1. Given that

$$\mathbf{E} = \hat{x}sin(y) + \hat{y}cos(x)$$

Let us calculate the following,

a) Curl of $\mathbf{E}$,

$$\nabla \times \mathbf{E} = \begin{Bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ sin(y) & cos(x) & 0 \end{Bmatrix} = (\frac{\partial}{\partial x}(cos(x)) - \frac{\partial}{\partial y}(sin(y)))\hat{z} = (-sin(x) - cos(y))\hat{z}.$$

Curl of curl of $\mathbf{E}$,

$$\nabla \times (\nabla \times \mathbf{E}) = \begin{Bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -sin(x) - cos(y) \end{Bmatrix} = sin(y)\hat{x} + cos(x)\hat{y}.$$

b) Applying Gauss law(in differential form)

$$\nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E} = \rho,$$

we find that

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = 0.$$

2. Given an electrostatic potential V(x,y,z) $= x^2 - 2V$ in certain region of space, let us calculate the following,

a) Electrostatic field $\mathbf{E}$,

$$\mathbf{E} = -\nabla V = -\frac{\partial}{\partial x}(x^2 - 2)\hat{x} - \frac{\partial}{\partial y}(x^2 - 2)\hat{y} - \frac{\partial}{\partial z}(x^2 - 2)\hat{z} = -2x\hat{x}[\frac{V}{m}].$$

b) Curl of $\mathbf{E}$,

$$\nabla \times \mathbf{E} = \begin{Bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x & 0 & 0 \end{Bmatrix} = 0.$$

c) Divergence of $\mathbf{E}$,

$$\nabla \cdot \mathbf{E} = \frac{\partial}{\partial x}(-2x) = -2.$$

d) Charge density ρ,

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = -2\epsilon_0[\frac{C}{m^3}].$$

3. In electrostatics, we generate a curl-free vector field $\mathbf{E}(x, y, z)$ if we take the gradient of a scalar function $V(x, y, z)$. Therefore, $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = 0$.

The same answer can be reached by calculating:

$$\mathbf{E} = -\nabla V = -2x\hat{x} + z\hat{y} + y\hat{z}$$

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x & z & y \end{vmatrix} = 0$$

4. Starting with the left-hand side of $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, we write

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla \times \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & x+y & 2 \end{vmatrix} \\ &= \nabla \times (2\hat{z}) = 0. \end{aligned}$$

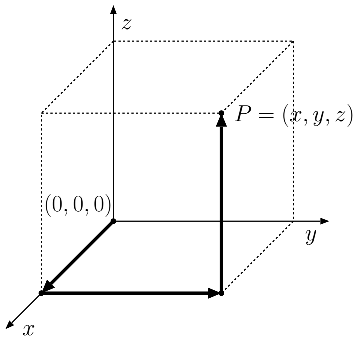
Solving the right-hand side of the equation, we obtain

$$\begin{aligned} \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} &= \nabla (\nabla \cdot ((x-y)\hat{x} + (x+y)\hat{y} + 2\hat{z})) - \nabla^2 ((x-y)\hat{x} + (x+y)\hat{y} + 2\hat{z}) \\ &= \nabla (2) - \mathbf{0} = \mathbf{0}. \end{aligned}$$

Consequently, we can see that $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ is verified since both sides have the same solutions.

5.

a) The electrostatic potential V at any point $P = (x, y, z)$ can be calculated by performing a vector line integral by using the path shown in the below figure.



Therefore, we can write

$$\begin{aligned} V(P) - V(0) &= -\int_0^P \mathbf{E} \cdot d\mathbf{l} \\ &= -\int_0^x E_x(x,0,0)dx - \int_0^y E_y(x,y,0)dy - \int_0^z E_z(x,y,z)dz \\ &= -\frac{43}{2}\,\text{V}. \end{aligned}$$

Given that $V(0) = 0\,\text{V}$, the electrostatic potential at $P = (1,2,3)$ is $V(1,2,3) = -\frac{43}{2}\,\text{V}$.

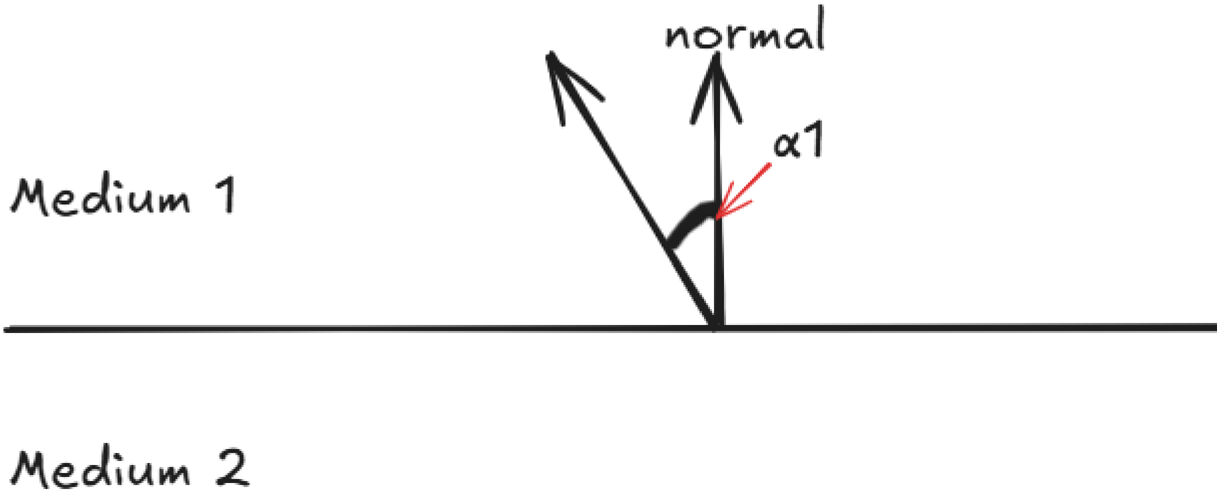
b) Using differential form of Gauss' Law,

$$\rho = \nabla \cdot \mathbf{D} = \epsilon_o(4 + 4z)$$

At (0,0,0), $\rho = 4\epsilon_o\,\frac{C}{m^3}$ and at (1,2,3), $\rho = 16\epsilon_o\,\frac{C}{m^3}$.

6.

a) In this part, consider the following geometry:



We know that the boundary conditions for the electric field at an interface are $\hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = \rho_s$ and $\hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) = 0$. We also know that the electric field magnitude$E_1$is 10 V/m at $\alpha_1 = 30°$from the normal. Therefore, we can compute the components of the field in medium 1.

$$E_{1n} = E_1 \cos(30°) = 5\sqrt{3}V/m \approx 8.66V/m$$

$$E_{1t} = E_1 \sin(30°) = 5V/m$$

Since the tangential component of the electric field is continuous at a boundary, we know that $E_{1t} = E_{2t} = 5V/m$. From the boundary condition on the normal component of the displacement field, $E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{5\sqrt{3}}{4} V/m \approx 2.165V/m$. Therefore, $E_2 = \sqrt{E_{2n}^2 + E_{2t}^2} V/m \approx 5.449V/m$.

The direction of the field in medium 2, α_2, is given by $\tan^{-1}(\frac{E_{2t}}{E_{2n}}) \approx 66.7°$.

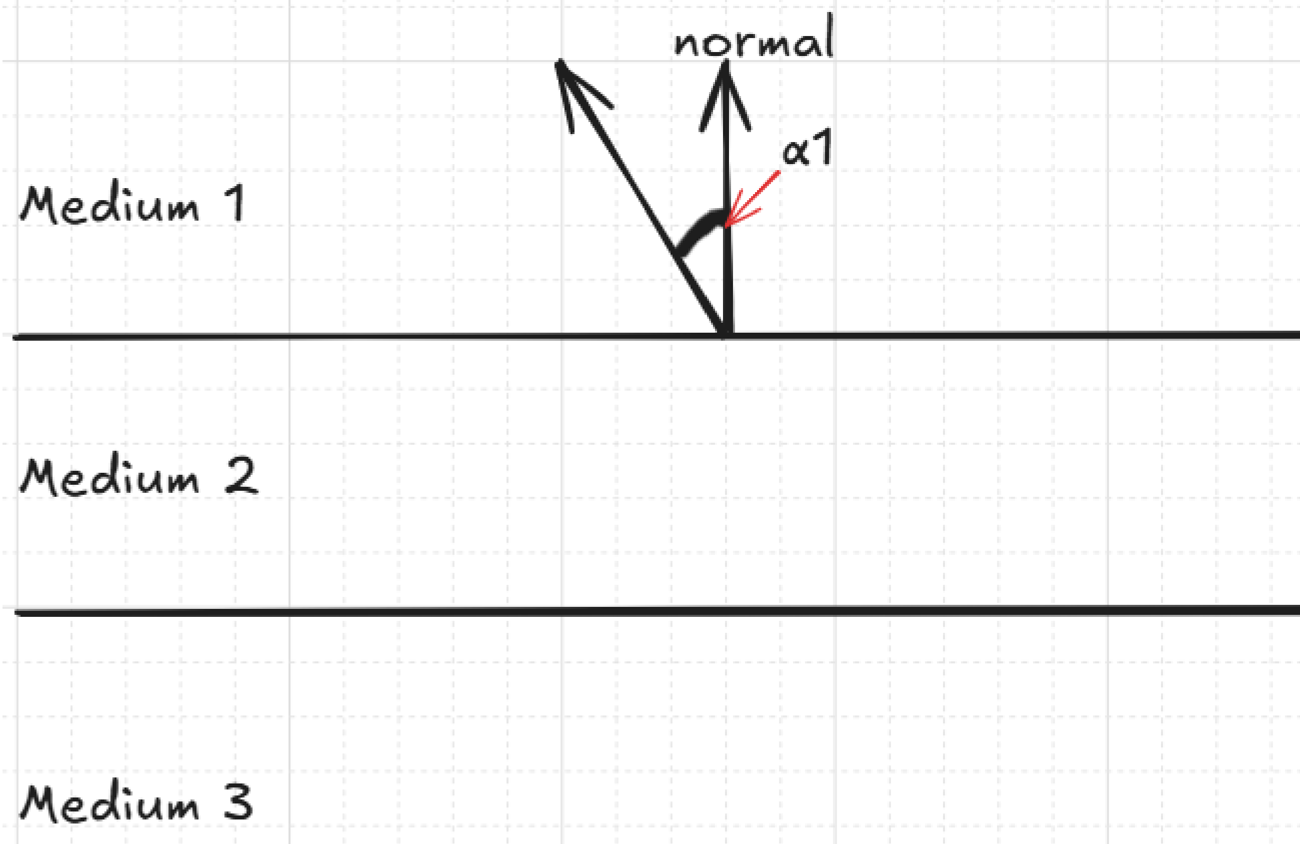
b) Again, consider the geometry from part a).

Now, $|\mathbf{E}_2| = 10\ V/m$, so we must compute its components in order to determine the charge density on the boundary.

Still tangential $E_{2t} = E_{1t} = 5\ V/m$, so normal component $E_{2n} = \sqrt{10^2 - 5^2} \approx 8.66\ V/m$

Evaluating the boundary condition $\rho_s = \epsilon_1 E_{1n} - \epsilon_2 E_{2n} \approx \epsilon_0(10\cos(30°) - 4\epsilon_0 \cdot 8.66 \approx -3 \times 8.66\epsilon_0 \approx 25.98\epsilon_0\ C/m^2$.

c) Now, the geometry has changed from the previous two parts.



It is stated that the boundaries between the media are charge-free, and that medium 2 is now a slab of finite thickness, while medium 3 is free space. To find α_3, the angle from normal of the electric field in medium 3, we must again use boundary conditions.

Since $\rho_{s,23} = 0$, we know that $\epsilon_2 E_{2n} = \epsilon_3 E_{3n}$and $E_{2t} = E_{3t}$ due to the continuity of tangential electric fields at boundaries. Since $\epsilon_2 E_{2n} = \epsilon_3 E_{3n}$, $E_{3n} = \frac{\epsilon_2}{\epsilon_3} E_{2n} = 4 \cdot E_{2n}$. From part a), $E_{2n} = \frac{5\sqrt{3}}{4} V/m$ and $E_{2t} = 5V/m$.

Then we can find E_{3n}as $4 \cdot \frac{5\sqrt{3}}{4} V/m = 5\sqrt{3} V/m$, which is also the value of E_{1n}!

Finally, since $E_{3t} = E_{2t} = 5V/m$, $\alpha_3 = \tan^{-1}(\frac{E_{3t}}{E_{3n}}) = 30°$.