = -\nabla V(z) = \begin{cases} -A_1 & , \quad 0 < z < d \\ -A_2 & , \quad d < z < z_0 \end{cases}$$



Given that $V = 0$ at $z = 0$, we will have $B_1 = 0$. Similarly, since $V = V_p$ at $z = z_0$, we get

$$V_p = A_2 z_o + B_2$$

The boundary conditions state that the electrostatic potential field must be continuous along the interface between the two dielectrics. Thus, we can write

$$A_2 d + B_2 = A_1 d \Rightarrow A_2 d + B_2 - A_1 d = 0$$

for the potential at $z = d$.

Applying the other boundary condition stating that there must be no change between the normal components of the displacement vector $\mathbf{D}$ within the two dielectrics due to the fact that there are no mobile free carriers along it, we can write

$$\begin{aligned} \hat{n} \cdot (\mathbf{D_1} - \mathbf{D_2}) &= (-\hat{z}) \cdot (\epsilon_1(-A_1\hat{z}) - \epsilon_2(-A_2\hat{z})) \\ 0 &= \epsilon_1 A_1 - \epsilon_2 A_2 \end{aligned}$$

Using the last three equations, we find

$$\begin{aligned} A_1 &= \frac{\epsilon_2 V_p}{z_0\epsilon_1 + d(\epsilon_2 - \epsilon_1)} \\ A_2 &= \frac{\epsilon_1 V_p}{z_0\epsilon_1 + d(\epsilon_2 - \epsilon_1)} \\ B_2 &= \frac{d(\epsilon_2 - \epsilon_1) V_p}{z_0\epsilon_1 + d(\epsilon_2 - \epsilon_1)} \end{aligned}$$

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from which the electric potential can be written as

$$V(z) = \begin{cases} \frac{\epsilon_2 V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} z & , \quad 0 < z < d \\ \frac{\epsilon_1 V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} z + \frac{d(\epsilon_2-\epsilon_1)V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} & , \quad d < z < z_0 \end{cases}$$

b) The electric field inside the dielectrics is given by

$$\mathbf{E}(z) = -\nabla V(z) = \begin{cases} -\frac{\epsilon_2 V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} \hat{z} & , \quad 0 < z < d \\ -\frac{\epsilon_1 V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} \hat{z} & , \quad d < z < z_0 \end{cases}$$

For $0 < z < d$, we have

$$V_p = \frac{z_0\epsilon_1 + d(\epsilon_2 - \epsilon_1)}{\epsilon_2} E_z$$

Given that $z_0 = 4d = 2m, \epsilon_1 = 3\epsilon_0, \epsilon_2 = \epsilon_0$ and $\mathbf{E}(0 < z < d) = -5\hat{z}$, we find that

$$V_p = 25V$$

c) Given that $z_0 = 4d = 2m, \epsilon_1 = 3\epsilon_0, \epsilon_2 = \epsilon_0$ and $V_p = 25V$, we have

$$\mathbf{E}(z) = \begin{cases} -5\hat{z} & , \quad 0 < z < d \\ -15\hat{z} & , \quad d < z < z_0 \end{cases}$$

The surface charge density at $z = z_0$ is given by

$$\begin{aligned} \rho_s(z_0) &= \mathbf{D} \cdot \hat{n}|_{z=z_0} \\ &= D_z^+(z_0) - D_z^-(z_0) \end{aligned}$$

where $D_z^+(z_0) = 0$. Therefore, we can write

$$\rho_s(z_0) = -D_z^-(z_0) = -\epsilon_2 E_z(z_0) = 15\epsilon_0 \frac{C}{m^2}$$

d) Laplace's equation

$$\nabla^2 V = 0$$

results from the assumption that the permittivity is constant in space. In our case, however, there are two dielectrics in the region $0 < z < z_o$, which implies that the medium is not homogeneous within this region. Thus, $V(z)$ does not satisfy Laplace's equation at all points in the region (in fact, in this particular case, the equation is not satisfied only at $z = d$).

e) Knowing that $C = \frac{Q}{V}$ where $Q = \rho_s A$ and $V = |E_{1z}| d + |E_{2z}| (z_o - d)$, we have

$$C = \frac{Q}{V} = \frac{\rho_s A}{|E_{1z}| d + |E_{2z}| (z_o - d)}.$$

Using Maxwell's boundary conditions, we find that $\rho_s = \epsilon_2 |E_{2z}| = \epsilon_1 |E_{1z}|$. Then, the capacitance can be re-expressed as

$$C = \frac{A}{\frac{d}{\epsilon_1} + \frac{z_0 - d}{\epsilon_2}} = \frac{\epsilon_1\epsilon_2 A}{\epsilon_1 z_o + (\epsilon_2 - \epsilon_1)d}.$$

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Note: The same result can be obtained by combining the capacitances of each dielectric in series. Given that

$$C_1 = \epsilon_1 \frac{A}{d} \qquad \text{and} \qquad C_2 = \epsilon_2 \frac{A}{z_o - d},$$

we obtain

$$C = \left(C_1^{-1} + C_2^{-1}\right)^{-1} = \left(\frac{d}{\epsilon_1 A} + \frac{z_o - d}{\epsilon_2 A}\right)^{-1} = \left(\frac{\epsilon_1 z_o + (\epsilon_2 - \epsilon_1)\, d}{\epsilon_1 \epsilon_2 A}\right)^{-1} = \frac{\epsilon_1 \epsilon_2 A}{\epsilon_1 z_o + (\epsilon_2 - \epsilon_1)\, d}.$$

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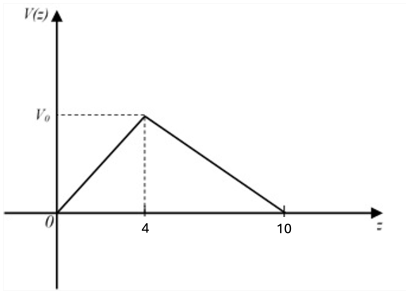
2. The plates at $z = 0$ and $z = 10m$ are both grounded and have equipotentials, i.e. $V = 0$. Referring to the hint, we can show the potential change by the distance as in the below figure,



where the lines are drawn straight since both media are homogeneous. Hence, applying the Laplace's equation, the potential in each slab will be given by

$$V(z) = \begin{cases} \frac{1}{4}V_0 z & , 0 < z < 4\,\text{m} \\ -\frac{V_0}{6}z + \frac{5}{3}V_0 & , 4\,\text{m} < z < 10\,\text{m}. \end{cases}$$

From $\mathbf{E} = -\nabla V$, we find

$$\mathbf{E} = -\nabla V = \begin{cases} -\frac{1}{4}V_0 \hat{z}\,\frac{\text{V}}{\text{m}} & , 0 < z < 4\,\text{m} \\ \frac{V_0}{6}\,\hat{z}\,\frac{\text{V}}{\text{m}} & , 4\,\text{m} < z < 10\,\text{m}. \end{cases}$$

Making use of the boundary condition for the interface at $z = 4\,\text{m}$, we write

$$\begin{aligned} \hat{z} \cdot \left(\mathbf{D}^+_{z=4} - \mathbf{D}^-_{z=4}\right) &= 8\epsilon_0 \frac{\text{C}}{\text{m}^2}, \\ 3\epsilon_0 \frac{V_o}{6} - \left(-\frac{1}{4}\epsilon_0 V_0\right) &= 8\epsilon_0 \frac{\text{C}}{\text{m}^2} \end{aligned}$$

from which we find $V_o = \frac{32}{3} \approx 10.67\,\text{V}$. Now, we can apply the same boundary condition to the interfaces at $z = 0$ and $z = 10\,\text{m}$, respectively. Due to the fact that $\mathbf{D} = 0$ for the exterior region, we write

$$D_{z<4} = \rho_{z=0} \Rightarrow -\frac{1}{4}\epsilon_0 V_0 = \rho_{z=0}$$

from which we find the surface charge density at $z = 0$ as $\rho_{z=0} = -\frac{8}{3}\epsilon_0\,\frac{\text{C}}{\text{m}^2}$. Likewise,

$$-D_{z>4} = \rho_{z=10} \Rightarrow -3\epsilon_0 \frac{V_0}{6} = \rho_{z=10}$$

from which we find the surface charge density at $z = 10\,\text{m}$ as $\rho_{z=10} = -\frac{16}{3}\epsilon_0\,\frac{\text{C}}{\text{m}^2}$.

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3.

a) In vacuum, the displacement vector is $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$. Thus, the displacement field between the plates is

$$\mathbf{D} = 3\epsilon_o \hat{z}\, \frac{\mathrm{C}}{\mathrm{m}^2},$$

from which we obtain the polarization field as $\mathbf{P} = 0\, \frac{\mathrm{C}}{\mathrm{m}^2}$.

b) Since $\mathbf{D} = \mathbf{0}$ for $z < 0$, i.e. $\mathbf{D}\,|_{z=0^-}$ the surface charge on the plate at $z = 0$ is

$$\rho_s|_{z=0} \quad = \quad \hat{z} \cdot (\mathbf{D}\,|_{z=0^+} - \mathbf{D}\,|_{z=0^-}\,) = 3\epsilon_o\, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

c) If the gap is filled with a dielectric of permittivity $\epsilon = 81\epsilon_o$ without changing the surface charge density then the displacement field will remain the same, i.e.,

$$\mathbf{D} = 3\epsilon_o\, \hat{z}\, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

But, the electric field is now

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{27}\, \hat{z}\, \frac{\mathrm{V}}{\mathrm{m}}.$$

Consequently, the polarization field becomes

$$\mathbf{P} = \mathbf{D} - \epsilon_o \mathbf{E} = \frac{80}{27}\epsilon_o\, \hat{z}\, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

d) If the medium in the gap has a finite conductivity, then it will also have $\mathbf{E} = 0$ in "steady-state".Thus, $\mathbf{D} \to 0$ and $\mathbf{P} \to 0$. Because, the mobile free charges within the medium in the gap will be pushed and pulled to pile up at the surfaces until the surface charge density generates a secondary field that cancels out the fields wtihin the medium. In this particular case, the salt water shorts out the original field between the plates.

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4. The solution of the problem will be given region by region.

- The region defined by $r \leq a$ is occupied by a conductor with $\sigma = 10^6\,\mathrm{S/m}$, therefore, we can directly write

$$\mathbf{D} = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}, \qquad \mathbf{E} = \mathbf{0}\,\frac{\mathrm{V}}{\mathrm{m}}, \qquad \mathbf{P} = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

  for this particular region. In steady-state, charges can accumulate only on the surface of conducting materials. Since this material holds a net charge per unit length $Q = 2\,\frac{C}{m}$, the surface charge density at radius r = a is

$$\rho_s|_{r=a} = \frac{Q}{Circumference} = \frac{2}{2\pi a} = \frac{1}{\pi a}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

- The region defined by $a < r < b$ is occupied by a dielectric with $\epsilon = 10\epsilon_o$. Applying Gauss's law $\oint \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV = Q_{enc}$ and considering $\mathbf{D} = D_r\hat{r}$, we find

$$D.(2\pi rL) = Q_{enc} = 2 \times L$$

$$\mathbf{D} = \frac{2}{2\pi r}\hat{r} = \frac{1}{\pi r}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}$$

  for $a < r < b$. Hence, the electric field $\mathbf{E}$ is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{1}{10\epsilon_o\pi r}\hat{r}\,\frac{V}{\mathrm{m}},$$

  from which we can obtain the polarization $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{D} - \epsilon_o\mathbf{E} = (\frac{1}{\pi r} - \frac{1}{10\pi r})\hat{r} = \frac{9}{10\pi r}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

- As the region defined by $b \leq r \leq c$ has the same material properties as region $r \leq a$, we can write

$$\mathbf{D} = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}, \qquad \mathbf{E} = \mathbf{0}\,\frac{\mathrm{V}}{\mathrm{m}}, \qquad \mathbf{P} = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

  Then, the surface charge density at $r = b$ is given by

$$\begin{aligned}\rho_s|_{r=b} &= \hat{r}.(\mathbf{D}\,|_{r=b^+} - \mathbf{D}\,|_{r=b^-})\,|_{r=b} \\ &= \hat{r}\cdot\left(-\frac{1}{\pi r}\hat{r}\right)\Big|_{r=b} = -\frac{1}{\pi b}\,\frac{\mathrm{C}}{\mathrm{m}^2}.\end{aligned}$$

- The region defined by $c \leq r \leq d$ is occupied by a dielectric with $\epsilon = 2\epsilon_o$. Since the net charge per unit length is $2 - 4 = -2\,\frac{C}{m}$, the surface charge density at radius r = c is

$$\rho_s|_{r=c} = \frac{Q}{\text{Circumference}} = -\frac{2}{2\pi c} = -\frac{1}{\pi c}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

  Applying Gauss's law $\oint \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV = Q$ and considering $\mathbf{D} = D_r\hat{r}$, we find

$$\mathbf{D} = -\frac{2}{2\pi r}\hat{r} = -\frac{1}{\pi r}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}$$

  for $c < r < d$. Hence, the electric field $\mathbf{E}$ is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = -\frac{1}{2\epsilon_o\pi r}\hat{r}\,\frac{V}{\mathrm{m}},$$

  from which we can obtain the polarization $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{D} - \epsilon_o\mathbf{E} = (-\frac{1}{\pi r} + \frac{1}{2\pi r})\hat{r} = -\frac{1}{2\pi r}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

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- Region defined by r > d is free space. Applying Gauss' law and noting that the total charge per unit length enclosed is $-2\,\frac{C}{m}$, we get $\oint \mathbf{D}\cdot d\mathbf{S} = Q = -2\times L\,\frac{C}{m}$ where $d\mathbf{S} = \hat{r}2\pi r$. Therefore, we can write

$$\mathbf{D} = -\frac{1}{\pi r}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}, \qquad \mathbf{E} = -\frac{1}{\pi\epsilon_0 r}\hat{r}\,\frac{\mathrm{V}}{\mathrm{m}}, \qquad \mathbf{P}_4 = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

Consequently, the surface charge density at $r = d$ is given by

$$\begin{aligned} \rho_s|_{r=d} &= \hat{r}.(\mathbf{D}\,|_{r=d^+} - \mathbf{D}\,|_{r=d^-})\,|_{r=d} \\ &= 0\,\frac{\mathrm{C}}{\mathrm{m}^2}. \end{aligned}$$

To find capacitance per unit length, we use $C = \frac{Q}{V}$, where V can be found using $V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{r}$. From previous problems and slides, we know that the field between the center conductor and the shield is given by $\mathbf{E}(r) = \hat{r}\frac{Q/L}{2\pi\epsilon r}V/m$ due to Gauss' Law. We can integrate this field to find $V = V(b) - V(a) = -\int_a^b \frac{Q/L}{2\pi\epsilon r}\hat{r}\cdot\hat{r}dr$, which yields $V = \frac{Q/L}{2\pi\epsilon}\ln(b/a)$. Substituting this into $C = \frac{Q}{V}$, we find that $C = \frac{2\pi\epsilon L}{\ln(b/a)}$F. Dividing by L, we find the capacitance per unit length to be $C = \frac{2\pi\epsilon}{\ln(b/a)}$F/m.

Finding conductance per unit length, while somewhat similar, is more involved. Assume that the current in the coaxial cable has the distribution $\mathbf{J} = \hat{r}\frac{I}{A}A/m^2$,where $A = 2\pi rL$ (surface area). We may then proceed by finding $\mathbf{E}$ using Ohm's Law: $\mathbf{E} = \hat{r}\frac{I}{2\pi rL\sigma_{diel}}$. Finally, we need to integrate the electric field between the inner and outer conductors using $V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{r}$. The resulting integral evaluates to $V = \frac{I}{2\pi L\sigma_{diel}}\ln(b/a)$. Therefore, conductance $G = \frac{1}{R} = \frac{I}{V} = \frac{2\pi L\sigma_{diel}}{\ln(b/a)}$S. Dividing by L, we find the conductance per unit length to be $\frac{2\pi\sigma_{diel}}{\ln(b/a)}$S/m. However, since we are considering perfect dielectrics for the sake of this problem, $\sigma_{diel} = 0$ S/m, so conductance per unit length is also 0 S/m.

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